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ON GENERATING SETS OF FINITE ALGEBRAS

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Abstract. We consider here some notions and results about sets of generators of finite algebras motivated by the case of finite groups. We illustrate these notions by simple examples closed to lattices and semigroups. Next, we examine these notions in the case of groups with operators.

1. General algebras

All algebras considered here, often A, are finite and contain the unique one element subalgebra called *trivial subalgebra.* This subalgebra will be denoted here by E. Our assumption is satisfied, for example, in any pseudovariety of algebras with at least one 0-ary operation (see [2], p. 235), in particular in groups, Ω-groups, semigroups with 0, monoids and in lattices with 0 as a 0-ary operation.

If $X \subseteq A$ is a subset then $\langle X \rangle$ is the subalgebra of A generated by X. In particular $E = \langle \emptyset \rangle$. Subalgebras invariant under automorphisms will be named *characteristic.*

An element $a \in A$ is a *nongenerator* if it can be rejected from every generating set of A containing this element. If A is any algebra then let $\Phi(A)$ denotes the set of all nongenerators of A, named usually the *Frattini subalgebra of* A. If, in particular, $A = E$, then $\Phi(A) = A$. In any other case $\Phi(A) \neq A$, because A is finite. The following observation is known

Proposition 1.1 (see [2]). *If A is any algebra then $\Phi(A)$ is the intersection of all maximal subalgebras of A. Thus $\Phi(A)$ is a characteristic subalgebra of A.*

Let A be an algebra. Then a subalgebra $B \subseteq A$ is *join irreducible* (see [5]) if it is not a join of its proper subalgebras. An element $a \in A$ will be named

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join irreducible, or *a j-element,* if $\langle a\rangle$ is a join irreducible subalgebra of A. For j-elements we have the following result

Proposition 1.2. *Let A be an algebra and φ be a homomorphism of A onto an algebra B.*

(1) *A is generated by j-elements.*
(2) *If $u \in A$ is a j-element, then $\varphi(u)$ is a j-element in B.*
(3) *If $x \in B$ is a j-element, then there exists a j-element $u \in A$ such that $\langle x\rangle = \langle\varphi(u)\rangle$.*

Proof. The first claim can be proved as in the lattice case (see [5, Exercise 1.6.13]).

(2) If $u \in A$ is a j-element then $\varphi(u)$ is a j-element, by homomorphism theorems.

(3) Now, let $x \in B$ be a j-element and $C = \varphi^{-1}(\langle x\rangle)$. By the first claim applied to C there are j-elements $u_1, \dots u_m \in C$ such that $C = \langle u_1, \dots, u_m\rangle$. Then

$$\langle x\rangle = \varphi(C) = \langle \varphi\langle u_1\rangle, \dots, \varphi\langle u_m\rangle\rangle.$$

Thus, by assumption on x, $\langle x\rangle = \langle\varphi(u_i)\rangle$ for some $1 \le i \le m$. ∎

Let us say that A is a *CP-algebra* if its every element is a j-element. The abbreviation 'CP' is often used in group theory instead of "the centralizer property". j-elements in groups are precisely elements of prime power order. Thus, CP-groups are groups containing only elements of such orders. They were studied for many years and are classified (see [3, 7] and references there).

Example 1.3. Let $(L, \vee, \wedge, 0)$ be a lattice. One can see that L is a j-algebra if and only if it has exactly one coatom, and L is a CP-algebra if and only if it is a chain. On the other hand, the lower semilattice $(L, \wedge, 0)$ is always a CP-algebra.

2. Independences

Various concepts of independences and bases are discussed by many authors. For some examples see [4, 13, 15] and [18]. In this note, we concentrate on independences related to sets of generators and defined in spirit of [18], where independent sets are subsets of expected bases and bases are expected to be maximal independent subsets. Thus, we will use notation and terminology motivated by the case of groups, where this approach is quite well developed (see [1, 6, 8, 10, 14, 17]).

A subset $X \subseteq A$ is said here to be:

- *g-independent* if $\langle Y, \Phi(A)\rangle \neq \langle X, \Phi(A)\rangle$ for every proper subset $Y \subset X$;
- *a g-base of A,* if X is a g-independent generating set of A.

Every algebra A has a g-base and it can be selected from any generating set of A. Thus, we can consider the following g-invariants:

$$s_g(A) = \sup_X |X| \quad \text{and} \quad i_g(A) = \inf_X |X|, \tag{1}$$

where X runs over all g-bases of A.

