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Research Article

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On Weil homomorphism in locally free sheaves over structured spaces

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Abstract: Inspired by the work of Heller and Sasin [1], we construct in this paper Weil homomorphism in a locally free sheaf $\mathcal{W}$ of Φ-fields [2] over a structured space. We introduce the notion of G-consistent, linear connection on this sheaf, what allows us to clearly define Chern, Pontrjagin and Euler characteristic classes. We also show proper equalities between those classes.

Keywords: Structured space, Locally free sheaf, Linear connection, Curvature, Weil homomorphism, Invariant forms

MSC: 14F05, 53B05, 57R20

1 Preliminaries

Let M be a topological space and $\mathcal{C}$ be a sheaf of algebras of real continuous functions on M with $\tau = \tau_{\mathcal{C}}$, where $\tau_{\mathcal{C}}$ is the weakest topology in which all functions from $\mathcal{C}$ are continuous. The triple $(M, \tau, \mathcal{C})$ is called a structured space [1].

Let Φ denote a function on M such that for any point $p \in M$, the image $\Phi(p)$ is an n-dimensional vector space over $\mathbb{R}$.

A mapping $\xi : U \to \coprod_{p \in U} \Phi(p)$ is called a local Φ-field on $U \in \tau$ if $\xi(p) \in \Phi(p)$ for $p \in U$.
Let $\mathcal{W}$ be a sheaf of local Φ-fields such that:

1. $\mathcal{W}$ is a locally free $\mathcal{C}$-module of rank $n \in \mathbb{N}$;
2. For every local basis $e = (e_1, \dots, e_n)$ of $\mathcal{W}$ on $U \in \tau$ the sequence $e_1(p), \dots, e_n(p)$ is a vector basis of $\Phi(p)$ for any $p \in U$.

In the sequel, any $\mathcal{C}$-module $\mathcal{W}$ satisfying the conditions 1–2 is said to be a locally free sheaf of Φ-fields over the structured space $(M, \tau, \mathcal{C})$.

Let $\mathcal{X}$ be the sheaf of morphisms $X : \mathcal{C} \to \mathcal{C}$ such that for any $U \in \tau$

$$X(U) : \mathcal{C}(U) \to \mathcal{C}(U)$$

is a derivation of an $\mathbb{R}$-algebra $\mathcal{C}(U)$. The sheaf $\mathcal{X}$ will be called the sheaf of derivations.

By $A^k(\mathcal{X}, \mathcal{C}); k \geq 1$ we will denote the sheaf of morphisms $\omega : \mathcal{X}^k \to \mathcal{C}$ such that

$$\omega(U) : \mathcal{X}^k(U) \to \mathcal{C}(U)$$

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is $\mathcal{C}(U)$-k-linear for every $U \in \tau$. For $k = 0$ we set $A^0(\mathcal{X}, \mathcal{C}) := \mathcal{C}$.

The $\mathcal{C}$-module $A^k(\mathcal{X}, \mathcal{C})$ will be called the sheaf of differential k-forms on $(M, \tau, \mathcal{C})$ and an element $\omega \in A^k(\mathcal{X}, \mathcal{C})$ we will call a differential k-form (or shortly a k-form) on $(M, \tau, \mathcal{C})$.

In the sheaf $A^k(\mathcal{X}, \mathcal{C})$ we define the exterior derivative d as a morphism $d : A^k(\mathcal{X}, \mathcal{C}) \to A^{k+1}(\mathcal{X}, \mathcal{C})$, $k \geq 0$, in the following way [3]:

1. If $k = 0$ and $\alpha \in A^0(\mathcal{X}(U), \mathcal{C}(U)) := \mathcal{C}(U)$, then $(d\alpha)(X) := X(\alpha)$ for any $X \in \mathcal{X}(U)$, $U \in \tau$.
2. If $k > 0$ and $\omega(U) : \mathcal{X}^k(U) \to \mathcal{C}(U)$ then

$$\begin{aligned}(d\omega(U))(X_1, \ldots, X_{k+1}) = & \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) \\ & + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1})\end{aligned}$$

   for any $X_1, \ldots, X_{k+1} \in \mathcal{X}(U)$, $U \in \tau$, where $\hat{X}_i, \hat{X}_j$ are omitted.

By $\mathcal{H}^k$, $k = 0, 1, 2, \ldots$ we denote the k-th de Rham cohomology group of the space $(M, \tau, \mathcal{C})$

$$\begin{aligned}\mathcal{H}^k &= \ker d_k / \operatorname{im} d_{k-1} \quad \text{for } k \geq 1, \\ \mathcal{H}^0 &= \mathcal{C}.\end{aligned}$$

Let us put

$$\mathcal{H}^* := \sum_{k \geq 0} \mathcal{H}^k.$$

$\mathcal{H}^*$ is a graded algebra if we accept the following definition:

$$[\omega_1] \cup [\omega_2] := [\omega_1 \wedge \omega_2]$$

for any $\omega_1, \omega_2 \in A^*(\mathcal{X}, \mathcal{C}) := \sum_{k \geq 0} A^k(\mathcal{X}, \mathcal{C})$.

Let $\mathcal{W}$ be a locally free sheaf of local Φ-fields. By $A^k(\mathcal{X}, \mathcal{W})$; $k \geq 1$ we will denote the sheaf of morphisms $\omega : \mathcal{X}^k \to \mathcal{W}$ such that

$$\omega(U) : \mathcal{X}^k(U) \to \mathcal{W}(U)$$

is $\mathcal{C}(U)$-k-linear for every $U \in \tau$. For $k = 0$ we set $A^0(\mathcal{X}, \mathcal{W}) := \mathcal{W}$.

The $\mathcal{C}$-module $A^k(\mathcal{X}, \mathcal{W})$ will be called $\mathcal{W}$-valued sheaf of differential k-forms on $(M, \tau, \mathcal{C})$ and an element $\omega \in A^k(\mathcal{X}, \mathcal{W})$ will be called a differential $\mathcal{W}$-valued k-form on $(M, \tau, \mathcal{C})$.

2 Connection as a morphism of sheaves

Definition 2.1. *A linear connection in the sheaf $\mathcal{W}$ is a morphism of sheaves $D : \mathcal{W} \to A^1(\mathcal{X}, \mathcal{W})$ satisfying the conditions:*

1. *D is an $\mathbb{R}$-linear local morphism, i.e.*

$$\begin{aligned}&D(U) : \mathcal{W}(U) \to A^1(\mathcal{X}(U), \mathcal{W}(U)), \\ &D(U)(\lambda\xi_1 + \mu\xi_2) = \lambda D(U)\xi_1 + \mu D(U)\xi_2\end{aligned}$$

   for any $\xi_1, \xi_2 \in \mathcal{W}(U)$, $\lambda, \mu \in \mathbb{R}$, $U \in \tau$.
2. *D satisfies the Leibniz rule*

$$D(U)(\alpha\xi) = (d(U)\alpha) \otimes \xi + \alpha D(U)\xi$$

   for any $\xi \in \mathcal{W}(U)$ and $\alpha \in \mathcal{C}(U)$, $U \in \tau$.

