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RETRACTS OF ULTRAHOMOGENEOUS STRUCTURES IN THE CONTEXT OF KATĚTOV FUNCTORS

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Abstract. In this paper, we characterize retracts of a wide class of Fraïssé limits using the tools developed in a recent paper by W. Kubiś and the present author, which we refer to as Katětov functors. This approach enables us to conclude that in many cases, a structure is a retract of a Fraïssé limit if and only if it is algebraically closed in the surrounding category.

1. Introduction

In 1927, a paper [9] by P. S. Urysohn was published (posthumously) in which he constructed what we nowadays refer to as the Urysohn space – a complete separable metric space which is ultrahomogeneous and embeds all separable metric spaces. Some sixty years later, M. Katětov published a paper [6] in which a new, more streamlined construction of the Urysohn space was presented. The elegance of Katětov's construction caught the eye of the scientific community and started several new lines of research. One such spin-off is presented in [8], where we apply the idea of the Katětov's construction of the Urysohn space to a wide range of Fraïssé limits, showing thus that the Katětov's construction draws its strength from its strong categorical properties.

Let us now briefly outline the Katětov's construction of the Urysohn space [6] in the case of the rational Urysohn space. Let X be a metric space with rational distances. A *Katětov function over* X is every function $\alpha : X \to \mathbb{Q}$ such that

$$|\alpha(x) - \alpha(y)| \leqslant d(x, y) \leqslant \alpha(x) + \alpha(y)$$

for all $x, y \in X$. Let $K(X)$ be the set of all Katětov functions over X. The sup metric turns $K(X)$ into a metric space. There is a natural isometric

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embedding $X \hookrightarrow K(X)$ which takes $a \in X$ to $d(a, \cdot) \in K(X)$. Hence we get a chain of embeddings

$$X \hookrightarrow K(X) \hookrightarrow K^2(X) \hookrightarrow K^3(X) \hookrightarrow \cdots$$

whose colimit is easily seen to be the rational Urysohn space.

It was first observed in [1] that the construction K is actually functorial with respect to embeddings. However, more is true: if $\mathcal{A}$ is the category of all finite metric spaces with rational distances and nonexpansive maps, and $\mathcal{C}$ is the category of all countable metric spaces with rational distances and nonexpansive maps, then K can be turned into a functor from $\mathcal{A}$ to $\mathcal{C}$ straightforwardly [8]. This observation is then expanded to a general setting, where $\mathcal{A}$ is a category of all "finitely generated structures" and $\mathcal{C}$ is the category of colimits of ω-chains of objects from $\mathcal{A}$ and K is a functor $\mathcal{A} \to \mathcal{C}$ with certain properties. Our main result in [8] is that the existence of such a functor $K : \mathcal{A} \to \mathcal{C}$, which we refer to as the *Katětov functor*, implies that $\mathcal{A}$ is an amalgamation class, and that its Fraïssé limit can be constructed in the fashion of the Katětov's construction. Details are outlined in Section 2.

In this paper, we characterize retracts of a wide class of Fraïssé limits which can be obtained by the Katětov construction. Another characterization of retracts of Fraïssé limits in terms of categorical properties of those objects was presented in [7]. In this paper, however, we generalize the main result of [4] and provide the characterization of retracts of a large class of Fraïssé limits using the tools developed in [8], which then enable us to conclude that in many cases a structure is a retract of a Fraïssé limit if and only if it is algebraically closed in the surrounding category. This is the content of Section 3.

2. Preliminaries

Let $\Delta = \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$ be a first-order language where $\mathcal{R}$ is a set of relational symbols, $\mathcal{F}$ is a set of functional symbols and $\mathcal{C}$ is a set of constant symbols. We say that Δ is a *purely relational language* if $\mathcal{F} = \mathcal{C} = \emptyset$. For a Δ-structure A and $X \subseteq A$, by $\langle X \rangle_A$ we denote the substructure of A generated by X. We say that A is *finitely generated* if $A = \langle X \rangle_A$ for some finite $X \subseteq A$.

For a countable relational structure A, the class of all finitely generated structures that embed into A is called the *age* of A and we denote it by $\operatorname{age}(A)$. A class $\mathcal{K}$ of finitely generated structures is an *age* if there is a countable structure A such that $\mathcal{K} = \operatorname{age}(A)$. It is easy to see that a class $\mathcal{K}$ of finitely generated structures is an age if

- $\mathcal{K}$ is an abstract class (that is, closed for isomorphisms);
- there are at most countably many pairwise nonisomorphic structures in $\mathcal{K}$;
- $\mathcal{K}$ has the hereditary property (HP): if $B \in \mathcal{K}$ and $A \hookrightarrow B$ then $A \in \mathcal{K}$;

- $\mathcal{K}$ has the joint embedding property (JEP): for all $A, B \in \mathcal{K}$ there is a $C \in \mathcal{K}$ such that $A \hookrightarrow C$ and $B \hookrightarrow C$;

An age $\mathcal{K}$ is a *Fraïssé class* (= amalgamation class) if $\mathcal{K}$ satisfies the amalgamation property (AP): for all embeddings $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$ where $A, B, C \in \mathcal{K}$ there is a $D \in \mathcal{K}$ and embeddings $u : B \hookrightarrow D$ and $v : C \hookrightarrow D$ such that $u \circ f = v \circ g$.