Remark 2.1. One can consider g-independence and g-bases without using the Frattini subalgebra, as was mentioned in [1]. This approach gives the same notion of g-base but a different notion of g-independent set, not consistent with this from [18].

The proofs of the following two propositions are straightforward.

Proposition 2.2. *Let A be an algebra. The following conditions are equivalent:*

(1) $i_g(A) = 0$;
(2) $s_g(A) = 0$;
(3) $A = E$;
(4) *A has no proper subalgebras.*

In every other case $1 \leq i_g(A) \leq s_g(A) < \infty$.

Proposition 2.3. *Let A be an algebra. The following conditions are equivalent:*

(1) $s_g(A) = 1$;
(2) *A has exactly one maximal subalgebra;*
(3) *A is nontrivial and join irreducible;*
(4) *$\Phi(A) \neq A$ and A is generated by any element from $A \backslash \Phi(A)$.*

As in the group theory, algebras A with $i_g(A) = s_g(A)$ will be named *$\mathcal{B}$-algebras*. Also an algebra A will have *the basis property* if every its subalgebra (in particular A itself) is a $\mathcal{B}$-algebra.

$\mathcal{B}$-groups and groups with the basis property are completely described (see [1, 10, 14]). These classes are very narrow. This was one of our reasons to propose in [10] a modification of these notions. On the level of algebras our modified notions can be formulated in the following way: A subset $X \subseteq A$ will be said here:

- *j-independent* if X is a g-independent set of j-elements;
- a *j-generating set* if X is a generating set of j-elements;
- a *j-base* of A, if X is a j-independent generating set of A.

By Proposition 1.2 every algebra is generated by j-elements. Hence, it has a j-base and a j-base can be selected from any set of j-generators. Thus, as in Formula (1) the following j-invariants can be considered:

$$(2) \qquad s_j(A) = \sup_X |X| \quad \text{and} \quad i_j(A) = \inf_X |X|,$$

where X runs over all j-bases of A. Connecting formulas (1) and (2) we obtain:

$$(3) \qquad i_g(A) \le i_j(A) \le s_j(A) = s_g(A).$$

Indeed: $i_g(A) \le i_j(A)$ and $s_j(A) \le s_g(A)$, because any j-base of A is a g-base. Also, if $B = \{x_1, \ldots, x_n\}$ is a g-base of A then, with the help of Proposition 1.2 applied to algebras $\langle x_i \rangle$ for $1 \le i \le n$, one can construct a j-base, say $\{y_1, \ldots, y_m\}$, of A with m elements, where $m \ge n$. Hence $s_j(A) \ge s_g(A)$, thus $s_j(A) = s_g(A)$. The inequality $i_j(A) \le s_j(A)$ is evident.

Let us agree, analogously to [10], that an algebra A has *property* $\mathcal{B}_j$ if $s_j(A) = i_j(A)$ and A has the *j-basis property* if all its subalgebras are $\mathcal{B}_j$-algebras. From the above definitions, Propositions 2.2 and 2.3 and from Formula (3), we obtain the following

Proposition 2.4. *Let A be an algebra.*

(1) *If A is join irreducible then it is a $\mathcal{B}$-algebra, with one element g-base.*
(2) *If A is a $\mathcal{B}$-algebra then it is a $\mathcal{B}_j$-algebra.*
(3) *If A is a $\mathcal{B}_j$-algebra and a CP-algebra, then it is a $\mathcal{B}$-algebra.*
(4) *A has the basis property if and only if it has the j-basis property and is a CP-algebra.*

In contrast to algebras with the basis property, a $\mathcal{B}$-algebra need not be a CP-algebra, even for groups, (see [10], Example 4.5). This example is a $\mathcal{B}$-group with the j-basis property, but is not a CP-group.

Example 2.5. Let L be a lattice with exactly one coatom, but not a chain. Then, by Example 1.3, L is a $\mathcal{B}$-algebra, with the j-basis property, but is not a CP-algebra.

Our assumption on existence of trivial subalgebras allows considering factors of direct products not only as homomorphic images, but also as subalgebras in a natural way. If A, B are algebras and $E = \langle e \rangle$ is the trivial algebra, then $A \simeq A \times E$ by $a \to (a, e)$. Similarly $B \simeq E \times B$. We can also apply Proposition 1.2. Hence, as for groups, we obtain (see [9, Proposition 2.5])

Proposition 2.6. *Let A and B be algebras. Then*

(1) $s_g(A \times B) \ge s_g(A) + s_g(B)$;
(2) $i_g(A \times B) \ge \max(i_g(A), i_g(B))$;
(3) $s_j(A \times B) \ge s_j(A) + s_j(B)$;
(4) $i_j(A \times B) \ge \max(i_j(A), i_j(B))$.

