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Multiplicity formulas for representations of transformation groupoids

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Abstract: We study the representations of transitive transformation groupoids with the aim of generalizing the Mackey theory. Using the Mackey theory and a bijective correspondence between the imprimitivity systems and the representations of a transformation groupoid we derive the irreducibility theory. Then we derive the direct sum decomposition for representations of a groupoid together with the formula for the multiplicity of subrepresentations. We discuss a physical interpretation of this formula. Finally, we prove the claim analogous to the Peter–Weyl theorem for a noncompact transformation groupoid. We show that the representation theory of a transitive transformation groupoids is closely related to the representation theory of a compact groups.

Keywords: Groupoids, Induced representations, Imprimitivity systems

MSC: 22A22, 22A30, 22D30

1 Introduction

The groupoid representation theory was initiated by Westman [1] and investigated by many authors [2–8].

Many articles by Heller, Sasin and Pysiak i.a. [9–12] were devoted to the model unifying gravity theory with quantum mechanics. In this model the transformation groupoid of the principal bundle of Lorentz frames over the spacetime was applied to describe symmetries of the physical theories.

In the present paper we solve two fundamental problems of representation theory for transformation groupoids. These problems are:

1. to describe the elementary objects of the set of inequivalent unitary groupoid representations (these elementary objects are called irreducible representations)
2. to obtain a formula for the decomposition of any unitary representation into the elementary objects (this formula is called the generalized Fourier–Plancherel transform)

Both problems are solved in Theorem 5.4. Previous theorems and lemmas lead us to the proof of this theorem but are also interesting from the point of view of representation theory.

In the beginning of the article we have made a review of the basic concepts of representation theory of groups and groupoids. Note that some of the definitions are formulated in a non-standard, but equivalent way. We restrict our research to the case of the transitive transformation groupoid $\Gamma = K\backslash G \times G$ for a locally compact group G and its compact subgroup K. In the proofs of theorems we use the Mackey theorem and the Landsman theorem concerning the relationship between the representations of the transformation groupoid and imprimitivity systems

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on G [4, 5]. This combination of theorems is an original idea that allows us to describe the groupoid representations by representations of the compact group K. We show an important fact about the decomposition of unitary representations of the transformation groupoid. In the case of a locally compact group an unitary representation has a decomposition into the direct integral of irreducible representations. But for the transformation groupoid we have the decomposition of unitary representations into the direct sum of irreducible representations.

We have shown that the theory of unitary representations of our transformation groupoid is closely related to the theory of unitary representations of a compact group. For this reason one can easily and completely describe the groupoid representations.

Lemma 3.2 gives the condition on equivalence of two representations of the transformation groupoid. Lemmas 3.3 and 3.4 give the sufficient and necessary condition for the irreducibility of groupoid representation. Lemma 3.5 describes the decomposition of an unitary groupoid representation into the countable sum of irreducible components. Theorems 4.1 and 4.2 give the formulas on multiplicities. Theorem 4.1 is a certain type of the Frobenius theorem on duality applied to groupoids. Theorem 4.2 determines the multiplicity of an irreducible representation in the unitary groupoid representation. We prove that this multiplicity is equal to the dimension of space of intertwining operators of certain type. In Theorem 5.4 we show how the generalized Fourier–Plancherel transformation for the groupoids looks like. It is also a generalization of the Peter–Weyl theorem for the noncompact transformation groupoid. We achieve this by showing some facts and theorems about the equivalence between imprimitivity systems, the equivalence and irreducibility of groupoid representations, the dimension of irreducible groupoid representations, the decompositions into irreducible components. We present the application of our theory for the locally compact group $SL_2(\mathbb{C})$ and its compact subgroup $SU(2)$. This example can be interpreted in quantum mechanics. We apply the results of the article to describe elementary particles as the transformation groupoid representations.

According to Rieffel's investigation [13] the space of intertwining Hilbert–Schmidt operators for noncompact group vanishes. The theorem of Peter–Weyl type for a noncompact group is not satisfied. The advantage of our approach is that the generalization of the Peter–Weyl theorem for groupoids does not need compactness. Considering the representations of a transformation groupoid instead of the representation of a noncompact group, we bypass the above mentioned problem of the group G being noncompact. All of our results concerning representations of a transformation groupoid are described in the language of the inducing representation of a compact group. Therefore, we work at the groupoid level using the representation theory of compact groups.