Let $e = (e_1, \dots, e_n)$ be a local $\mathcal{C}(U)$-basis of the $\mathcal{C}(U)$-module $\mathcal{W}(U)$, $U \in \tau$. Decomposing the 1-forms $De_i \in A^1(\mathcal{X}(U), \mathcal{W}(U))$ for $i = 1, \dots, n$ with respect to e we get:

$$De_i = \theta_i^j(D, e)e_j, \quad i, j = 1, \dots, n, \tag{1}$$

where $\theta_i^j(D, e) \in A^1(\mathcal{X}(U), \mathcal{C}(\mathcal{U}))$ for $i, j = 1, \dots, n$.

The connection D satisfies the consistency condition: if $V \subset U$ then we have

$$D(V)(s_{|V}) = (D(U)(s))_{|V}.$$

The matrix $\theta(D, e) = (\theta_i^j(D, e))$ of the 1-forms $\theta_i^j(D, e)$ for $i, j = 1, \dots, n$ is called the matrix of the linear connection D with respect to the local $\mathcal{C}(U)$-basis e.

Now, putting

$$\Theta(D, e) = d\theta(D, e) + \theta(D, e) \wedge \theta(D, e) \tag{2}$$

we obtain the matrix called the matrix of curvature 2-forms of the linear connection D with respect to the local $\mathcal{C}(U)$-basis e. When the connection D in the sheaf $\mathcal{W}$ is fixed then the matrices $\theta(D, e)$ and $\Theta(D, e)$ will be denoted simply by $\theta(e)$ and $\Theta(e)$, respectively.

Let $\mathrm{GL}(n, \mathcal{C})$ be the general linear group sheaf of non-singular matrices with elements in the sheaf $\mathcal{C}$ (compare with [4, 5]). Then for $g \in \mathrm{GL}(n, \mathcal{C}(U))$ and the local $\mathcal{C}(U)$-basis $e = (e_1, \dots, e_n)$, $U \in \tau$, we define a new local $\mathcal{C}(U)$-basis by the formula

$$eg = (g_1^i e_i, \dots, g_n^i e_i), \quad i = 1, 2, \dots n.$$

One can verify the following transformation laws:

$$g\theta(eg) = dg + \theta(e)g, \tag{3}$$

$$\Theta(eg) = g^{-1}\Theta(e)g. \tag{4}$$

Let $\mathcal{L}(\mathcal{W}, \mathcal{W})$ be the sheaf of endomorphisms of the $\mathcal{C}$-module $\mathcal{W}$. By $A^k(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W})); k \geq 1$ we will denote the sheaf of morphisms $\omega : \mathcal{X}^k \to \mathcal{L}(\mathcal{W}, \mathcal{W})$ such that

$$\omega(U) : \mathcal{X}^k(U) \to \mathcal{L}(\mathcal{W}, \mathcal{W})(U)$$

is $\mathcal{C}(U)$-k-linear for every $U \in \tau$. For $k = 0$ we set $A^0(\mathcal{X}, \mathcal{W}) := \mathcal{L}(\mathcal{W}, \mathcal{W})$.

The $\mathcal{C}$-module $A^k(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ will be called $\mathcal{L}(\mathcal{W}, \mathcal{W})$-valued sheaf of differential k-forms on $(M, \tau, \mathcal{C})$ and an element $\omega \in A^k(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ will be called a differential $\mathcal{L}(\mathcal{W}, \mathcal{W})$-valued k-form on $(M, \tau, \mathcal{C})$.

Any $\mathcal{L}(\mathcal{W}, \mathcal{W})$-valued k-form determines the matrix of k-forms $\omega(e) = (\omega_j^i(e))$ in a local $\mathcal{C}(U)$-basis e, where $\omega_j^i(e) \in A^k(\mathcal{X}(U), \mathcal{C}(U))$ are given by the decomposition

$$\omega(U)(X_1, \dots, X_k)(e_j) = \omega_j^i(e)(X_1, \dots, X_k)e_i \tag{5}$$

for all $X_1, \dots, X_k \in \mathcal{X}(U)$ and $i, j = 1, \dots, n$.

For the k-form ω we have the transformation law

$$\omega(eg) = g^{-1}\omega(e)g,$$

where $g \in \mathrm{GL}(n, \mathcal{C}(U))$.

The covariant derivative of a k-form $\omega \in A^k(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ is a $(k+1)$-form $D\omega \in A^{k+1}(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ regarded as a morphism of sheaves

$$D\omega : \mathcal{X}^{k+1} \to \mathcal{L}(\mathcal{W}, \mathcal{W})$$

such that its matrix with respect to a local $\mathcal{C}(U)$-basis e is defined by

$$(D\omega)(e) = (d(U))(\omega(e)) + \theta(e) \wedge \omega(e) + (-1)^k \omega(e) \wedge \theta(e). \tag{6}$$

The exterior product of a k-form $\chi \in A^k(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ and an l-form $\psi \in A^l(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ is a $(k+l)$-form $\chi \wedge \psi \in A^{k+l}(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ such that its matrix with respect to a local $\mathcal{C}(U)$-basis e is of the form

$$(\chi \wedge \psi)(e) = \chi(e) \wedge \psi(e).$$

The Lie bracket is given by the formula

$$[\chi, \psi] = \chi \wedge \psi - (-1)^{kl} \psi \wedge \chi. \tag{7}$$

One can prove the following identities

$$D(\chi \wedge \psi) = D\chi \wedge \psi + (-1)^k \chi \wedge D\psi \tag{8}$$

for $\chi \in A^k(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$, $\psi \in A^l(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$,

$$D\Theta = 0 \quad \text{(Bianchi identity)}. \tag{9}$$

3 Chern–Weil homomorphism in locally free sheaf

Let $I_k(M_n(\mathbb{R}))$ be an $\mathbb{R}$-linear space of all $\mathrm{GL}(n, \mathbb{R})$-invariant polynomials of degree k on the set of $n \times n$ matrices with real entries $M_n(\mathbb{R})$, i.e. polynomials $P : M_n(\mathbb{R}) \times \ldots \times M_n(\mathbb{R}) \to \mathbb{R}$ which satisfy the condition

$$P(gA_1g^{-1}, \ldots, gA_kg^{-1}) = P(A_1, \ldots, A_k)$$

for arbitrary $A_1, \ldots, A_k \in M_n(\mathbb{R})$ and $g \in \mathrm{GL}(n, \mathbb{R})$.

Let $E^i_j \in M_n(\mathbb{R})$ be the standard basis of the space $M_n(\mathbb{R})$ for $i, j = 1, \ldots, n$. Using the Einstein convention, each matrix $A = (a^i_j) \in \mathrm{GL}(n, \mathbb{R})$ can be uniquely expressed as a linear combination

$$A = a^j_i E^i_j, \quad i, j = 1, \ldots, n.$$

If $A_l = (\overset{l}{a}{}^{i_l}_{j_l})$, $l = 1, \ldots, k$ are some matrices from $M_n(\mathbb{R})$ then for any $P \in I_k(M_n(\mathbb{R}))$ we have

$$P(A_1, \ldots, A_k) = \lambda^{i_1 \ldots i_k}_{j_1 \ldots j_k} \overset{1}{a}{}^{j_1}_{i_1} \ldots \overset{k}{a}{}^{j_k}_{i_k},$$

where $\lambda^{i_1 \ldots i_k}_{j_1 \ldots j_k} := P(E^{i_1}_{j_1}, \ldots, E^{i_k}_{j_k})$.