Proof. If $X \subset A$ and $Y \subset B$ are g-independent subsets, then it is easy to check that the set $X \cup Y$ is g-independent in $A \times B$, and inequality (1) follows.

On the other hand, if $Z \subset A \times B$ is a g-independent subset then its projections, say $X \subseteq A$ and $Y \subseteq B$, are generating subsets. Thus, one can select from them g-bases of A and B, respectively. Hence the inequality (2) follows. Inequalities for j-independent sets can be proved in a similar way. ∎

Example 2.7. Let $(L, \vee, \wedge, 0)$ be a nontrivial lattice and $\hat{L}$ be the lattice L with additional unary operations μ_x being multiplications by x, for every $x \in L$. Thus, subalgebras in $\hat{L}$ are precisely ideals of the lattice L. Also $a \in L$ is a j-element of $\hat{L}$ if and only if the ideal (interval) $[0, a] \subseteq L$ is join irreducible.

For $A = \hat{L}$, the Frattini subalgebra $\Phi(A)$ is the intersection of all maximal ideals of L. We also have: $i_g(A) = 1$ because $\{1\} \subset L$ is a g-base of L, while $i_j(A) > 1$ if L has more then one coatom. Hence, if $B = A \times A$ then it can be seen that $i_g(B) = 1$, while $i_j(B) > 1$.

3. Groups and Ω-groups

Some possible generalizations of results on independences considered here were indicated in previous sections. To indicate some others, we restrict ourselves in this section to groups and Ω-groups. For notation, terminology and results used in this section one can consult for example [16]. Ω-groups will also be understood as in [16]. Thus, operators from an arbitrary set Ω will be always endomorphisms of underlying groups, group reducts of considered Ω-groups. When particular properties of Ω will not be essential, we will often write 'algebras' instead of 'Ω-groups'.

Theorem 6.1 from [10] can be generalized to Ω-groups in the following way

Theorem 3.1. (see [1, 10]) *The classes of $\mathcal{B}$-algebras, $\mathcal{B}_j$-algebras, algebras with the basis property and algebras with the j-basis property are homomorphically closed.*

Proof. Let Ω be a set of operators, A, B be Ω-groups and let φ be a homomorphism of A onto B with the kernel N. Let $C \subseteq A$ be an Ω-subgroup of minimal cardinality, with $\varphi(C) = B$. Then, as in the case of groups, $A = CN$.

Assume that A is a $\mathcal{B}$-algebra, but suppose that $\{x_1, \ldots, x_d\}$ and $\{y_1, \ldots, y_e\}$ are two g-bases of B, where $d < e$. Then there are elements $u_1, \ldots, u_d$ and $v_1, \ldots, v_e$ in C such that $\varphi(u_i) = x_i$ for every $i = 1, \ldots, d$ and $\varphi(v_i) = y_i$ for every $i = 1, \ldots, e$. Then, from our choice of C, we obtain that $C = \langle u_1, \ldots, u_d\rangle = \langle v_1, \ldots, v_e\rangle$.

Let us take a subset $\{w_1, \dots, w_r\} \subset N$ of minimal cardinality, such that $A = \langle w_1, \dots, w_r, C \rangle$. Then it is easy to check, that the sets $\{w_1, \dots, w_r, u_1, \dots, u_d\}$ and $\{w_1, \dots, w_r, v_1, \dots, v_e\}$ are g-bases of A, contrary to our assumption on A.

With the help of Proposition 1.2 one can check, that every j-base of B can be lifted to a j-base of C of the same cardinality. Now one can easily finish the proof of the theorem. ∎

A nontrivial direct product of groups is a group with property $\mathcal{B}$ or with the basis property if and only if it is a p-group (see [1, 8]). On the other hand, there are many nontrivial direct products, which are groups with property $\mathcal{B}_j$, or with j-basis property, but being not p-groups (see [10, 11, 12]). For Ω-groups the situation is much more complicated. In Example 3.5 we will show an Ω-group based on a p-group, which is not a $\mathcal{B}$-algebra. On the other hand, there are many nontrivial direct products of Ω-groups with property $\mathcal{B}_j$, or with the j-basis property, but not being p-groups. To explain this, and for some other reasons, an Ω-group G will be called *coprimely indecomposable* if it is not a direct product of nontrivial Ω-groups with coprime orders.