2 Preliminaries

We consider the topological groupoid Γ with base X. We assume that Γ and X are locally compact Hausdorff spaces. For definitions of these concepts we refer to [6]. In the following we assume that G is a locally compact group acting from the right, transitively and continuously on X and K is the stabilizer of a given point of X. Also we assume that K is a compact subgroup of the group G. For this reason [14, p.98] there is a homeomorphism between X and $K\backslash G$. For simplicity, we set $X = K\backslash G$. We consider the transformation groupoid $\Gamma = (X \times G, X, s, t, 1, \circ, ^{-1})$ where according to [6]

- the source map $s\colon X \times G \to X; s(x, g) = x$
- the target map $t\colon X \times G \to X; t(x, g) = xg$
- the unity map $1\colon X \to X \times G; 1_x = (x, 1)$
- the set of composable pairs $(X \times G)^{\langle 2\rangle} = \{((x_1, g_1), (x_2, g_2)) \in (X \times G)^2 \mid x_2 = x_1 g_1\}$
- the composition map $\circ\colon (X \times G)^{\langle 2\rangle} \to X \times G; (x_2, g_2) \circ (x_1, g_1) = (x_1, g_2 g_1)$
- the inversion map $^{-1}\colon X \times G \to X \times G; (x, g) \mapsto (x, g)^{-1} = (xg, g^{-1})$.

Additionally, we assume that there is a G-invariant measure μ defined on X [15, p.160]. It must be noted that there always exists a G-quasi-invariant measure on X and all statements and theorems in this work remain true in this case [15].

We consider the continuous representations of groupoids. For this reason, as in [2] we introduce the definition of groupoid representations that satisfies the condition of continuity. Let $\mathcal{H} = (H_x)_{x \in X}$ be a continuous Hilbert

bundle over X that is separable fiberwise. The space of continuous sections of this bundle that vanish at infinity will be denoted by $\Delta = C_0(X, \mathcal{H})$. This space is a Hilbert $C_0(X)$-module [2].

Definition 2.1. *Let $\Gamma = (\Gamma, X, s, t, 1, \circ, ^{-1})$ be a topological groupoid over locally compact Hausdorff space X and let $\mathcal{H} = (H_x)_{x\in X}$ be a continuous Hilbert bundle over X. A unitary continuous representation of Γ on $\mathcal{H}$ is the collection $(\mathcal{U}_\gamma)_{\gamma\in\Gamma}$ of unitary operators $\mathcal{U}_\gamma : H_{s(\gamma)} \to H_{t(\gamma)}$ such that*
- *$\mathcal{U}_{1_x} = \mathrm{id}_{H_x}$ for any $x \in X$*
- *$\mathcal{U}_{\gamma_2\circ\gamma_1} = \mathcal{U}_{\gamma_2} \circ \mathcal{U}_{\gamma_1}$ for any $(\gamma_1, \gamma_2) \in \Gamma^{\langle 2\rangle}$*
- *the map $\Gamma \ni \gamma \mapsto \langle \mathcal{U}_\gamma \xi(s(\gamma)), \eta(t(\gamma))\rangle \in \mathbb{C}$ is continuous for any $\xi, \eta \in \Delta$.*

We will often use the shorter notation $\mathcal{U}$ instead of $(\mathcal{U}, \mathcal{H})$.

We consider the irreducible representations of groupoid Γ and its subrepresentations.

Definition 2.2. *Let $((\mathcal{U}_\gamma)_{\gamma\in\Gamma}, (H_x)_{x\in X})$ be a unitary continuous representation of a topological groupoid $\Gamma = (\Gamma, X, s, t, 1, \circ, ^{-1})$ over locally compact Hausdorff space X on a continuous Hilbert bundle $\mathcal{H}$ and let $\hat{\mathcal{H}} = (\hat{H}_x)_{x\in X}$ be a Hilbert subbundle of $\mathcal{H}$. Then*
- *$\hat{\mathcal{H}}$ is called invariant if it satisfies the condition $\mathcal{U}_\gamma(\hat{H}_{s(\gamma)}) \subset \hat{H}_{t(\gamma)}$ for any $\gamma \in \Gamma$*
- *any restriction of $((\mathcal{U}_\gamma)_{\gamma\in\Gamma}, (H_x)_{x\in X})$ to a given invariant subbundle $\hat{\mathcal{H}}$ is called a subrepresentation of representation $((\mathcal{U}_\gamma))_{\gamma\in\Gamma}, (H_x)_{x\in X})$*
- *$(\mathcal{U}_\gamma)$ is called irreducible if for any invariant Hilbert subbundle $(\hat{H}_x)_{x\in X}$ either $\hat{H}_x = 0$ for any $x \in X$ or $\hat{H}_x = H_x$ for any $x \in X$.*

Definition 2.3 (cf. [2])**.** *Let $((\mathcal{U}_\gamma)_{\gamma\in\Gamma}, (H_x)_{x\in X})$ and $((\bar{\mathcal{U}}_\gamma)_{\gamma\in\Gamma}, (\bar{H}_x)_{x\in X})$ be two unitary continuous representations of the groupoid Γ. Then*
- *the intertwining family for representations $(\mathcal{U}_\gamma)_{\gamma\in\Gamma}$ and $(\bar{\mathcal{U}}_\gamma)_{\gamma\in\Gamma}$ is the family of linear maps $l_x\colon H_x \to \bar{H}_x$ for all $x \in X$ satisfying two conditions*
  - *$l_{t(\gamma)} \circ \mathcal{U}_\gamma = \bar{\mathcal{U}}_\gamma \circ l_{s(\gamma)}$ for any $\gamma \in \Gamma$*
  - *the map $X \ni x \mapsto \langle l_x\xi(x), \eta(x)\rangle \in \mathbb{C}$ is continuous for any sections $\xi \in C_0(X, \mathcal{H})$ and $\eta \in C_0(X, \bar{\mathcal{H}})$.*