Let $\eta_1 \in A^{d_1}(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W})), \ldots, \eta_k \in A^{d_k}(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ be skew-symmetric forms of degrees $d_1, \ldots, d_k$, respectively. We can consider these forms as morphisms of sheaves $\mathcal{X}^{d_i}$ and a $\mathcal{C}$-sheaf $\mathcal{L}(\mathcal{W}, \mathcal{W})$, $\eta_i : \mathcal{X}^{d_i} \to \mathcal{L}(\mathcal{W}, \mathcal{W})$ for $i = 1, \ldots, k$. Let $\eta_1(e), \ldots, \eta_k(e)$ be the local representation of these forms in the local basis $e = (e_1, \ldots, e_n)$ on $U \in \tau$.

Given an invariant polynomial $P \in I_k(M_n(\mathbb{R}))$ we define the form $P_U(\eta_1, \ldots, \eta_k)$ of degree $d_1 + \ldots + d_k$ on U by

$$P_U(\eta_1, \ldots, \eta_k) := \lambda^{i_1, \ldots, i_k}_{j_1, \ldots, j_k} \overset{1}{\eta}{}^{j_1}_{i_1}(e) \wedge \ldots \wedge \overset{k}{\eta}{}^{j_k}_{i_k}(e). \tag{10}$$

It is easy to check that this definition is independent of the choice of a local basis.

For any $V \in \tau$, $V \subset U$ we have the equality

$$P_U(\eta_1, \ldots, \eta_k)_{|V} = P_V(\eta_{1|V}, \ldots, \eta_{k|V}).$$

There exists exactly one $(d_1 + \ldots + d_k)$-form

$$P(\eta_1, \ldots, \eta_k) \in A^{d_1 + \ldots + d_k}(\mathcal{X}, \mathcal{C})$$

regarded as a morphism $P(\eta_1, \ldots, \eta_k) : \mathcal{X}^{d_1 + \ldots + d_k} \to \mathcal{C}$ such that

$$P(\eta_1, \ldots, \eta_k)_{|U} = P_U(\eta_1, \ldots, \eta_k).$$

For any $(\eta_1, \ldots, \eta_k) \in A^{d_1}(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W})) \times \ldots \times A^{d_k}(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ the mapping $(\eta_1, \ldots, \eta_k) \mapsto P(\eta_1, \ldots, \eta_k) \in A^{d_1 + \ldots + d_k}(\mathcal{X}, \mathcal{C})$ is a $\mathcal{C}$-linear morphism of sheaves.

Lemma 3.1. *Let $\eta_i \in A^{d_i}(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$, $i = 1, \ldots, k$ be a d_i-form and let $\psi \in A^1(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ be a 1-form. Then for any $P \in I_k(M_n(\mathbb{R}))$ we have the identities [6]*

(i) $\sum_{i=1}^{k} (-1)^{d_1+\ldots+d_{i-1}} P(\eta_1, \ldots, [\psi, \eta_i], \ldots, \eta_k) = 0$,

(ii) $dP(\eta_1, \ldots, \eta_k) = \sum_{i=1}^{k} (-1)^{d_1+\ldots+d_{i-1}} P(\eta_1, \ldots, D\eta_i, \ldots, \eta_k)$,

where D is a linear connection in the sheaf $\mathcal{W}$ and $d_0 := 0$.

Let $A^*(\mathcal{X}, \mathcal{M})$ be the $\mathcal{C}$-sheaf of all morphisms of sheaves $\omega : \mathcal{X}^* \to \mathcal{M}$, where the sheaf $\mathcal{M}$ is either $\mathcal{C}$ or $\mathcal{L}(\mathcal{W}, \mathcal{W})$. Denote by $A^*(\mathcal{X}, \mathcal{M})[t]$ the set of all mappings such that

$$\omega(t) = \omega_0 + \omega_1 t + \ldots + \omega_n t^n,$$

where $\omega_0, \omega_1, \ldots, \omega_n \in A^*(\mathcal{X}, \mathcal{M})$, $t \in \mathbb{R}$, $n \in \mathbb{N}$. Of course $A^*(\mathcal{X}, \mathcal{M})[t]$ is a vector space over $\mathbb{R}$.

Let us define $\frac{d}{dt}, \int_a^b \in \operatorname{End} A^*(\mathcal{X}, \mathcal{M})[t]$, $a, b \in \mathbb{R}$, via

$$\left(\tfrac{d}{dt}\omega\right)(t) := \omega_1 + 2\omega_2 t + \ldots + n\omega_n t^{n-1} \quad \text{for } t \in \mathbb{R},$$

$$\int_a^b \omega := \omega_0(b-a) + \omega_1 \tfrac{b^2-a^2}{2} + \ldots + \omega_n \tfrac{b^{n+1}-a^{n+1}}{n+1}.$$

If $\omega \in A^*(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ and D is linear connection in the sheaf $\mathcal{W}$ then we define

$$(D\omega)(t) := D\omega_0 + (D\omega_1)t + \ldots + (D\omega_n)t^n,$$

where $\omega = \omega_0 + \omega_1 t + \ldots + \omega_n t^n$, $t \in \mathbb{R}$.

If $\omega \in A^*(\mathcal{X}, \mathcal{C})$, we additionally define

$$(d\omega)(t) := d\omega_0 + d\omega_1 t + \ldots + d\omega_n t^n.$$

It is easy to see that

$$d\left(\int_a^b \omega\right) = \int_a^b d\omega$$

for any $\omega \in A^*(\mathcal{X}, \mathcal{C})$ and $a, b \in \mathbb{R}$.

These simple algebraic operations are useful for the proof of the following fundamental theorem, which is a generalization of the Weil theorem.

Theorem 3.2. *Let $\mathcal{W}$ be a locally free sheaf of Φ-fields on M of rank n and let D be a linear connection in $\mathcal{W}$. Suppose that $P \in I_k(M_n(\mathbb{R}))$. Then*

(a) *The $2k$-form $P(\Theta)$ is closed, i.e. $dP(\Theta) = 0$.*

(b) *The cohomology class $[P(\Theta)] \in \mathcal{H}^{2k}$ is independent of the connection D.*

Proof. (a). From Lemma 3.1 we obtain

$$dP(\Theta) = P(D\Theta, \Theta, \ldots, \Theta) + \ldots + P(\Theta, \ldots, D\Theta, \ldots, \Theta) + \\ + \ldots + P(\Theta, \ldots, \Theta, D\Theta) = 0,$$

what vanishes by equation (9) and thus $P(\Theta)$ is a closed $2k$-form.

(b). Put $\eta := \tilde{D} - D$. It is easy to see that $\eta \in A^1(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$. Next, consider the one-parameter family of connections in $\mathcal{W}$

$$D_t = (1-t)D + t\tilde{D} \quad \text{for } t \in \mathbb{R}.$$

Hence

$$D_t = D + t(\tilde{D} - D) = D + t\eta.$$

It is easy to see that

$$\theta_t(e) = \theta(e) + t\eta(e), \quad \text{where} \quad \eta(e) = \tilde{\theta}(e) - \theta(e),\ t \in \mathbb{R}.$$

Therefore

$$\begin{aligned}\Theta_t(e) &= d\theta_t(e) + \theta_t(e) \wedge \theta_t(e)\\ &= d(\theta(e) + t\eta(e)) + (\theta(e) + t\eta(e)) \wedge (\theta(e) + t\eta(e))\\ &= d\theta(e) + \theta(e) \wedge \theta(e) + tD\eta(e) + t^2\eta(e) \wedge \eta(e)\\ &= \Theta(e) + tD\eta(e) + t^2\eta(e) \wedge \eta(e)\end{aligned}$$

for an arbitrary local basis e.