A countable structure L is *ultrahomogeneous* if every isomorphism between two finitely generated substructures of L extends to an automorphism of L. More precisely, L is ultrahomogeneous if for all $A, B \in \mathrm{age}(L)$, embeddings $j_A : A \hookrightarrow L$ and $j_B : B \hookrightarrow L$, and for every isomorphism $f : A \to B$ there is an automorphism f^* of L such that $j_B \circ f = f^* \circ j_A$.

The main result of the Fraïssé theory [5] is that the age of a countable ultrahomogeneous structure is a Fraïssé class, and vice versa, for every Fraïssé class $\mathcal{K}$ there is a unique (up to isomorphism) countable ultrahomogeneous structure A such that $\mathcal{K} = \mathrm{age}(A)$. We say that A is the *Fraïssé limit* of $\mathcal{K}$.

A countable structure L is *$\mathcal{C}$-morphism-homogeneous*, if every $\mathcal{C}$-morphism between two finitely generated substructures of L extends to a $\mathcal{C}$-endomorphism of L. More precisely, L is $\mathcal{C}$-morphism-homogeneous if for all $A, B \in \mathrm{age}(L)$, embeddings $j_A : A \hookrightarrow L$ and $j_B : B \hookrightarrow L$, and for every $\mathcal{C}$-morphism $f : A \to B$ there is a $\mathcal{C}$-endomorphism f^* of L such that $j_B \circ f = f^* \circ j_A$. In particular, if $\mathcal{C}$ is the category of all countable Δ-structures with all homomorphisms between them, instead of saying that L is $\mathcal{C}$-morphism-homogeneous, we say that L is *homomorphism-homogeneous* [3].

Let $\mathcal{C}$ be a category of Δ-structures. A *chain in* $\mathcal{C}$ is a chain of objects and embeddings of the form $C_1 \hookrightarrow C_2 \hookrightarrow C_3 \hookrightarrow \cdots$. Note that although there may be other kinds of morphisms in $\mathcal{C}$, a chain always consists of objects and embeddings. For a $C \in \mathrm{Ob}(\mathcal{C})$ let $\mathrm{Aut}(C)$ denote the permutation group consisting of all automorphisms of C, and let $\mathrm{End}_{\mathcal{C}}(C)$ denote the transformation monoid consisting of all $\mathcal{C}$-morphisms $C \to C$.

Standing assumption. Throughout the paper, we assume the following. Let Δ be a first-order language, let $\mathcal{C}$ be a category of countable Δ-structures and some appropriately chosen class of morphisms that includes, but is not limited to, embeddings. Let $\mathcal{A}$ be the full subcategory of $\mathcal{C}$ spanned by all finitely generated structures in $\mathcal{C}$. We also assume that the following holds:

- $\mathcal{C}$ has colimits of chains: for every chain $C_1 \hookrightarrow C_2 \hookrightarrow \cdots$ in $\mathcal{C}$ there is an $L \in \mathrm{Ob}(\mathcal{C})$ which is a colimit of this diagram in $\mathcal{C}$;
- every $C \in \mathrm{Ob}(\mathcal{C})$ is a colimit of some chain $A_1 \hookrightarrow A_2 \hookrightarrow \cdots$ in $\mathcal{A}$;

- $\mathcal{A}$ has only countably many isomorphism types; and
- $\mathcal{A}$ has (HP) and (JEP).

We say that $C \in \mathrm{Ob}(\mathcal{C})$ is a *one-point extension* of $B \in \mathrm{Ob}(\mathcal{C})$ if there is an embedding $j : B \hookrightarrow C$ and an $x \in C \backslash j(B)$ such that $C = \langle j(B) \cup \{x\} \rangle_C$. In that case, we write $j : B \dot{\hookrightarrow} C$ or simply $B \dot{\hookrightarrow} C$.

Definition 2.1. [8] A functor $K^0 : \mathcal{A} \to \mathcal{C}$ is a *Katětov functor* if:

- K^0 preserves embeddings, that is, if $f : A \to B$ is an embedding in $\mathcal{A}$, then $K^0(f) : K^0(A) \to K^0(B)$ is an embedding in $\mathcal{C}$; and
- there is a natural transformation $\eta^0 : \mathrm{ID} \to K^0$ such that for every one-point extension $A \dot{\hookrightarrow} B$ where $A, B \in \mathrm{Ob}(\mathcal{A})$, there is an embedding $g : B \hookrightarrow K^0(A)$ satisfying

$$\begin{array}{ccc} A & \xhookrightarrow{\eta^0_A} & K^0(A) \\ \dot{\downarrow} & \nearrow_{g} & \\ B & & \end{array} \tag{1}$$

Theorem 2.2. [8] *If there exists a Katětov functor $K^0 : \mathcal{A} \to \mathcal{C}$, then there is a functor $K : \mathcal{C} \to \mathcal{C}$ such that:*

- *K is an extension of K^0 (that is, K and K^0 coincide on $\mathcal{A}$);*
- *there is a natural transformation $\eta : \mathrm{ID} \to K$ which is an extension of η^0 (that is, $\eta_A = \eta^0_A$ whenever $A \in \mathrm{Ob}(\mathcal{A})$);*
- *K preserves embeddings.*

We also say that K is a Katětov functor and from now on denote both K and K^0 by K, and both η and η^0 by η.