  The space of all intertwining families for representations $(\mathcal{U}_\gamma), (\bar{\mathcal{U}}_\gamma)$ is denoted by $L_\Gamma((H_x), (\bar{H}_x))$.
- *two representations $((\mathcal{U}_\gamma)_{\gamma\in\Gamma}, (H_x)_{x\in X})$ and $((\bar{\mathcal{U}}_\gamma)_{\gamma\in\Gamma}, (\bar{H}_x)_{x\in X})$ are called equivalent, if there exist a intertwining family $l_x\colon H_x \to \bar{H}_x$ such that l_x is a Hilbert space isomorphism for all $x \in X$. The fact that two representations are equivalent is expressed via $((\mathcal{U}_\gamma)_{\gamma\in\Gamma}, (H_x)_{x\in X}) \sim ((\bar{\mathcal{U}}_\gamma)_{\gamma\in\Gamma}, (\bar{H}_x)_{x\in X})$.*

We now move to considering the representations of a group G induced by a unitary representation τ of a subgroup K of G in a Hilbert space V [16, p.143].

This representation of G is realized in the Hilbert space of functions $H^\tau = L^2_\tau(G, V)$ consisting measureable functions $f\colon G \to V$ such that $f(kg) = \tau(k)f(g)$ for any $k \in K$ and $g \in G$ such that $\int_X \langle f(kg), f(kg)\rangle_V \, d\mu(x) < \infty$. The measure μ is invariant on $X = K\backslash G$. The linear operators that determine the induced representation $U^\tau(g)\colon H^\tau \to H^\tau$ are defined by the following condition:

$$\text{for any } g, \tilde{g} \in G \text{ and } f \in H^\tau \; U^\tau(g)(f)(\tilde{g}) = f(\tilde{g}g).$$

We present the notion of an imprimitivity system that was introduced by Mackey [17]. Our definitions and theorems are based on the works of Landsman and Taylor [4, 16].

Definition 2.4. *Let U be a unitary representation of a group G on a Hilbert space H. Let also X be a locally compact Hausdorff right G-space and P be a spectral measure from Borel family on X taking values in the space of orthogonal projections on H. Then*
- *the quadruple (G, U, X, P) is called imprimitivity system, if $P(X) = I$ and further $U(g) \circ P(B) \circ U(g^{-1}) = P(Bg^{-1})$ for any $g \in G$ and Borel $B \subseteq X$*
- *the imprimitivity system (G, U, X, P) is called transitive, if G acts transitively on the space X. In this case we have a homeomorphism of X to homogeneous space $K\backslash G$ where K is a closed subgroup of G.*

We introduce the definition of equivalent imprimitivity systems.

Definition 2.5. *Let (G, U, X, P) and $(G, \bar{U}, X, \bar{P})$ be two imprimitivity systems. The systems (G, U, X, P) and $(G, \bar{U}, X, \bar{P})$ are equivalent, if representations (U, H) and $(\bar{U}, \bar{H})$ are equivalent, i.e. there exists a Hilbert space isomorphism $L\colon H \to \bar{H}$ such that $\bar{U}(g) \circ L = L \circ U(g)$ for any $g \in G$ and $L \circ P(E) = \bar{P}(E) \circ L$ for any Borel $E \subseteq X$.*

We introduce the definition of imprimitivity subsystem.

Definition 2.6. *Let (G, U, X, P) be an imprimitivity system. The imprimitivity system (G, U_0, X, P_0) is called an imprimitivity subsubsystem of (G, U, X, P) if (U_0, H_0) is a subrepresentation of (U, H) and $P_0(E) = P(E)|_{H_0}$ for any Borel $E \subseteq X$.*

For any induced representation U^τ there exists a canonical imprimitivity system (G, U^τ, X, P^τ) [18, p.171] that is given by:

$$(P(E)^\tau f)(g) = \chi_E(Kg) f(g) \text{ for any Borel } E \subseteq X,\ f \in H^\tau,\ g \in G.$$

We will need the following theorem [16, p.144], [17].

Theorem 2.7 (Mackey)**.** *For any transitive imprimitivity system $(G, U, X = K\backslash G, P)$ there exist a unitary representation τ of group K on the Hilbert space V and there exist a unitary isomorphism $A\colon H \to H^\tau$ such that $A \circ U(g) \circ A^{-1} = U(g)^\tau$ and $A \circ P(B) \circ A^{-1} = P^\tau(B)$ for any $g \in G$ and Borel $B \subseteq X$.*

In other words, every transitive imprimitivity system (G, U, X, P) is equivalent to a canonical imprimitivity system (G, U^τ, X, P^τ) for a unitary representation (τ, V) of a group K. We denote by $\tilde{A}$ the map which assigns to a transitive imprimitivity system (G, U, X, P) the unitary representation (τ, V) of K.