The curvature form Θ_t of D_t has the form

$$\Theta_t = \Theta + tD\eta + t^2\eta \wedge \eta \quad \text{for } t \in \mathbb{R}. \tag{11}$$

Hence

$$D\Theta_t := D\Theta + tD^2\eta + t^2 D(\eta \wedge \eta) \quad \text{for } t \in \mathbb{R}.$$

It follows from (7) and (8) that

$$D(\eta \wedge \eta) = D\eta \wedge \eta + (-1)^1 \eta \wedge D\eta = [D\eta, \eta].$$

Of course, $[\eta \wedge \eta, \eta] = (\eta \wedge \eta) \wedge \eta - \eta \wedge (\eta \wedge \eta) = 0$. Hence we have

$$\begin{aligned}D\Theta_t &= D\Theta + t^2[\Theta, \eta] + t^2[D\eta, \eta] + t^3[\eta \wedge \eta, \eta]\\ &= t[\Theta + t^2 D\eta + t^2\eta \wedge \eta, \eta] = t[\Theta_t, \eta]\end{aligned} \tag{12}$$

for $t \in \mathbb{R}$.

Now, we shall show the identity

$$\tfrac{d}{dt} P(\Theta_t) = dQ(\eta, \Theta_t),\ t \in \mathbb{R}, \tag{13}$$

where $Q(\chi, \psi) = kP(\chi, \psi, \ldots, \psi)$ for any $\chi, \psi \in A^*(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$.

From (13) we obtain

$$\tfrac{d}{dt}\Theta_t = \tfrac{d}{dt}(\Theta + tD\eta + t^2\eta \wedge \eta) = D\eta + 2t\eta \wedge \eta. \tag{14}$$

From Lemma 3.1 and the symmetry of the k-form P we obtain

$$\tfrac{d}{dt} P(\Theta_t) = kP\left(\tfrac{d}{dt}\Theta_t, \Theta_t, \ldots \Theta_t\right).$$

Hence and from (14) we have

$$\begin{aligned}\tfrac{d}{dt} P(\Theta_t) &= kP(D\eta + 2t\eta \wedge \eta, \Theta_t, \ldots \Theta_t)\\ &= kP(D\eta, \Theta_t, \ldots \Theta_t) + 2ktP(\eta \wedge \eta, \Theta_t, \ldots \Theta_t)\\ &= Q(D\eta, \Theta_t) + 2tQ(\eta \wedge \eta, \Theta_t).\end{aligned} \tag{15}$$

From the symmetry of the k-form P and from Lemma 3.1, (ii), we obtain

$$\begin{aligned}dQ(\eta, \Theta_t) &= d(kP(\eta, \Theta_t, \ldots \Theta_t))\\ &= kP(D\eta, \Theta_t, \ldots \Theta_t) - k(k-1)P(\eta, D\Theta_t, \Theta_t, \ldots \Theta_t).\end{aligned} \tag{16}$$

Applying the identity (i) from Lemma 3.1 with $\psi = \eta$ and $\chi_i = \theta_i$, $i = 1, \dots, k$, $t \in \mathbb{R}$, we obtain

$$P([\eta,\eta], \Theta_t, \dots \Theta_t) - (k-1)P(\eta, [\eta, \Theta_t], \Theta_t, \dots \Theta_t) = 0. \tag{17}$$

From (12), (16), (17) and by the identity

$$[\eta,\eta] = \eta \wedge \eta - (-1)\eta \wedge \eta = 2\eta \wedge \eta$$

one can compute

$$dQ(\eta, \Theta_t) = Q(D\eta, \Theta_t) + 2tQ(\eta \wedge \eta, \Theta_t).$$

Hence and on the strength of (15) we obtain (13).

By integrating identity (13) we have

$$\int_0^1 \tfrac{d}{dt} P(\Theta_t) = \int_0^1 dQ(\eta, \Theta_t), \; t \in \mathbb{R}.$$

Hence

$$P(\Theta_1) - P(\Theta_0) = d\left(\int_0^1 Q(\eta, \Theta_t)\right)$$

or, for cohomology classes,

$$[P(\Theta_1)] = [P(\Theta_0)].$$ □

The mapping $w : I^*(M_n(\mathbb{R})) \to \mathcal{H}^*$ given by $P \mapsto [P(\Theta)]$ is a well-defined homomorphism of graded algebras, where Θ is the curvature of a linear connection in $\mathcal{W}$. The mapping w is called the Weil homomorphism.

4 G-consistency

We will consider now classical subgroups G of Lie groups $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ such as: $\mathrm{O}(n)$-subgroup of orthogonal matrices, $\mathrm{SO}(n)$-subgroup of special orthogonal matrices, $\mathrm{U}(n)$-subgroup of unitary matrices with their proper Lie algebras $\mathfrak{g}$: $\mathfrak{o}(n)$ for $\mathrm{O}(n)$ and $\mathrm{SO}(n)$, $\mathfrak{u}(n)$ for $\mathrm{U}(n)$.

Let $(E_i^j), i, j = 1, \dots, n$ be the standard basis of the linear space $M_n(\mathbb{R})$. Let $G(n;\mathcal{C})$ be a sheaf of one of a main classical group (as above) of matrices with entries in the sheaf $\mathcal{C}$

$$G(n;\mathcal{C}(U)) = \{g : U \to G \mid g = g_j^i E_i^j, g_j^i \in \mathcal{C}(U) \quad \text{for } i, j = 1, \dots, n\}.$$

Let $\mathfrak{g}(n;\mathcal{C})$ be the sheaf of its matrix algebra

$$\mathfrak{g}(n;\mathcal{C}(U)) = \{\omega : U \to \mathfrak{g} \mid \omega = \omega_j^i E_i^j, \omega_j^i \in \mathcal{C}(U) \quad \text{for } i = 1, \dots, n\}.$$

By the symbol $G(\mathcal{A}^k)$ we denote a sheaf of one of a main classical group of matrices with entries in $\mathcal{A}^k := A^k(\mathcal{X}, \mathcal{C})$:

$$G(\mathcal{A}^k(U)) = \{\omega : \mathcal{X}^k \to G(n;\mathcal{C}(U)) \mid \omega = \omega_j^i E_i^j, \omega_j^i \in A^k(U)\},$$

and by the symbol $\mathfrak{g}(\mathcal{A}^k)$ we denote a set of its matrix algebra

$$\mathfrak{g}(\mathcal{A}^k(U)) = \{\omega : \mathcal{X}^k \to \mathfrak{g}(n;\mathcal{C}(U)) \mid \omega = \omega_j^i E_i^j, \omega_j^i \in A^k(U)\},$$

where $i, j = 1, \dots, n$.

Definition 4.1. *A k-form $\eta \in A^k(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ is G-consistent, if for $U \in \tau$ there exists a local $\mathcal{C}(U)$ basis $e = (e_1, \ldots, e_n)$ such that*

$$\eta(e) \in \mathfrak{g}(\mathcal{A}^k(U)).$$

By the symbol $A^k_G(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ we will denote a sheaf of G-consistent k-forms from $A^k(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$.

If $\eta \in A^k_G(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$, then there exists an open cover $\mathcal{U}$ of the space M such that an arbitrary set $U \in \mathcal{U}$ is a domain of the local basis e_U and $\eta(e_U) \in \mathfrak{g}(\mathcal{A}^k(U))$. The following consistency condition is true

$$\eta(e_U)_{|U \cap U'} = g^{-1}\eta(e_{U'})_{|U \cap U'} g \quad \text{for } U' \in \mathcal{U}, \tag{18}$$

where $g \in G(n; \mathcal{C}(U \cap U'))$.