Let $K : \mathcal{C} \to \mathcal{C}$ be a Katětov functor. A *Katětov construction* [8] is a chain of the form:

$$C \xhookrightarrow{\eta_C} K(C) \xhookrightarrow{\eta_{K(C)}} K^2(C) \xhookrightarrow{\eta_{K^2(C)}} K^3(C) \hookrightarrow \cdots$$

where $C \in \mathrm{Ob}(\mathcal{C})$. We denote the colimit of this chain by $K^\omega(C)$. An object $L \in \mathrm{Ob}(\mathcal{C})$ *can be obtained by the Katětov construction starting from C* if $L = K^\omega(C)$. We say that L *can be obtained by the Katětov construction* if $L = K^\omega(C)$ for some $C \in \mathrm{Ob}(\mathcal{C})$.

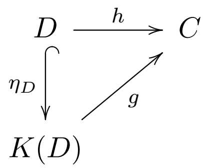
Note that K^ω is actually a functor $\mathcal{C} \to \mathcal{C}$. It is easy to show that K^ω preserves embeddings. Moreover, the canonical embeddings $\eta^\omega_A : A \hookrightarrow K^\omega(A)$ constitute a natural transformation $\eta^\omega : \mathrm{ID} \to K^\omega$. Thus, we have that $K^\omega : \mathcal{C} \to \mathcal{C}$ is a Katětov functor as well [8].

Theorem 2.3. [8] *If there exists a Katětov functor $K : \mathcal{A} \to \mathcal{C}$, then $\mathcal{A}$ is an amalgamation class, it has a Fraïssé limit L in $\mathcal{C}$, and L can be obtained by the Katětov construction starting from an arbitrary $C \in \mathrm{Ob}(\mathcal{C})$. Moreover, L is $\mathcal{C}$-morphism-homogeneous.*

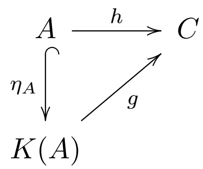
Consequently, if the Katětov functor is defined on a category of countable Δ-structures and all homomorphisms between Δ-structures, the Fraïssé limit of $\mathcal{A}$ is both ultrahomogeneous and homomorphism-homogeneous.

3. K-closed sets and retracts of Fraïssé limits

We say that $C \in \mathrm{Ob}(\mathcal{C})$ is *K-closed* if for every $D \in \mathrm{Ob}(\mathcal{C})$ and every $\mathcal{C}$-morphism $h : D \to C$, there exists a $\mathcal{C}$-morphism $g : K(D) \to C$ such that $g \circ \eta_D = h$:

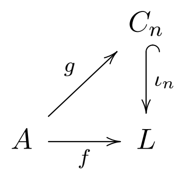


We say that $C \in \mathrm{Ob}(\mathcal{C})$ is *locally K-closed* if for every $A \in \mathrm{Ob}(\mathcal{A})$ and every $\mathcal{C}$-morphism $h : A \to C$, there exists a $\mathcal{C}$-morphism $g : K(A) \to C$ such that $g \circ \eta_A = h$:



We say that a Katětov functor $K : \mathcal{A} \to \mathcal{C}$ is *locally finite* if $K(A) \in \mathrm{Ob}(\mathcal{A})$ for every $A \in \mathrm{Ob}(\mathcal{A})$.

Lemma 3.1. [8] *Let $C_1 \hookrightarrow C_2 \hookrightarrow \cdots$ be a chain in $\mathcal{C}$ and let L be the colimit of the chain with the canonical embeddings $\iota_k : C_k \hookrightarrow L$. Then for every $A \in \mathrm{Ob}(\mathcal{A})$ and every morphism $f : A \to L$, there is an $n \in \mathbb{N}$ and a morphism $g : A \to C_n$ such that $f \circ g = \iota_n$. Moreover, if f is an embedding, then so is g.*



We say that $C \in \mathrm{Ob}(\mathcal{C})$ is a *natural retract* of $K(C)$ if there is a morphism $r : K(C) \to C$ such that $r \circ \eta_C = \mathrm{id}_C$. The following is an easy observation.

Lemma 3.2. *An object $C \in \mathrm{Ob}(\mathcal{C})$ is K-closed if and only if C is a natural retract of $K(C)$.*

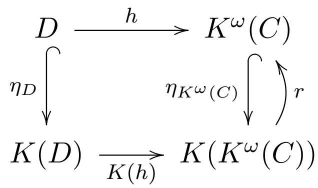
Proof. ($\Rightarrow$) Assume that C is K-closed. Then for $\mathrm{id}_C : C \to C$ there is a $g : K(C) \to C$ such that $g \circ \eta_C = \mathrm{id}_C$. Hence C is a natural retract of $K(C)$.