3 Decomposition of transformation groupoid representations

We present the main theorem of [5]. This is a basic tool in proofs in the rest of the work.

Theorem 3.1 ([5], [4])**.** *There is a bijection I between the set of unitary representations of transformation groupoid $\Gamma = X \times G$ and the set of imprimitivity systems of the group G.*

The imprimitivity system (G, U, X, P) where U is a representation of the group G and P is a spectral measure we denote $(G, U, X, P) = I((\mathcal{U}, \mathcal{H}))$ where $(\mathcal{U}, \mathcal{H})$ is a groupoid representations. We are going to use this theorem together with the Mackey theorem to show some facts concerning the irreducibility of groupoid representations in connection with the irreducibility of inducing representations of subgroup K. We show the relation between equivalent representations of a transformation groupoid and equivalent imprimitivity systems.

Lemma 3.2. *For any two representations $((\mathcal{U}_\gamma)_{\gamma\in\Gamma}, (H_x)_{x\in X})$ and $((\bar{\mathcal{U}}_\gamma)_{\gamma\in\Gamma}, (\bar{H}_x)_{x\in X})$ of the transformation groupoid $\Gamma = X \times G$ if $((\mathcal{U}_\gamma)_{\gamma\in\Gamma}, (H_x)_{x\in X}) \sim ((\bar{\mathcal{U}}_\gamma)_{\gamma\in\Gamma}, (\bar{H}_x)_{x\in X})$ then imprimitivity systems $I((\mathcal{U}, \mathcal{H}))$ and $I((\bar{\mathcal{U}}, \bar{\mathcal{H}}))$ are equivalent.*

Proof. The fact that representations $((\mathcal{U}_\gamma)_{\gamma\in\Gamma}, (H_x)_{x\in X})$ and $((\bar{\mathcal{U}}_\gamma)_{\gamma\in\Gamma}, (\bar{H}_x)_{x\in X})$ are equivalent means that there is a intertwining family $(l_x)_{x\in X}$ where $l_x\colon H_x \to \bar{H}_x$ is a Hilbert spaces isomorphism for any $x \in X$. Let $L = \int_\oplus l_x d\mu(x)$ i.e. L is a direct integral of l_x. Let $H = \int_\oplus H_x d\mu(x)$ and $\bar{H} = \int_\oplus \bar{H}_x d\mu(x)$ be direct integrals of H_x and $\bar{H}_x$ (cf. [19], [18, p.85]). It turns out that $L\colon H \to \bar{H}$ gives the equivalence of imprimitivity systems. Indeed $\bar{U} \circ L = L \circ U$ what means $\int_\oplus \bar{U}(x, g) \cdot l_x d\mu(x) = \int_\oplus l_{xg} \cdot U(x, g) d\mu(x)$. This identity together with the fact that $\bar{P}(E) \circ L = L \circ P(E)$ for any Borel $E \subseteq X$ gives us equivalence of imprimitivity systems. □

We denote the inducing representation of the group K as $J(\mathcal{U},\mathcal{H})$ thus $J(\mathcal{U},\mathcal{H})=(\tau,V)$ where $(\mathcal{U},\mathcal{H})$ is a unitary representation of groupoid. Notice that J is bijection as a composition of two bijections $\tilde{A}$ from Theorem 2.7 and I from Theorem 3.1, $J=\tilde{A}\circ I$.

Lemma 3.3. *If $J(\mathcal{U},\mathcal{H})$ is irreducible then $(\mathcal{U},\mathcal{H})$ is irreducible.*

Proof. Assume that we have a groupoid representation $(\mathcal{U},\mathcal{H})$ that is not irreducible. This means that there is a nontrivial subrepresentation $(\mathcal{U}_1,\mathcal{H}_1)$ of the representation $(\mathcal{U},\mathcal{H})$. Then the imprimitivity system (G,U_1,X,P_1) corresponding to the representation $\mathcal{U}_1$ is a subsystem of imprimitivity of the system (G,U,X,P) corresponding to the representation $\mathcal{U}$. In particular, the representation (U_1,H_1) of G is a nontrivial subrepresentation of the representation (U,H) of G where $U_1=U|_{H_1}$ and $H_1\subset H$. The Mackey theorem says that the imprimitivity system (G,U_1,X,P_1) is unitary equivalent to the canonical imprimitivity system $(G,U_1^{\tau_1},X,P_1^{\tau_1})$ where (τ_1,V_1) is a subrepresentation of representation (τ,V). Similarly, the imprimitivity system (G,U,X,P) is unitary equivalent to the canonical imprimitivity system (G,U^{τ},X,P^{τ}). In particular, it means that U^{τ_1} is a subrepresentation of U^{τ}. The Hilbert space H^{τ_1} consists of functions $F_1\colon G\to V_1$ such that $F_1(kg)=\tau_1(k)F_1(g)$ and the Hilbert space H^{τ} consists of functions $F\colon G\to V$ such that $F(kg)=\tau(k)F(g)$. The space V_1 is a subspace of V. On the other hand for $F_1\in H^{\tau_1}$ we have $F_1(kg)=\tau_1(k)F_1(g)$ and $F_1(kg)=\tau(k)F_1(g)$ and therefore $\tau_1=\tau|_{V_1}$. For this reason τ_1 is a nontrivial subrepresentation of representation τ of K. We conclude from this that representation τ is not irreducible. □

We now show the inverse implication.