Lemma 4.2. *Let $\{\eta(e_U)\}_{U \in \mathcal{U}}$ be a family of matrices of k-forms $\eta(e_U) \in A^k_G(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ satisfying the consistency condition (18), where $\mathcal{U}$ is an open cover of the space M. Then there exists exactly one G-consistent k-form $\eta : \mathcal{X}^k \to \mathcal{L}(\mathcal{W}, \mathcal{W})$ such that $\eta(e_U)$ is its matrix in the basis e_U.*

Proof. We define $\eta_U : \mathcal{X}^k(U) \to \mathcal{L}(\mathcal{W}(\mathcal{U}), \mathcal{W}(\mathcal{U}))$ by the formula

$$\eta_U(X_1, \ldots, X_k)(e_j) = \eta^i_j(e_U)(X_1, \ldots, X_k)e_i \quad \text{for } X_1, \ldots, X_k \in \mathcal{X}(U).$$

It is easy to see the consistency condition

$$\eta(e_U)_{|U \cap U'} = \eta(e_{U'})_{|U \cap U'}.$$

There exists exactly one k-form η such that $\eta_{|U} = \eta_U$ for $U \in \mathcal{U}$. □

Definition 4.3. *A linear connection $D : \mathcal{W} \to A^1(\mathcal{X}, \mathcal{W})$ is called G-consistent if for any $U \in \tau$ there exists a local $\mathcal{C}(U)$-basis e such that*

$$\theta(D, e) \in \mathfrak{g}(\mathcal{A}^1(U)),$$

where $\theta(D, e)$ is the matrix of the linear connection D with respect to the local basis e.

Definition 4.4. *For any k-form $\eta \in A^k_G(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ a local vector basis e is said to be G-admissible, if $\eta(e) \in \mathfrak{g}(\mathcal{A}^k(U))$.*

For any G-consistent forms $\eta_i \in A^{d_i}_G(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$, $i = 1, \ldots, k$ and a symmetric k-form $P \in I_k(\mathfrak{g})$ there exists exactly one $(d_1 + \ldots + d_k)$-form $P(\eta_1, \ldots, \eta_k) \in A^{d_1 + \ldots + d_k}(\mathcal{X}, \mathcal{C})$ such that

$$P(\eta_1, \ldots, \eta_k)_{|U} = P_U(\eta_1, \ldots, \eta_k).$$

For any $(\eta_1, \ldots, \eta_k) \in A^{d_1}(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W})) \times \ldots \times A^{d_k}(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ the mapping $(\eta_1, \ldots, \eta_k) \mapsto P(\eta_1, \ldots, \eta_k) \in A^{d_1 + \ldots + d_k}(\mathcal{X}, \mathcal{C})$ is $\mathcal{C}(M)$-linear.

The following lemma corresponding to lemma 3.1 in the case of G-consistent forms is true.

Lemma 4.5. *Let $\eta_i \in A^{d_i}_G(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$, $i = 1, \ldots, k$ together with $\psi \in A^1_G(\mathcal{X}, \mathcal{L}(\mathcal{W}, \mathcal{W}))$ be G-consistent forms. Then for any $P \in I_k(M_n(\mathbb{R}))$ we have the identities*
(i) $\sum_{i=1}^k (-1)^{d_1 + \ldots + d_{i-1}} P(\eta_1, \ldots, [\psi, \eta_i], \ldots, \eta_k) = 0$,
(ii) $dP(\eta_1, \ldots, \eta_k) = \sum_{i=1}^k (-1)^{l_1 + \ldots + l_{i-1}} P(\eta_1, \ldots, D\eta_i, \ldots, \eta_k)$,
where D is G-consistent linear connection in the sheaf $\mathcal{W}$ and $d_0 := 0$.

One can check that in the case of G-consistent forms the following generalization of the Weil theorem is also true.

Theorem 4.6. *Let $\mathcal{W}$ be a locally free sheaf of Φ-fields on M of rank n, $n \in \mathbb{N}$ and let D let be G-consistent linear connection in $\mathcal{W}$. Suppose that $P \in I_k(M_n(\mathbb{R}))$. Then*
(a) *the $2k$-form $P(\Theta)$ is closed, i.e. $dP(\Theta) = 0$;*
(b) *the cohomology class $[P(\Theta)] \in \mathcal{H}^{2k}$ is independent of the connection D.*

5 Characteristic classes

Similarly to the theory of characteristic classes on differential spaces [7] we can consider characteristic classes in a more general case.

Let $(M, \tau, \mathcal{C})$ be a structured space. Let $\mathcal{W}$ be a locally free sheaf of rank $n \in \mathbb{N}$ of local sections Φ, where $\Phi(p)$ is an n-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, $p \in M$. Let $\mathrm{GL}(n, \mathcal{C})$ be the general linear group sheaf of degree $n \in \mathbb{N}$ of non-singular matrices with entries in $\mathcal{C}$, $M(n, \mathcal{C})$ be the full matrix algebra sheaf of all matrices of degree $n \in \mathbb{N}$ with entries in $\mathcal{C}$.

5.1 Pontrjagin class

Let $\mathcal{W}$ be locally free sheaf of Φ-fields on the structured space $(M, \tau, \mathcal{C})$, of rank $n = 2m$, $m \in \mathbb{N}$, where $\Phi(p)$ is an n-dimensional vector space over $\mathbb{R}$, $p \in M$.

Let the sheaf $\mathcal{W}$ be equipped with a metric tensor $g : \mathcal{W} \times \mathcal{W} \to \mathcal{C}$, i.e. a $\mathcal{C}$-bilinear morphism such that

1. $g(s_1, s_2) = g(s_2, s_1) \ \ \forall s_1, s_2 \in \mathcal{W}$,
2. $(g(s_1, s_2) = 0 \ \ \forall s_1 \in \mathcal{W}) \Rightarrow s_2 = 0$.

The operator of Riemannian metric $g : \mathcal{W} \times A^1(\mathcal{X}, \mathcal{W}) \to \mathcal{C}$ in the sheaf $\mathcal{W}$ is given by the formula

$$g(s, \omega)(X) = g(s, \omega(X)), \quad s \in \mathcal{W}, \omega \in A^1(\mathcal{X}, \mathcal{W}), X \in \mathcal{X}.$$

Let $D : \mathcal{W} \to A^1(\mathcal{X}, \mathcal{W})$ be a linear connection in the sheaf $\mathcal{W}$ which is consistent with the metric g [8], i.e.

$$dg(s_1, s_2) = g(Ds_1, s_2) + g(s_1, Ds_2) \quad \forall s_1, s_2 \in \mathcal{W}. \tag{19}$$

The metric tensor g allows to introduce a local orthonormal basis in the sheaf $\mathcal{W}$. In this orthonormal basis the sheaf $\mathrm{GL}(n, \mathcal{C})$ can be reduced to the subsheaf $\mathrm{O}(n, \mathcal{C})$ of the orthogonal group of matrices with entries in $\mathcal{C}$.

Let further $D : \mathcal{W} \to A^1(\mathcal{X}, \mathcal{W})$ be $\mathrm{O}(n, \mathcal{C})$-consistent connection in $\mathcal{W}$. Let $\Theta(e) = (\Theta_i^j(e))$ be the matrix of curvature 2-forms of the connection D in the local basis $e = (e_1, \dots, e_n)$.