($\Leftarrow$) Assume now that there is a morphism $r : K(C) \to C$ such that $r \circ \eta_C = \mathrm{id}_C$. Take any $D \in \mathrm{Ob}(\mathcal{C})$ and any morphism $h : D \to C$. Then $\eta_C \circ h = K(h) \circ \eta_D$ because η is natural. After multiplying from left by r we have that $h = (r \circ K(h)) \circ \eta_D$. ∎

Lemma 3.3. *Let $K : \mathcal{A} \to \mathcal{C}$ be a Katětov functor, let L be the Fraïssé limit of $\mathcal{A}$ and assume that L is a natural retract of $K(L)$.*

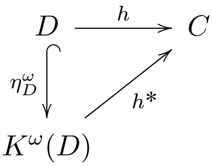
(a) *For every $C \in \mathrm{Ob}(\mathcal{C})$ we have that $K^\omega(C)$ is K-closed.*
(b) *If C is K-closed, then C is K^ω-closed. In particular, $K^\omega(C)$ is K^ω-closed for every $C \in \mathrm{Ob}(\mathcal{C})$.*
(c) *Assume that K is a locally finite Katětov functor. If C is locally K-closed, then C is locally K^ω-closed.*

Proof. (a) Since $K^\omega(C) \cong L$, the assumption yields that $K^\omega(C)$ is a natural retract of $K(K^\omega(C))$, so let $r : K(K^\omega(C)) \to K^\omega(C)$ be a morphism such that $r \circ \eta_{K^\omega(C)} = \mathrm{id}_{K^\omega(C)}$. Take any morphism $h : D \to K^\omega(C)$. Then:

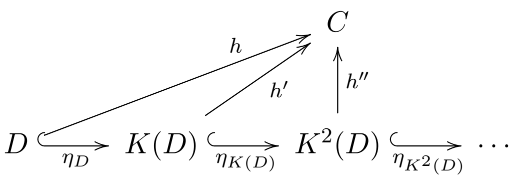


because η is natural. Therefore, $h = (r \circ K(h)) \circ \eta_D$.

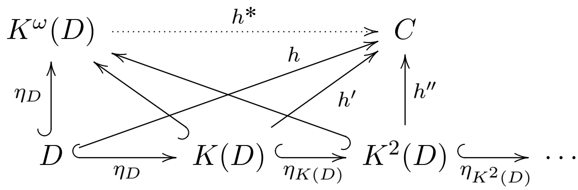
(b) Take any morphism $h : D \to C$ and let us show that there exists a morphism $h^* : K^\omega(D) \to C$ such that $h^* \circ \eta^\omega_D = h$:



By iterating the fact that C is K-closed, we get a commutative diagram:



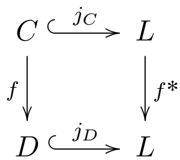
Since $K^\omega(D)$ is the colimit of the chain $D \hookrightarrow K(D) \hookrightarrow K^2(D) \hookrightarrow \cdots$, there is a unique morphism $h^* : K^\omega(D) \to C$ such that the diagram below commutes:



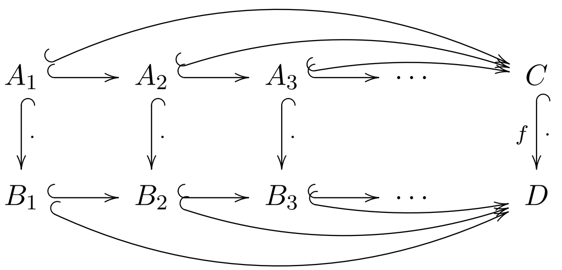
This completes the proof.

(c) Analogous to (b). ∎

Before we move on to the characterization of retracts of Fraïssé limits of categories that admit a Katětov functor and have the Fraïssé limit L which is a natural retract of $K(L)$, we shall use Lemma 3.3 to improve Theorem 2.3. Recall that a countable structure L is *$\mathcal{C}$-morphism-homogeneous* if every $\mathcal{C}$-morphism between two *finitely generated* substructures of L extends to a $\mathcal{C}$-endomorphism of L. Analogously, we say that a countable structure L is *totally $\mathcal{C}$-morphism-homogeneous* if every $\mathcal{C}$-morphism between two *arbitrary* substructures of L extends to a $\mathcal{C}$-endomorphism of L. More precisely, L is totally $\mathcal{C}$-morphism-homogeneous if for all $C, D \in \mathrm{Ob}(\mathcal{C})$ and embeddings $j_C : C \hookrightarrow L$ and $j_D : D \hookrightarrow L$, and for every $\mathcal{C}$-morphism $f : C \to D$, there is a $\mathcal{C}$-endomorphism f^* of L such that $j_D \circ f = f^* \circ j_C$.