Lemma 3.4. *Let the group representation $(\tau,V)=J(\mathcal{U},\mathcal{H})$ have a nontrivial subrepresentation (τ_1,V_1). Then the representation $(\mathcal{U},\mathcal{H})$ of the groupoid $\Gamma=X\times G$ has a nontrivial subrepresentation $(\mathcal{U}_1,\mathcal{H}_1)=J^{-1}(\tau_1,V_1)$.*

Proof. Let (U^{τ},H^{τ}) be the induced representation of group G and (G,U^{τ},X,P^{τ}) be the canonical imprimitivity system corresponding to the groupoid representation $(\mathcal{U},\mathcal{H})$. Then $(G,U^{\tau_1},X,P^{\tau_1})$ is a imprimitivity subsystem of the system (G,U^{τ},X,P^{τ}) (reasoning as in the proof of Lemma 3.2). The imprimitivity system $(G,U^{\tau_1},X,P^{\tau_1})$ corresponds to the groupoid representation $(\mathcal{U}_1,\mathcal{H}_1)=J^{-1}(\tau_1,V_1)$. We show that this is nontrivial subrepresentation of representation $(\mathcal{U},\mathcal{H})$ of Γ. By the general spectral theorem (similarly as in the proof of the Mackey theorem on imprimitivity) we have the equalities $H^{\tau_1}=\int_{\oplus}\mathcal{H}_{1,x}d\mu(x)$, $U^{\tau_1}=\int_{\oplus}\mathcal{U}_1(x,g)d\mu(x)$ and $H^{\tau}=\int_{\oplus}\mathcal{H}_x d\mu(x)$, $U^{\tau}=\int_{\oplus}\mathcal{U}(x,g)d\mu(x)$. Because (U^{τ_1},H^{τ_1}) is a subrepresentation of (U^{τ},H^{τ}) it follows that $\mathcal{H}_{1,x}\subset\mathcal{H}_x$ and $U^{\tau_1}(g)=U^{\tau}(g)|_{H^{\tau_1}}$ and $\mathcal{U}_1(x,g)=\mathcal{U}(x,g)|_{\mathcal{H}_{1,x}}$ for any $x\in X, g\in G$. But this means that $\mathcal{U}_1$ is a subrepresentation of $\mathcal{U}$. □

On the strength of Lemma 3.4, every irreducible representation $(\mathcal{U},\mathcal{H})$ of Γ is finite dimensional i.e. $\dim H_x<+\infty$ for any $x\in X$.

Now we prove the result concerning decomposition.

Lemma 3.5. *Consider the representation $(\tau,V)=J(\mathcal{U},\mathcal{H})$. Let $(\tau_q,V_q)_{q\in Q}$, where Q is countable, be irreducible representations of group K such that $\tau=\oplus_{q\in Q}\tau_q$. Then $\mathcal{U}=\oplus_{q\in Q}\mathcal{U}_q$ and $\mathcal{H}=\oplus_{q\in Q}\mathcal{H}_q$ where $(\mathcal{U}_q,\mathcal{H}_q)$ are irreducible representations of groupoid Γ that satisfy $(\tau_q,V_q)=J(\mathcal{U}_q,\mathcal{H}_q)$.*

Proof. We consider the induced representation (U^{τ},H^{τ}) of G where $(\tau,V)=J(\mathcal{U},\mathcal{H})$. By one of well known theorems about induced representations [13] we have $U^{\tau}=\oplus_{q\in Q}U^{\tau_q}$ and $H^{\tau}=\oplus_{q\in Q}H^{\tau_q}$. Furthermore as in Lemmas 3.3 and 3.4 we define $(\mathcal{U}_q,\mathcal{H}_q)=J^{-1}(\tau_q,V_q)$. Just as in the proofs of these lemmas, every space H^{τ_q} is a direct integral $H^{\tau_q}=\int_{\oplus}\mathcal{H}_{q,x}d\mu(x)$ and $\mathcal{U}_q(x,g)=\mathcal{U}(x,g)|_{\mathcal{H}_{q,x}}$. This proves the decomposition $\mathcal{U}=\oplus_{q\in Q}\mathcal{U}_q$. □

4 Multiplicities

Before we show what consequences the above lemmas have on the multiplicities, let us introduce the following notation.