Lemma 5.1. *Let $e = (e_1, \dots, e_n)$ be a local orthonormal basis of Φ-fields of the sheaf $\mathcal{W}$ i.e. $g(e_i, e_j) = \delta_{ij}$. If the connection D in the sheaf $\mathcal{W}$ is consistent with the Riemannian metric g, then its connection matrix $\theta(e) = (\theta_i^j(e))$ is skew-symmetric in this basis.*

Proof. If the connection D is consistent with the metric g, then from (19) we have

$$\begin{aligned} dg(e_i, e_j) &= g(De_i, e_j) + g(e_i, De_j) = g(\theta_i^k(e) e_k, e_j) + g(e_i, \theta_i^k(e) e_k) \\ &= \theta_i^j(e) + \theta_j^i(e). \end{aligned}$$

On the other hand

$$dg(e_i, e_j) = d\delta_{ij} = 0,$$

hence

$$\theta_i^j(e) + \theta_j^i(e) = 0. \qquad \square$$

Corollary 5.2. *If the matrix $\theta(e)$ of connection 1-forms in the orthonormal basis e of the sheaf $\mathcal{W}$ is skew-symmetric, then also the matrix $\Theta(e)$ of curvature 2-forms of the connection D is skew-symmetric in this basis, $\Theta(e) \in \mathfrak{o}(A^2(U))$.*

Proof. Let $e = (e_1, \dots, e_n)$ be a local orthonormal basis of Φ-fields of the sheaf $\mathcal{W}$. We have $\Theta(e) = d\theta(e) + \theta(e) \wedge \theta(e)$. We see that if $\theta_j^i(e) = -\theta_i^j(e)$ for $i, j = 1, \dots, n$, then also $d\theta(e)$ is a skew-symmetric matrix. It is enough to show the skew-symmetry of the matrix $\theta(e) \wedge \theta(e)$. We have:

$$(\theta \wedge \theta)_k^i(e) = \theta_j^i(e) \wedge \theta_k^j(e) = -\theta_i^j(e) \wedge (-\theta_j^k(e)) = \theta_i^j(e) \wedge \theta_j^k(e)$$

$$= -(\theta_j^k(e) \wedge \theta_i^j(e)) = -(\theta \wedge \theta)_i^k(e)$$

for any $i, j, k = 1, \dots, n$. From the above we obtain that the matrix $\Theta(e) = d\theta(e) + \theta(e) \wedge \theta(e)$ is skew-symmetric as a sum of two skew-symmetric matrices in the orthonormal basis. □

If A is skew-symmetric matrix, $A \in \mathfrak{o}(2m)$, it can be block diagonalized by an element $S \in \mathrm{O}(2m)$ [9]

$$A \to S^T A S = \Lambda = \begin{pmatrix} 0 & \lambda_1 & & & & & & \\ -\lambda_1 & 0 & & & & & 0 & \\ & & 0 & \lambda_2 & & & & \\ & & -\lambda_2 & 0 & & & & \\ & & & & \ddots & & & \\ & 0 & & & & & 0 & \lambda_m \\ & & & & & & -\lambda_m & 0 \end{pmatrix},$$

where $\lambda_1, \dots, \lambda_m$, $m \in \mathbb{N}$, are eigenvalues of A. Next, the block diagonalized matrix Λ can be diagonalized by an element $B \in GL(2m, \mathbb{C})$ as

$$\Lambda \to B\Lambda B^{-1} = \begin{pmatrix} i\lambda_1 & & & & & & & \\ & -i\lambda_1 & & & & & 0 & \\ & & i\lambda_2 & & & & & \\ & & & -i\lambda_2 & & & & \\ & & & & \ddots & & & \\ & 0 & & & & & i\lambda_m & \\ & & & & & & & -i\lambda_m \end{pmatrix}.$$

The above procedure can be applied for the matrix $\Theta(e)$ of curvature 2-forms of the connection D, which is skew-symmetric in the orthonormal basis $e = (e_1, \dots, e_n)$, $\Theta(e) \in \mathfrak{o}(A^2(U))$. For $\lambda_k = \Theta_k(e)$, where $\Theta_k(e)$ are eigenvalues (being 2-forms) of the matrix $\Theta(e)$ for $k = 1, \dots, m$ we obtain the following diagonalized matrix

$$\tilde{\Theta}(e) = \mathrm{diag}(i\Theta_1(e), -i\Theta_1(e) \dots, i\Theta_m(e), -i\Theta_m(e)).$$

We define the **total Pontrjagin class** $p(\mathcal{W})$ of the locally free sheaf $\mathcal{W}$ of rank $n = 2m$ as a class represented by the $4m$-form $p(\Theta)$

$$\begin{aligned} p(\Theta)_{|U} &= p(\Theta(e)) = \det\left(I_n + \frac{\tilde{\Theta}(e)}{2\pi}\right) = \\ &= \det\begin{pmatrix} 1 + \frac{i}{2\pi}\Theta_1(e) & & & & 0 \\ & 1 - \frac{i}{2\pi}\Theta_1(e) & & & \\ & & \ddots & & \\ & & & 1 + \frac{i}{2\pi}\Theta_m(e) & \\ 0 & & & & 1 - \frac{i}{2\pi}\Theta_m(e) \end{pmatrix} = \\ &= 1 + \left[\left(\frac{\Theta_1(e)}{2\pi}\right)^2 + \dots + \left(\frac{\Theta_m(e)}{2\pi}\right)^2\right] + \\ &+ \left[\left(\frac{\Theta_1(e)}{2\pi}\right)^2 \wedge \left(\frac{\Theta_2(e)}{2\pi}\right)^2 + \dots + \left(\frac{\Theta_{m-1}(e)}{2\pi}\right)^2 \wedge \left(\frac{\Theta_m(e)}{2\pi}\right)^2\right] + \\ &+ \left(\frac{\Theta_1(e)}{2\pi}\right)^2 \wedge \dots \wedge \left(\frac{\Theta_m(e)}{2\pi}\right)^2, \end{aligned}$$

where U is a domain of the local vector basis $e = (e_1, \dots, e_n)$, $n = 2m$ and p is $O(n, \mathcal{C}(U))$-invariant polynomial.

Since the $4k$-form $p_k(\Theta)$ is closed, it defines the k**-th Pontrjagin class** $p_k(\mathcal{W}) = [p_k(\Theta)] \in \mathcal{H}^{4k}$.