Proposition 3.2. *Every Ω-group is a direct product of coprimely indecomposable Ω-groups with coprime orders. This decomposition is unique up to the order of factors and is identical with analogous decomposition of underlying group.*

Proof. Let G be an Ω-group. By induction on the order of G one can easily show, that $G = G_1 \times \ldots \times G_n$ for some $n \geq 1$, where G_i are coprimely indecomposable groups with coprime orders. This decomposition is an Ω-group decomposition, because operations from Ω are group endomorphisms. Indeed: if for an Ω-group G we have $G = G_1 \times G_2$ where G_i are groups of coprime orders, then both G_i are invariant under all endomorphisms of G, hence are Ω-subgroups.

Now, let for an Ω-group G, $G = G_1 \times G_2$ where G_i are groups of coprime orders. If $H \leq G$ is an Ω-subgroup, then one can check that $H = (H \cap G_1) \times (H \cap G_2)$. From these observations the result follows. ∎

Using this observation we can generalize Theorem 6.3 from [10] to Ω-groups in the following way

Theorem 3.3. *Let G_1 and G_2 be Ω-groups with coprime orders.*

(1) *G_1 and G_2 have property $\mathcal{B}_j$ if and only if $G_1 \times G_2$ has property $\mathcal{B}_j$.*
(2) *G_1 and G_2 have the j-basis property if and only if $G_1 \times G_2$ has the j-basis property.*

From [1, 8, 10] we know that $\mathcal{B}$-groups, $\mathcal{B}_j$-groups and groups with the basis property are solvable. This result need not be true for Ω-groups.

Example 3.4. Let S be a nonabelian simple group and Ω be the set of all inner automorphisms of S. Then S with the set of operators Ω mentioned above as the Ω-group has property $\mathcal{B}$ and even has the basis property, having no proper Ω-subgroups.

Classes of $\mathcal{B}$-groups, groups with the basis property and groups with j-basis property are completely described with the help of direct products of p-groups and some well defined $\{p, q\}$-groups, with coprime orders (see [1, 10, 12]).

It would be interesting to obtain some characterizations of $\mathcal{B}$-algebras, $\mathcal{B}_j$-algebras, and algebras with the basis (j-basis) property in some important pseudovarieties of algebras, in particular in Ω-groups.

The following two examples show that $\mathcal{B}$-groups or $\mathcal{B}_j$-groups need not to be Ω-groups with property $\mathcal{B}$ or $\mathcal{B}_j$-respectively for a suitable set Ω.

Example 3.5. Let p be an odd prime and C_p be a cyclic group of order p. Put $A = C_p \times C_p$ and let σ be a nontrivial automorphism of C_p. Then A with usual multiplication and an operator ω such that $\omega(a, b) = (a, \sigma(b))$ is an Ω-group. Then every element $c = (a, b) \in A$ satisfying $a \neq 1 \neq b$ generates A is a Ω-group, but c is not a j-element. All other elements of A are j-elements and $A = \langle (a, 1), (1, b) \rangle$. Thus, Ω-group A is not a $\mathcal{B}$-algebra, but A is a $\mathcal{B}_j$-algebra. Obviously, A is a $\mathcal{B}$-group.

Example 3.6. Let $P = \langle a, b \mid a^7 = 1 = b^7, ab = ba \rangle$ be a group and $Q = \langle x \mid x^3 = 1 \rangle$ be a cyclic group. Let $G = P \rtimes Q$ be a semidirect product, where $a^x = a^2$ and $b^x = b^4$. Then G has trivial Frattini subgroup. Since $\{a, b, x\}$ and $\{ab, x\}$ are g-bases of G, then G is not a $\mathcal{B}$-group.

Assume now that ω is an automorphism of G such that $\omega(a) = b$, $\omega(b) = a$ and $\omega(x) = ax^2$. Then we can consider G as an Ω-group, where $\Omega = \{\omega\}$. One can check that P is the unique proper Ω-subgroup of G. Hence, by Proposition 2.3, G is a $\mathcal{B}_j$-algebra. Moreover, P is the Frattini subalgebra of G and also P is a $\mathcal{B}_j$-algebra. So G is an algebra with the j-basis property.

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