Denotation. *For a given groupoid Γ, groupoid representation $(\mathcal{U},\mathcal{H})$, irreducible subrepresentation $(\mathcal{U}_1,\mathcal{H}_1)$, compact group K and its representation (τ,V) and its subrepresentation (τ_1,V_1)*

- *denote by $(\mathcal{U}:\mathcal{U}_1)_\Gamma$ the multiplicity of the occurence of irreducible representation $\mathcal{U}_1$ of Γ in the decomposition of representation $\mathcal{U}$ of Γ into irreducible components. If $\mathcal{U}_1$ does not appear in this decomposition then we set $(\mathcal{U}:\mathcal{U}_1)_\Gamma = 0$.*
- *denote by $(\tau:\tau_1)_K$ the multiplicity of the occurence of irreducible representation τ_1 of K in the decomposition of representation τ of K into irreducible components. If τ_1 does not appear in this decomposition then we set $(\tau:\tau_1)_K = 0$.*

The following equality between the multiplicities is satisfied.

Theorem 4.1. *Let $(\mathcal{U},\mathcal{H})$ be a representation of Γ and $(\mathcal{U}_1,\mathcal{H}_1)$ be an irreducible representation of Γ. For $(\tau,V) = J(\mathcal{U},\mathcal{H})$ and $(\tau_1,V_1) = J(\mathcal{U}_1,\mathcal{H}_1)$ we have that*

$$(\mathcal{U}:\mathcal{U}_1)_\Gamma = (\tau:\tau_1)_K.$$

Proof. This is an obvoius consequence of Lemmas 3.3, 3.4 and 3.5. □

The following theorem gives the equality of dimension of space of intertwining operators and the multiplicities.

Theorem 4.2. *Under the assumptions of the previous theorem*

$$\dim L_\Gamma(\mathcal{H},\mathcal{H}_1) = (\mathcal{U}:\mathcal{U}_1)_\Gamma = \dim L_K(V,V_1).$$

Proof. We showed that $(\mathcal{U}:\mathcal{U}_1)_\Gamma = (\tau:\tau_1)_K$ and the Rieffel–Frobenius theorem about duality for compact groups [13, p.164] gives the equality $(\tau:\tau_1)_K = \dim L_K(V,V_1)$. Indeed, in the Rieffel–Frobenius theorem the operators of the Hilbert–Schmidt type are considered. Notice that (τ_1,V_1) is an irreducible representation of a compact group K and therefore $\dim V_1 < \infty$. Moreover, every linear operator that takes values in the space V_1 is of the Hilbert–Schmidt type. For this reason, we get $(\mathcal{U}:\mathcal{U}_1)_\Gamma = \dim L_K(V,V_1)$.

We still want to prove the equality $\dim L_\Gamma(\mathcal{H},\mathcal{H}_1) = (\mathcal{U}:\mathcal{U}_1)_\Gamma$. To this end, we use a method derived from the proof of the Schur Lemma [20]. We assume that both $\mathcal{U}$ and $\mathcal{U}_1$ are irreducible. Let $\mathcal{U} = \mathcal{U}_1$. We intend to show that the dimension $\dim L_\Gamma(\mathcal{H},\mathcal{H}_1) = 1$. Let $\bar{L} = (l_x)_{x\in X} \in L_\Gamma(\mathcal{H},\mathcal{H}_1)$ and $\bar{L} \neq 0$. Then for each $\gamma \in \Gamma$ that satisfies $s(\gamma) = x$ and $t(\gamma) = y$ and moreover $h_1 \in H_x, h_2 \in H_y$ the following equality is fulfilled

$$\langle l_y^\star \mathcal{U}_\gamma h_1, h_2\rangle_y = \langle \mathcal{U}_\gamma h_1, l_y h_2\rangle_y = \langle h_1, \mathcal{U}_\gamma^{-1} l_y h_2\rangle_x = \langle h_1, l_x \mathcal{U}_\gamma^{-1} h_2\rangle_x = \langle l_x^\star h_1, \mathcal{U}_\gamma^{-1} h_2\rangle_x = \langle \mathcal{U}_\gamma l_x^\star h_1, h_2\rangle_y.$$

This means that the family $\bar{L}^\star = (l_x^\star)_{x\in X} \in L_\Gamma(\mathcal{H},\mathcal{H}_1)$ is an intertwining family. We create the self-adjoint elements of the form $B = \bar{L} + \bar{L}^\star = (l_x + l_x^\star)_{x\in X}$ and $C = i\cdot(\bar{L} - \bar{L}^\star) = (i\cdot(l_x - l_x^\star))_{x\in X}$. The spectral theorem for self-adjoint operators gives the spectral decompositions of the elements B and C. It turns out that the projections that appear in these decompositions of B and C form intertwining families. Due to the fact that $\bar{L} \neq 0$ we can choose a nonzero projection. This projection is the identity because the representation $\mathcal{U}$ is irreducible. For this reason B, C and consequently $\bar{L}$ are multiples of identity. Therefore $\dim L_\Gamma(\mathcal{H},\mathcal{H}_1) = 1$.