The characteristic forms which generate Pontrjagin classes are given by the formulas

$$p_1(\Theta) = \left(\frac{\Theta_1}{2\pi}\right)^2 + \ldots + \left(\frac{\Theta_m}{2\pi}\right)^2,$$
$$p_2(\Theta) = \left(\frac{\Theta_1}{2\pi}\right)^2 \wedge \left(\frac{\Theta_2}{2\pi}\right)^2 + \ldots + \left(\frac{\Theta_{m-1}}{2\pi}\right)^2 \wedge \left(\frac{\Theta_m}{2\pi}\right)^2,$$
$$\ldots$$
$$p_m(\Theta) = \left(\frac{\Theta_1}{2\pi}\right)^2 \wedge \ldots \wedge \left(\frac{\Theta_m}{2\pi}\right)^2.$$

5.2 Chern class

Let $\mathcal{W}$ be a locally free sheaf of rank $n \in \mathbb{N}$ of local sections Φ, where $\Phi(p)$ is an n-dimensional vector space over $\mathbb{C}$, $p \in M$. Let $\mathcal{W}$ be equipped with a Hermitian metric $h : \mathcal{W} \times A^1(\mathcal{X}, \mathcal{W}) \to \mathcal{C}$, which is given by the formula

$$h(s, \omega)(X) = h(s, \omega(X)), \quad s \in \mathcal{W}, \omega \in A^1(\mathcal{X}, \mathcal{W}), X \in \mathcal{X}.$$

Let $D : \mathcal{W} \to A^1(\mathcal{X}, \mathcal{W})$ be a linear connection in the sheaf $\mathcal{W}$ which is consistent with the Hermitian metric h

$$dh(s_1, s_2) = h(Ds_1, s_2) + h(s_1, Ds_2) \quad \forall s_1, s_2 \in \mathcal{W}. \tag{20}$$

The Hermitian metric allows to introduce a local orthonormal basis in the sheaf $\mathcal{W}$. In this orthonormal basis the sheaf $\mathrm{GL}(n, \mathcal{C})$ can be reduced to the subsheaf $\mathrm{U}(n, \mathcal{C})$ of the unitary group of matrices with entries in $\mathcal{C}$.

Let further $D : \mathcal{W} \to A^1(\mathcal{X}, \mathcal{W})$ be the $\mathrm{U}(n, \mathcal{C})$-consistent linear connection in $\mathcal{W}$. Let $\Theta(e) = (\Theta_i^j(e))$ be the matrix of curvature 2-forms of the connection D in the local basis $e = (e_1, \ldots, e_n)$.

Analogously as in the real case, there the following lemma holds.

Lemma 5.3. *Let $e = (e_1, \ldots, e_n)$ be a local orthonormal basis of Φ-fields of the locally free sheaf $\mathcal{W}$ i.e. $h(e_i, e_j) = \delta_{ij}$. If the connection D in the sheaf $\mathcal{W}$ is consistent with the Hermitian metric h then the connection matrix $\theta(e) = (\theta_i^j(e))$ is skew-Hermitian in this basis.*

Proof. If the connection D is consistent with the Hermitian metric h, then from (20) we have:

$$\begin{aligned} dh(e_i, e_j) &= h(De_i, e_j) + h(e_i, De_j) = h(\theta_i^k(e)e_k, e_j) + h(e_i, \theta_i^k(e)e_k) \\ &= \bar{\theta}_i^j(e) + \bar{\theta}_j^i(e). \end{aligned}$$

On the other hand

$$dh(e_i, e_j) = d\delta_{ij} = 0,$$

hence

$$\bar{\theta}_i^j(e) = -\bar{\theta}_j^i(e).$$ □

Corollary 5.4. *If the matrix $\theta(e)$ of connection 1-forms is skew-Hermitian in the orthonormal basis e of the sheaf $\mathcal{W}$, then also the matrix $\Theta(e)$ of curvature 2-forms of the connection D is skew-Hermitian in this basis, $\Theta(e) \in \mathfrak{u}(A^2(U))$.*

Proof. Let $e = (e_1, \ldots, e_n)$ be a local orthonormal basis of Φ-fields of the sheaf $\mathcal{W}$. We have $\Theta(e) = d\theta(e) + \theta(e) \wedge \theta(e)$. We see that if $\bar{\theta}_j^i(e) = -\bar{\theta}_i^j(e)$ for $i, j = 1, \ldots, n$, then also $d\theta(e)$ is a skew-Hermitian matrix. It is enough to show that the matrix $\theta(e) \wedge \theta(e)$ is skew-Hermitian. We have

$$\begin{aligned} (\theta \wedge \theta)_k^i(e) &= \theta_j^i(e) \wedge \theta_k^j(e) = -\bar{\theta}_i^j(e) \wedge (-\bar{\theta}_j^k(e)) = \bar{\theta}_i^j(e) \wedge \bar{\theta}_j^k(e) \\ &= -(\bar{\theta}_j^k(e) \wedge \bar{\theta}_i^j(e)) = -(\bar{\theta} \wedge \bar{\theta})_i^k(e) \end{aligned}$$

for any $i, j, k = 1, \ldots, n$. From the above we obtain that the matrix $\Theta(e) = d\theta(e) + \theta(e) \wedge \theta(e)$ is skew-Hermitian as a sum of two skew-Hermitian martices in the orthonormal basis. □

If the matrix $\Theta(e)$ of curvature 2-forms of the connection D is skew-Hermitian in the orthonormal basis $e = (e_1, \ldots, e_n)$, $\Theta(e) \in \mathfrak{u}(A^2(U))$, then there exists a matrix $B \in \mathrm{U}(n, \mathcal{C}(U))$ such that

$$B\Theta(e)B^{-1} = \mathrm{diag}(\Theta_1(e), \ldots, \Theta_n(e)),$$

where $\Theta_1(e), \ldots, \Theta_n(e)$ are 2-forms.

Let us denote

$$\tilde{\Theta}(e) = \mathrm{diag}(\Theta_1(e), \ldots, \Theta_n(e)).$$

The Chern class $c(\Theta(e))$ is determined by the polynomial function which is $\mathrm{U}(n, \mathcal{C}(U))$-invariant and is given by the formula

$$\hat{P}(\tilde{\Theta}(e)) = \det\left(I_n + \frac{i}{2\pi}\tilde{\Theta}(e)\right),$$

where $\tilde{\Theta}(e)$ is the diagonal matrix of curvature 2-forms of the $\mathrm{U}(n, \mathcal{C}(U))$-consistent linear connection D.

The total Chern class $c(\mathcal{W})$ of the sheaf $\mathcal{W}$ is represented by the $2n$-form $c(\Theta)$, $c(\mathcal{W}) = [c(\Theta)]$, and

$$\begin{aligned} c(\Theta)_{|U} &= c(\Theta(e)) = \hat{P}(\tilde{\Theta}(e)) = \det\left(I_n + \frac{i}{2\pi}\tilde{\Theta}(e)\right) = \\ &= \det\begin{pmatrix} 1 + \frac{i}{2\pi}\Theta_1(e) & & 0 \\ & \ddots & \\ 0 & & 1 + \frac{i}{2\pi}\Theta_n(e) \end{pmatrix} = \\ &= \prod_{j=1}^{n}(1 + \frac{i}{2\pi}\Theta_j(e)). \end{aligned}$$

According to the above notation the total Chern class of the sheaf $\mathcal{W}$ can be expressed as follows [10]

$$c(\mathcal{W}) = 1 + c_1(\mathcal{W}) + c_2(\mathcal{W}) + \ldots + c_n(\mathcal{W}).$$

If the $2k$-form $c_k(\Theta)$ is closed, it defines the **k-th Chern class**

$$c_k(\mathcal{W}) = [c_k(\Theta)] \in \mathcal{H}^{2k}.$$

The characteristic forms which generate Chern classes are given by the formulas

$$\begin{aligned} &c_0(\Theta) = 1, \\ &c_1(\Theta) = \frac{i}{2\pi}(\Theta_1 + \Theta_2 + \ldots + \Theta_n), \\ &c_2(\Theta) = \left(\frac{i}{2\pi}\right)^2(\Theta_1 \wedge \Theta_2 + \ldots + \Theta_{n-1} \wedge \Theta_n), \\ &\ldots \\ &c_n(\Theta) = \left(\frac{i}{2\pi}\right)^n \Theta_1 \wedge \Theta_2 \wedge \ldots \wedge \Theta_n. \end{aligned}$$