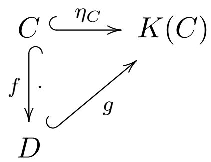


Lemma 3.4. *Let $C, D \in \mathrm{Ob}(\mathcal{C})$ be structures such that $f : C \dot{\hookrightarrow} D$ is an embedding of C into its one-point extension D and let $A_1 \hookrightarrow A_2 \hookrightarrow \ldots$ be a chain in $\mathcal{A}$ whose colimit is C. Then there exists a chain $B_1 \hookrightarrow B_2 \hookrightarrow \ldots$ in $\mathcal{A}$ whose colimit is D and B_i is a one-point extension of A_i for all i:*

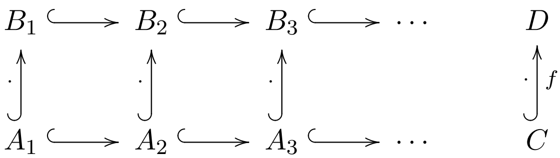


Proof. Without loss of generality, we can assume that $C \leqslant D$, and that $A_1 \leqslant A_2 \leqslant \ldots \leqslant C$, so that $C = \bigcup_{i \in \mathbb{N}} A_i$. Since D is a one-point extension of C, there exists an $x \in D \backslash C$ such that $D = \langle C \cup \{x\} \rangle_D$. Put $B_i = \langle A_i \cup \{x\} \rangle_D$. ∎

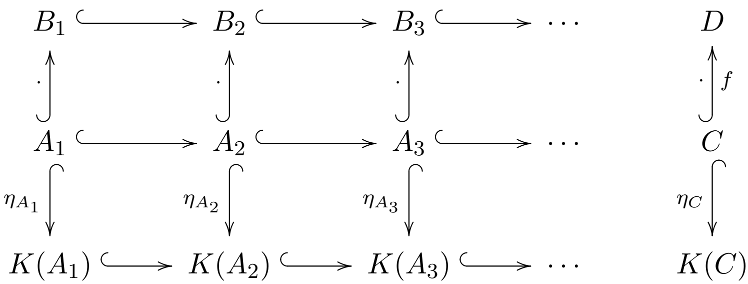
LEMMA 3.5. *For every one-point extension $f : C \dot{\hookrightarrow} D$ where $C, D \in \mathrm{Ob}(\mathcal{C})$, there exists a $g : D \hookrightarrow K(C)$ such that:*



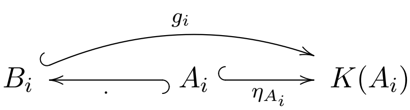
Proof. Let $A_1 \hookrightarrow A_2 \hookrightarrow \dots$ be a chain in $\mathcal{A}$ whose colimit is C, and let $B_1 \hookrightarrow B_2 \hookrightarrow \dots$ be a chain in $\mathcal{A}$ whose colimit is D such that



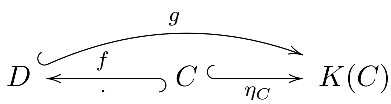
(which exists by Lemma 3.4). Since η is natural, we have



By Definition 2.1, for each i there is an embedding $g_i : B_i \hookrightarrow K(A_i)$ such that

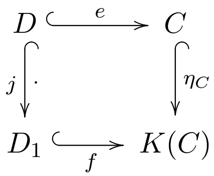


Therefore, in the colimit, we get an embedding $g : D \hookrightarrow K(C)$ such that

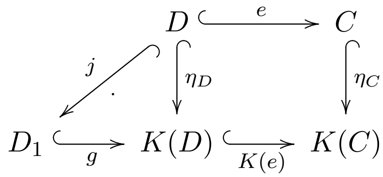


This completes the proof. ∎

Lemma 3.6. *For every embedding* $e : D \hookrightarrow C$ *and every one-point extension* $j : D \overset{\cdot}{\hookrightarrow} D_1$ *where* $C, D, D_1 \in \mathrm{Ob}(\mathcal{C})$*, there exists an embedding* $f : D_1 \hookrightarrow K(C)$ *such that:*

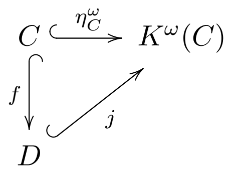


Proof. In the following diagram

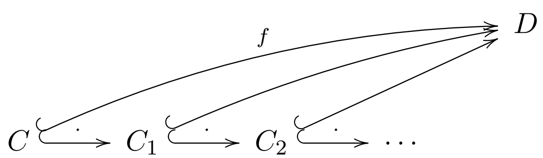


the triangle commutes because of Lemma 3.5, while the square commutes because η is natural. Put $f = K(e) \circ g$. ∎

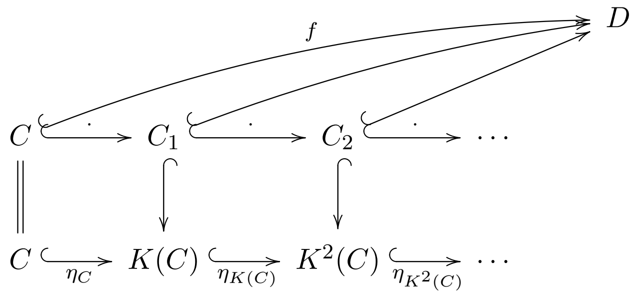
Lemma 3.7. *For every embedding* $f : C \hookrightarrow D$ *where* $C, D \in \mathrm{Ob}(\mathcal{C})$*, there exists an embedding* $j : D \hookrightarrow K^\omega(C)$ *such that*