Now we abandon the assumption of irreducibility of the representation $\mathcal{U}$. If the representation $(\mathcal{U},\mathcal{H})$ includes the irreducible representation $(\mathcal{U}_1,\mathcal{H}_1)$ of the multiplicity m, then $\mathcal{H}_1$ appears in the decomposition of the bundle $\mathcal{H}$ exactly m times. We consider a restriction of representation $\mathcal{U}$ to a copy of the bundle $\mathcal{H}_1$ and denote it by $\dot{\mathcal{H}}_1$. Notice that we have an orthogonal projection $P:\mathcal{H} \to \dot{\mathcal{H}}_1$. This projection is the identity on $\dot{\mathcal{H}}_1$ and therefore $P|_{\dot{\mathcal{H}}_1} = id_{\dot{\mathcal{H}}_1}$. In this way, using the first part of the proof, we have the equality $\dim L_\Gamma(\mathcal{H},\dot{\mathcal{H}}_1) = 1$. For this reason $\dim L_\Gamma(\mathcal{H},\mathcal{H}_1) = m$. □

We give a physical interpretation of the above theorems. For this purpose we use the example of the locally compact group $G = SL_2(\mathbb{C})$ and its compact subgroup $K = SU(2)$. We are going to interpret our theorems in the language of elementary particles and their properties. We describe the quantum mechanical momentum representation of a particle with the mass m. We consider the mass shell in the energy-momentum space $\mathcal{S} = \{(p_0, p_1, p_2, p_3) \in \mathbb{R}^4 \mid p_0^2 - (\sum_{i=1}^3 p_i^2) = m^2\}$. We identify the set S with the set $\bar{\mathcal{S}}$ of 2×2 Hermitian matrices with determinant equal to m^2 using the formula

$$\mathcal{S} \ni (p_0, p_1, p_2, p_3) \mapsto \begin{bmatrix} p_0 - p_3 & p_2 - i \cdot p_1 \\ p_2 + i \cdot p_1 & p_0 + p_3 \end{bmatrix} \in \bar{\mathcal{S}}.$$

The group $G = SL_2(\mathbb{C})$ acts from the right and transitively on the surface $\bar{\mathcal{S}}$ as follows $\bar{\mathcal{S}} \ni A \mapsto g^\star A g \in \bar{\mathcal{S}}$ for any $g \in G$ [6, 15, 16]. Notice that the stabilizer of this action of the group G at the point $(m, 0, 0, 0) \in \bar{\mathcal{S}}$ is equal to the subgroup K [6]. The transitivity of the action gives us a bijection $\mathcal{S} \simeq K\backslash G$, that is in fact a diffeomorphism, and therefore $K\backslash G \times G \simeq \mathcal{S} \times G$. Having $\Gamma = \mathcal{S} \times G$ and the above mentioned action of the group G on the set $\mathcal{S}$ we get the transitive transformation groupoid Γ. We consider a unitary representation $(\mathcal{U}, \mathcal{H})$ of this groupoid Γ in the Hilbert bundle $\mathcal{H}$. Using Theorem 3.1 we get that $I(\mathcal{U}, \mathcal{H})$ is the imprimitivity system obtained from this representation. Assume that the representation $(\tau, V) = J(\mathcal{U}, \mathcal{H})$ is a finite direct sum of irreducible representations $(\tau_i, V_i)_{i=1}^n$ of the group K. According to Lemma 3.5 we get that $\mathcal{U} = \oplus_{i=1}^n \mathcal{U}_i$ and $\mathcal{H} = \oplus_{i=1}^n \mathcal{H}_i$ where $(\mathcal{U}_i, \mathcal{H}_i)_{i=1}^n$ are irreducible representations of groupoid Γ such that $(\tau_i, V_i) = J(\mathcal{U}_i, \mathcal{H}_i)$ for any $i \in \{1, \ldots, n\}$. Let $(G, U^{\tau_i}, \mathcal{S}, P^{\tau_i})$ be the canonical imprimitivity system of induced representation (U^{τ_i}, H^{τ_i}) for any $i \in \{1, \ldots, n\}$. In analogy with [21] such imprimitivity system is called an elementary particle and every inducing representation (τ_i, V_i) of the group $K = SU(2)$ is called a spin of the corresponding elementary particle. The representation $\tau = \oplus_{i=1}^n \tau_i$ can be interpreted as the total spin of the system of particles.

5 Peter Weyl theorem for transformation groupoid

We are going to show that there is a very strong link between the theory of unitary representations of transformation groupoids and that of compact groups.