Proposition 5.5. *Let $\mathcal{W}$ be a locally free sheaf of local Φ-fields and let $D : \mathcal{W} \to A^1(\mathcal{X}, \mathcal{W})$ be a $\mathrm{U}(n, \mathcal{C}(U))$-consistent connection in this sheaf. Then the following properties of the Chern classes are true*

1. *Naturality*

$$c(f^*\mathcal{W}) = f^*c(\mathcal{W}),$$

where $f : M' \to M$ is a mapping between two structured spaces $(M, \tau_M, \mathcal{C}_M)$ and $(M', \tau_{M'}, \mathcal{C}_{M'})$;

2. *Whitney sum*

$$c(\mathcal{W}_1 \oplus \mathcal{W}_2) = c(\mathcal{W}_1) \wedge c(\mathcal{W}_2),$$

where $\mathcal{W}_1 \oplus \mathcal{W}_2 = \{s_1 + s_2 : s_1 \in \mathcal{W}_1, s_2 \in \mathcal{W}_2\}$ is the direct sum of sheaves $\mathcal{W}_1$ and $\mathcal{W}_2$, of ranks n and m, respectively, $n, m \in \mathbb{N}$.

5.3 Euler class

Let $(TM, T\mathcal{C})$ be the structured space tangent to $(M, \mathcal{C})$. Let $\mathcal{W}$ be a locally free sheaf of rank $n = 2m, m \in \mathbb{N}$ of local Φ-fields on the space TM, which is oriented and equipped with a Riemannian metric. The orientation allows to reduce the sheaf $\mathrm{GL}(n, \mathcal{C})$ to the subsheaf $\mathrm{GL}^+(n, \mathcal{C})$ of general linear group of matrices with positive determinants. Riemannian metric allows to reduce the sheaf $\mathrm{GL}^+(n, \mathcal{C})$ to the subsheaf $\mathrm{SO}(n, \mathcal{C})$ of special linear group.

Let $D : \mathcal{W} \to A^1(\mathcal{X}; \mathcal{W})$ be a $\mathrm{SO}(n, \mathcal{C})$-consistent linear connection in $\mathcal{W}$. The proper algebra sheaf of the sheaf $G = \mathrm{SO}(n, \mathcal{C})$ is $\mathfrak{g} = \mathfrak{o}(n, \mathcal{C})$.

The **total Euler class** $e(\mathcal{W})$ of the sheaf $\mathcal{W}$ of local sections of rank $n = 2m$ on the space TM we define as a class, which is represented by the form $e(\Theta)$

$$e(\Theta)_{|U} = e(\Theta(e)) = \mathrm{Pf}\left(\frac{\Theta(e)}{2\pi}\right),$$

where U is the domain of the local vector basis e and

$$\mathrm{Pf}(A) = \frac{1}{2^m m!} \sum_{\sigma \in S^{2m}} \mathrm{sgn}\ \sigma\ A_{\sigma(1)\sigma(2)} \dots A_{\sigma(2m-1)\sigma(2m)}$$

is the **Pfaffian of the matrix** $A \in \mathfrak{o}(n, \mathcal{C}(U))$, which is $\mathrm{SO}(n, \mathcal{C}(U))$-invariant.

The form which represents the Euler class can be expressed as follows

$$\begin{aligned} e(\Theta(e)) = \mathrm{Pf}(\frac{\Theta(e)}{2\pi}) &= \mathrm{Pf}(\frac{\bar{\Theta}(e)}{2\pi}) = \\ = \frac{(-1)^m}{(4\pi)^m m!} &\sum_{\sigma \in S^{2m}} \mathrm{sgn}\, \sigma \bar{\Theta}_{\sigma(1)\sigma(2)}(e) \wedge \dots \wedge \bar{\Theta}_{\sigma(2m-1)\sigma(2m)}(e), \end{aligned}$$

where the matrix $\Theta(e)$ of curvature 2-forms can be block diagonalized to the matrix $\bar{\Theta}(e) = (\bar{\Theta}_j(e)),\ j = 1, \dots, m$.

From the equalities between the proper characteristic forms in an arbitrary basis e of the sheaf $\mathcal{W}$ we obtain the equalities between that forms. As a consequence, the following equalities between characteristic classes are true

1. Chern and Pontrjagin classes
$$p_j(\mathcal{W}) = (-1)^j c_{2j}(\mathcal{W}^C),$$
where $\mathcal{W}^C$ is the complexification of the sheaf $\mathcal{W}$.
2. Euler and Pontrjagin classes
$$e(\mathcal{W}) \wedge e(\mathcal{W}) = p_m(\mathcal{W}).$$
3. Euler and Chern classes
$$e(\mathcal{W}) \wedge e(\mathcal{W}) = (-1)^m c_{2m}(\mathcal{W}^C).$$

From the Weil theorem we know that characteristic classes are independent of the choice of connection. Thanks to this we obtain the equalities between different characteristic classes. Taking connections consistent with different sheaves of the classical subgroups, we obtain the proper characteristic classes. That classes are independent of the choice of connection, but they are determined by the polynomials invariant of the proper subsheaf of $\mathrm{GL}(n, \mathcal{C})$.

In the following table we present characteristic classes, the proper sheaves of the main classical groups and sheaves of their algebras.

Table 1. Characteristic classes, the proper sheaves of the main classical groups and sheaves of their algebras

characteristic class	sheaf of group G	algebra sheaf $\mathfrak{g}$
Chern class	$\mathrm{U}(n, \mathcal{C})$	$\mathfrak{u}(n, \mathcal{C})$
Pontrjagin class	$\mathrm{O}(n, \mathcal{C})$	$\mathfrak{o}(n, \mathcal{C})$
Euler class	$\mathrm{SO}(n, \mathcal{C})$	$\mathfrak{o}(n, \mathcal{C})$

References

[1] Heller M., Sasin W., Structured spaces and their application to relativistic physics, J. Math. Phys., 1995, 36 (7), 3644-3662
[2] Sikorski R., Introduction to differential geometry (in Polish), PWN, Warsaw, 1972
[3] Sasin W., Żekanowski Z., On some sheaves over a differential space, Arch. Math. 4, Scripta Fac. Sci. Nat. UJ EP BRUNENSIS, 1982, 18, 193-199
[4] Mallios A., Geometry of vector sheaves. An axiomatic approach to differential geometry, vol.I: Vector sheaves. General Theory, Kluwer Academic Publishers, 1998
[5] Mallios A., Geometry of vector sheaves. An axiomatic approach to differential geometry, vol.II: Geometry. Examples and Applications, Kluwer Academic Publishers, 1998
[6] Sasin W., On Weil homomorphism in locally free sheaves over a differential space, Demonstratio Math., 1983, 16 (3), 699-711
[7] Falkiewicz E., An algebraic approach to Weil homomorphism on differential spaces, PhD thesis (in Polish), Warsaw, 2012
[8] Hirzebruch F., Topological methods in algebraic geometry, Springer Verlag, Berlin, Heidelberg, New York, 1978
[9] Nakahara M., Geometry, topology and physics, second edition, Institute of Physics Publishing, Bristol and Philadelphia, 2003
[10] Milnor J. W., Stasheff J. D., Characteristic classes, Princeton University Press and University of Tokyo Press, Princeton, New Yersey, 1974