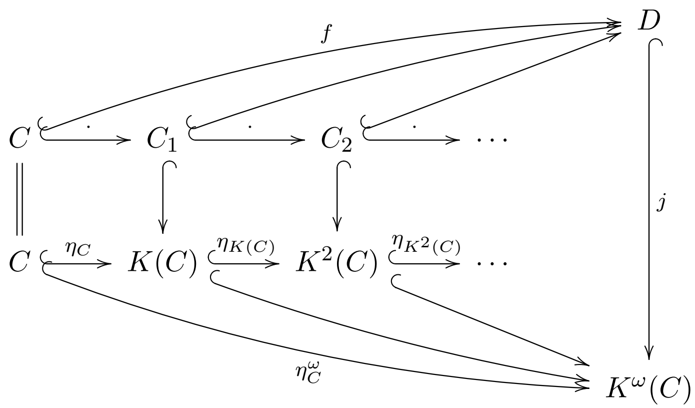
Proof. Because we work with categories of structures, it is easy to see that there exist $C_1, C_2 \ldots \in \mathrm{Ob}(\mathcal{C})$ such that



is a colimit diagram in $\mathcal{C}$. Then Lemmas 3.5 (which applies to the first square) and 3.6 (which applies to the remaining squares) yield that there exist embeddings $C_n \hookrightarrow K^n(C)$, $n \geqslant 1$ such that



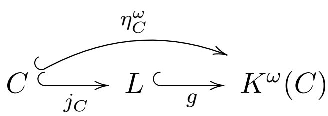
Since $K^\omega(C)$ is the colimit of the bottom chain and D is the colimit of the upper chain, there is an embedding $j : D \hookrightarrow K^\omega(C)$ such that



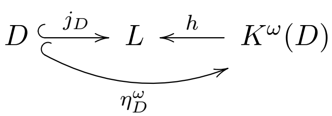
This completes the proof. ∎

Proposition 3.8. *Let $K : \mathcal{A} \to \mathcal{C}$ be a Katětov functor, let L be the Fraïssé limit of $\mathcal{A}$ and assume that L is a natural retract of $K(L)$. Then L is totally $\mathcal{C}$-morphism-homogeneous.*

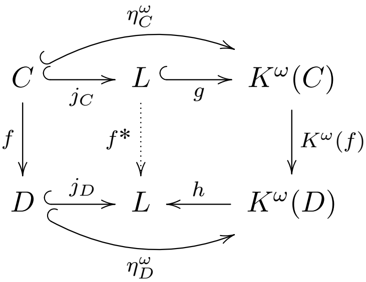
Proof. Take any $C, D \in \mathrm{Ob}(\mathcal{C})$ and embeddings $j_C : C \hookrightarrow L$ and $j_D : D \hookrightarrow L$, and and let $f : C \to D$ be a $\mathcal{C}$-morphism. According to Lemma 3.7, there exists a $g : L \to K^\omega(C)$ such that



On the other hand, $L \cong K^\omega(D)$ so L is K^ω-closed by Lemma 3.3 (b). Therefore, there exists a morphism $h : K^\omega(D) \to L$ such that



Hence,



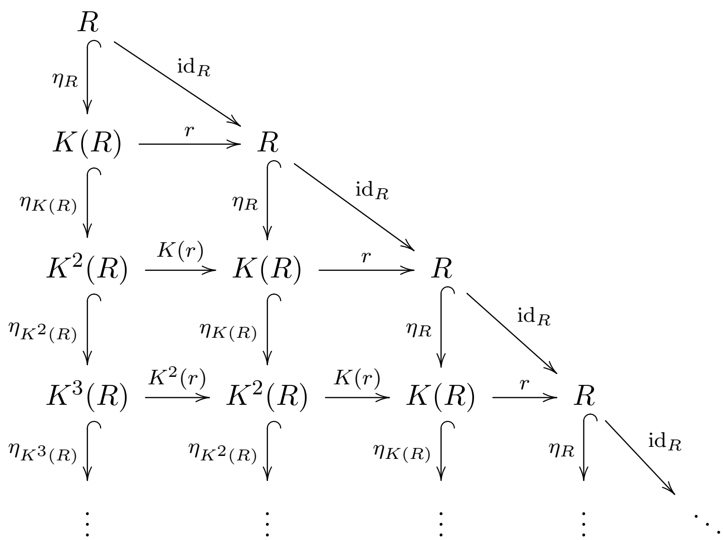
where the outer square commutes because η^ω is natural. Therefore,

$$f^* = h \circ K^\omega(f) \circ g$$

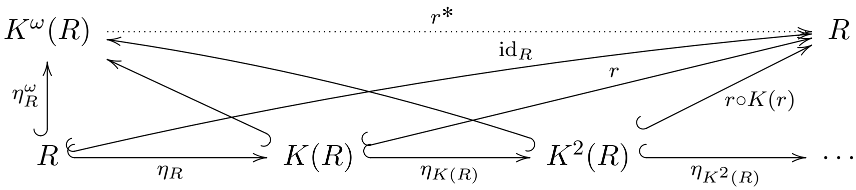
is a $\mathcal{C}$-endomorphism of L which extends f. ∎

Lemma 3.9. *Let L be the Fraïssé limit of $\mathcal{A}$. If $R \in \mathrm{Ob}(\mathcal{C})$ is a natural retract of $K(R)$ then R is a retract of L.*

Proof. Let $r : K(R) \to R$ be a morphism such that $r \circ \eta_R = \mathrm{id}_R$. The following diagram then commutes:



Therefore, there is a compatible cone with the tip at R and the morphisms id_R, r, $r \circ K(r)$, $r \circ K(r) \circ K^2(r)$ ... over the chain $R \hookrightarrow K(R) \hookrightarrow K^2(R) \hookrightarrow \cdots$. Since $K^\omega(R)$ is a colimit of the chain, there is a unique $r^* : K^\omega(R) \to R$ such that



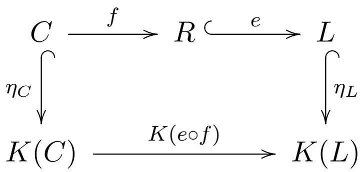
In particular, $r^* \circ \eta_R^\omega = \mathrm{id}_R$, so R is a retract of $K^\omega(R)$. But $K^\omega(R) \cong L$ by Theorem 2.3. ∎

Theorem 3.10. *Let $K : \mathcal{A} \to \mathcal{C}$ be a Katětov functor, let L be the Fraïssé limit of $\mathcal{A}$ and assume that L is a natural retract of $K(L)$. Then the following are equivalent:*

(1) *$R \in \mathrm{Ob}(\mathcal{C})$ is a retract of L;*
(2) *R is K-closed;*
(3) *R is locally K-closed.*

Proof. Let $\rho : K(L) \to L$ be a morphism such that $\rho \circ \eta_L = \mathrm{id}_L$.

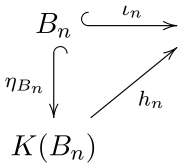
(1) ⇒ (2): Let R be a retract of L and let $e : R \hookrightarrow L$ and $r : L \to R$ be the corresponding embedding and retraction, so that $r \circ e = \mathrm{id}_R$. In order to show that R is K-closed, take any $C \in \mathrm{Ob}(\mathcal{C})$ and any $\mathcal{C}$-morphism $f : C \to R$. The following diagram commutes because η is natural:



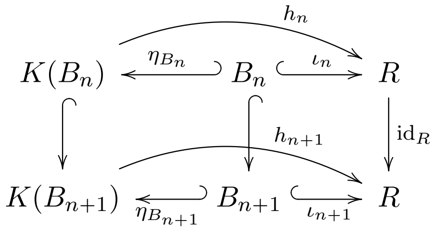
so $\eta_L \circ e \circ f = K(e \circ f) \circ \eta_C$. After multiplying by $r \circ \rho$ from the left we obtain $f = (r \circ \rho \circ K(e \circ f)) \circ \eta_C$.

(2) ⇒ (3): Obvious.

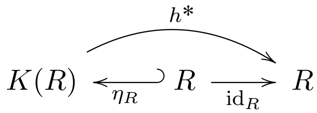
(3) ⇒ (1): Assume that R is locally K-closed, and let $B_1 \hookrightarrow B_2 \hookrightarrow B_3 \hookrightarrow \cdots$ be the chain in $\mathcal{A}$ whose colimit is R. Let us denote the canonical embeddings $B_n \hookrightarrow R$ by ι_n. Since R is locally K-closed, for every n there exists a $\mathcal{C}$-morphism $h_n : K(B_n) \to R$ such that $h_n \circ \eta_{B_n} = \iota_n$.



The following diagram clearly commutes for every n:



After taking the colimits of the vertical chains, we get the following colimit diagram:



Therefore, R is a retract of $K(R)$, so it is also a retract of L by Lemma 3.9. ∎

Just for the sake of illustration, we present an equivalent interpretation of local K-closedness. We are going to prove that under certain reasonable assumptions, locally K-closed objects in $\mathcal{C}$ correspond precisely to algebraically closed objects in $\mathcal{C}$.

For a tuple $\bar{c} = (c_1, \ldots, c_n)$ of elements of C and a morphism $f : C \to D$, let $f(\bar{c}) = (f(c_1), \ldots, f(c_n))$. Let $C \in \mathrm{Ob}(\mathcal{C})$ be a structure such that for every primitive positive Δ-formula Φ and for every embedding $e : C \hookrightarrow D$ where $D \in \mathrm{Ob}(\mathcal{C})$, if there exists a tuple $\bar{c}$ of elements of C such that $D \models \Phi(e(\bar{c}))$ then $C \models \Phi(\bar{c})$. We then say that $C \in \mathrm{Ob}(\mathcal{C})$ is *algebraically closed in* $\mathcal{C}$.

We say that a category $\mathcal{C}$ of Δ-structures is *locally finite* if every finitely generated object in $\mathcal{C}$ is finite.

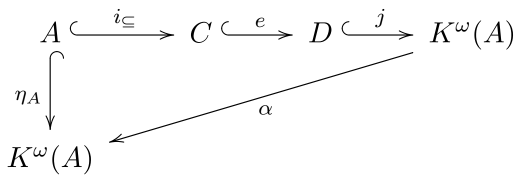
Theorem 3.11. *Let Δ be a finite first-order language. Let $\mathcal{C}$ be a locally finite category of Δ-structures and all Δ-homomorphisms, and let $\mathcal{A}$ be the full subcategory of $\mathcal{C}$ consisting of all finitely generated (and hence finite) structures in $\mathcal{C}$. Assume that there exists a locally finite Katětov functor $K : \mathcal{A} \to \mathcal{C}$ such that the Fraïssé limit L of $\mathcal{A}$ is a natural retract of $K(L)$. Then $C \in \mathrm{Ob}(\mathcal{C})$ is locally K-closed if and only if C is algebraically closed in $\mathcal{C}$.*

Proof. ($\Rightarrow$) Let $C \in \mathrm{Ob}(\mathcal{C})$ be a locally K-closed structure. Let Φ be a primitive positive Δ-formula, and assume that there exist a tuple $\bar{c}$ of elements of C, a structure $D \in \mathrm{Ob}(\mathcal{C})$ and an embedding $e : C \hookrightarrow D$ such that $D \models \Phi(e(\bar{c}))$.