To begin with, we consider the regular representation of compact group K. We have $(\tau, V) = (R, L^2(K))$ where $(R_l f)(k) = f(kl)$ for any $f \in L^2(K)$ and $k, l \in K$. Consider now the representation of a group G induced by the above representation of its compact subgroup K. This representation of K is induced by the trivial representation $(\mathrm{id}_{\mathbb{C}}, \mathbb{C})$ of the trivial subgroup $\{1\} \subseteq K$. The Mackey theorem about inducing in stages [22, p.109] implies that the representation of group G induced from K by the representation $(R, L^2(K))$ is equivalent to the representation of G induced from $\{1\}$ by the trivial representation. Notice that the representation which is equivalent to representation induced from $\{1\}$ is regular. Then we have $H^\tau = L^2_R(G, L^2(K)) = H^R$ and so any $F \in H^\tau$ is the map $F\colon G \to L^2(K)$ such that $(F(kg))(\bar{k}) = (F(g))(\bar{k}k)$ and $(U^\tau(\bar{g})F)(g) = F(g\bar{g})$ for any $g, \bar{g} \in G$ and $k, \bar{k} \in K$. In this way we have showed the following lemma.

Lemma 5.1. *The representation of a group G induced by a regular representation of a subgroup K is a regular representation of G.*

We introduce the definition of a regular representation of the transformation groupoid $\Gamma = X \times G$.

Definition 5.2. *The groupoid representation $(\mathcal{U}, \mathcal{H})$ is called regular, if $J(\mathcal{U}, \mathcal{H}) = (R, L^2(K))$.*

In the following, we recall the statement of the Peter–Weyl theorem for compact groups and then we state and prove its analogue for the transformation groupoid $\Gamma = X \times G$.

Theorem 5.3 (Peter–Weyl)**.** *For any compact group K we have the decomposition*

– $L^2(K) = \oplus_{i \in \hat{K}} n_i V_i$

– $R = \oplus_{i\in\hat{K}} n_i \tau_i$

where (τ_i, V_i) *are irreducible representations of* K, $n_i = \dim V_i$ *are multiplicities and* $\hat{K}$ *is the set of equivalence classes of irreducible representations of* K.

Notice that every irreducible representation of K occurs in this decomposition.

Theorem 5.4. *For a regular representation* $(\mathcal{U}, \mathcal{H})$ *of the groupoid* $\Gamma = X \times G$
– $\mathcal{U} = \oplus_{i\in\hat{K}} n_i \mathcal{U}_i$
– $\mathcal{H} = \oplus_{i\in\hat{K}} n_i \mathcal{H}_i$
where $(\mathcal{U}_i, \mathcal{H}_i)$ *are irreducible representations of* Γ *such that* $J(\mathcal{U}_i, \mathcal{H}_i) = (\tau_i, V_i)$ *and* (τ_i, V_i) *are irreducible representations of group* K, $n_i = \dim V_i$ *are multiplicities and* $\hat{K}$ *is the set of equivalence classes of irreducible representations of* K. *Every irreducible representation of groupoid* Γ *appears in this decomposition.*

Proof. Assume that we have a regular representation $(\mathcal{U}, \mathcal{H})$ of groupoid Γ. This means that $J(\mathcal{U}, \mathcal{H}) = (R, L^2(K))$. The theorem of Peter–Weyl applied to compact group K gives the decomposition $L^2(K) = \oplus_{i\in\hat{K}} n_i V_i$ and $R = \oplus_{i\in\hat{K}} n_i \tau_i$. These direct sums are countable and all representations (τ_i, V_i) are irreducible. Lemma 3.5 for regular representation of group K gives the decomposition of representation of groupoid Γ, namely $\mathcal{U} = \oplus_{i\in\hat{K}} m_i \mathcal{U}_i$ and $\mathcal{H} = \oplus_{i\in\hat{K}} m_i \mathcal{H}_i$ where $(\mathcal{U}_i, \mathcal{H}_i)_{i\in\hat{K}}$ are irreducible representations of groupoid Γ such that $(\tau_i, V_i) = J(\mathcal{U}_i, \mathcal{H}_i)$ and m_i is the multiplicity of $(\mathcal{U}_i, \mathcal{H}_i)$. Theorem 4.1 gives the equality $n_i = (\tau : \tau_i)_K = (\mathcal{U} : \mathcal{U}_i)_\Gamma = m_i$, which completes the proof. □

Let $\hat{\Gamma}$ be the set of equivalence classes of irreducible representations of Γ. It is clear that the map $J|_{\hat{\Gamma}} : \hat{\Gamma} \to \hat{K}; (\mathcal{U}, \mathcal{H}) \mapsto J(\mathcal{U}, \mathcal{H}) = (\tau, V)$ is bijection. Let us observe that the decomposition of representations of groupoid Γ is completely analogous to the decomposition of representation of group K. Therefore we can speak about close relationship of theory of representation of groupoid Γ and theory of representation of group K. In the paper [20] Amini proved the theorem about decomposition and the Peter–Weyl theorem for compact groupoids. Theorem 5.4 can be regarded as extending these results on a certain class of noncompact groupoids, namely the transitive transformation groupoids of the form $\Gamma = K\backslash G \times G$.

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