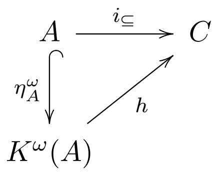
Let $A = \langle \bar{c} \rangle_C$ be the substructure of C generated by the entries of $\bar{c}$. Then $A \in \mathrm{Ob}(\mathcal{A})$ since A is finitely generated, and hence finite. Let $i_{\subseteq} : A \hookrightarrow C$

denote the inclusion of A into C so that $i_\subseteq(a) = a$ for all $a \in A$. Note that $D \models \Phi(e \circ i_\subseteq(\overline{c}))$.

Theorem 2.3 yields that $K^\omega(A)$ is a Fraïssé limit of $\mathcal{A}$. Hence $K^\omega(A)$ is universal for $\mathcal{C}$, so there is an embedding $j : D \hookrightarrow K^\omega(A)$, and it is ultrahomogeneous, so there is an automorphism α of $K^\omega(A)$ such that



Therefore, $K^\omega(A) \models \Phi(\alpha \circ j \circ e \circ i_\subseteq(\overline{c}))$, whence follows that $K^\omega(A) \models \Phi(\eta_A(\overline{c}))$ because $\alpha \circ j \circ e \circ i_\subseteq = \eta_A$. By Lemma 3.3 (c), we know that there is an $h : K^\omega(A) \to C$ such that



Since h is a homomorphism, we have that $C \models \Phi(h \circ \eta_A(\overline{c}))$, so $C \models \Phi(\overline{c})$ because $h \circ \eta_A = i_\subseteq$ and $i_\subseteq(\overline{c}) = \overline{c}$. This completes the proof that C is algebraically closed.

$(\Leftarrow)$ Assume that C is algebraically closed in $\mathcal{C}$ and take any morphism $f : A \to C$ where $A \in \mathrm{Ob}(\mathcal{A})$. Let $\overline{a} = \langle a_1, a_2, \ldots, a_n\rangle$ be an enumeration of all the elements of A and let $\overline{b} = \langle b_1, b_2, \ldots, b_m\rangle$ be an enumeration of all the elements of $K(A)\backslash\eta_A(A)$. Let

$$\Phi(\eta_A(\overline{a}), \overline{b}) = \bigwedge_{i=1}^{p} \varphi_i(\eta_A(\overline{a}), \overline{b})$$

be the positive diagram of $K(A)$, that is, the conjunction of all the positive atomic $(\Delta \cup K(A))$-formulas that hold in $K(A)$. Recall that a positive atomic Σ-formula is a formula of the form $R(s_1, \ldots, s_k)$ for some relational symbol $R \in \Sigma$ and constants $s_1, \ldots, s_k \in \Sigma$, or a Σ-formula of the form $s_0 = g(s_1, \ldots, s_l)$ for some functional symbol $g \in \Sigma$ and constants $s_0, s_1, \ldots, s_l \in \Sigma$. Since $\Delta \cup K(A)$ is finite, there are only finitely many positive atomic $(\Delta \cup K(A))$-formulas, so the above conjunction is indeed finite. Clearly,

$$K(A) \models (\exists\overline{b})\Phi(\eta_A(\overline{a}), \overline{b}),$$

so

$$K(C) \models (\exists\overline{b})\Phi(K(f) \circ \eta_A(\overline{a}), \overline{b})$$

since $K(f)$ is a Δ-homomorphism, and all Δ-homomorphisms preserve positive existential Δ-formulas. The following diagram commutes because η is a natural transformation:

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & C \\ {\scriptstyle \eta_A}\big\downarrow & & \big\downarrow{\scriptstyle \eta_C} \\ K(A) & \xrightarrow[K(f)]{} & K(C) \end{array}$$

whence follows that

$$K(C) \models (\exists \overline{b})\Phi(\eta_C \circ f(\overline{a}), \overline{b}).$$

Now, $\eta_C : C \hookrightarrow K(C)$ is an embedding, so the fact that C is algebraically closed implies that

$$C \models (\exists \overline{b})\Phi(f(\overline{a}), \overline{b}).$$

So, for some m-tuple $\overline{c}$ of elements of C we have that

$$C \models \Phi(f(\overline{a}), \overline{c}).$$

Therefore, the mapping $f^* : K(A) \to C$, which takes the n-tuple $\eta_A(\overline{a})$ to the n-tuple $f(\overline{a})$, and takes the m-tuple $\overline{b}$ to the m-tuple $\overline{c}$, is a Δ-homomorphism satisfying $f^* \circ \eta_A = f$. This completes the proof that C is locally K-closed. ■

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