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GREEN'S RELATIONS IN THE COMMUTATIVE CENTRALIZERS OF MONOUNARY ALGEBRAS

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Abstract. The paper deals with the monounary algebras for which the second centralizer equals the first centralizer. We describe Green's relations on the semigroup C, where C is the centralizer of such algebra.

1. Introduction

In the present paper, we deal with the semigroup formed as the centralizer of a monounary algebra.

For a given (partial) algebra $\mathcal{A}$, its centralizer is defined as the set of those mappings of $\mathcal{A}$ into $\mathcal{A}$ that commute with all basic operations of $\mathcal{A}$. Further, the second centralizer is the set of all transformations which commute with all elements of the (first) centralizer.

Centralizers of transformations appear in several areas of mathematical research. For example, they play a role in finding the group of automorphisms of a general semigroup [1]. They occur naturally in the theory of unary algebras and the knowledge of them is useful in studying homomorphisms of algebraic structures. A monounary algebra is a unary algebra with one operation. Monounary algebras have been investigated by several authors (see, e.g., monographs of B. Jónsson [6], J. G. Pitkethly and B. A. Davey [16], D. Jakubíková-Studenovská and J. Pócs [4]). Centralizers of full and partial transformations relative to various transformation semigroups have been investigated, e.g. by M. Novotný [15], O. Kopeček [14], J. Konieczny, S. Lipscomb and J. Araújo [2], [7], [8], [9], [12] and [13].

In semigroup theory, the notion of Green's relations is well known (see [3]). Green's relations provide one of the most important tools in studying semi-

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groups. In the papers [8], [12], [13], [10] and [11], the authors studied Green's relations on some transformation semigroups. For example, J. Konieczny [13] analyzes a case when various Green's relations coincide.

The paper [5] contains a description of all monounary algebras with the property that the first centralizer and the second centralizer coincide. Let us remark that this property is equivalent to the property that the (first) centralizer is commutative. Our aim is, applying these results, to characterize Green's relations on the semigroup $(C, \circ)$, where C is the centralizer of a given monounary algebra (A, f) such that C is commutative.

The commutativity of C implies that on C all Green relations coincide. Our aim is to describe conditions under which $\alpha, \beta \in C$ are in Green's relation, i.e., $\alpha \circ C = \beta \circ C$ (we write $\alpha \mathcal{R} \beta$). Also, we will characterize $\alpha, \beta \in C$ such that $\alpha \circ C \subseteq \beta \circ C$ (denoted $\alpha \leq_{\mathcal{R}} \beta$).

2. Preliminaries

In this section, some basic notions which will be used in the following sections are introduced.

Definition 2.1. For a nonempty set A, a mapping $f\colon A \to A$ is called a *unary operation* on A. The pair (A, f) is said to be a *monounary algebra*.

Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 2.2. Let (A, f) be a monounary algebra, $x, y \in A$. Put $f^0(x) = x$. If $n \in \mathbb{N}$ and $f^{n-1}(x)$ is defined, then we denote $f^n(x) = f\left(f^{n-1}(x)\right)$. Next, we put $x \sim y$ if there are $m, n \in \mathbb{N}_0$ such that $f^n(x) = f^m(y)$. Then $\sim$ is an equivalence on the set A and the elements of $A/\sim$ are called (*connected*) *components* of (A, f). Further, (A, f) is said to be *connected* if it has only one connected component.

Put $f^{-n}(x) = \{z \in A : f^n(z) = x\}$ for $n \in \mathbb{N}$.

Definition 2.3. An element c in a monounary algebra (A, f) is called *cyclic* if $f^k(c) = c$ for some $k \in \mathbb{N}$. The set of all cyclic elements of some connected component is called a *cycle*.

Definition 2.4. We say that the mappings $f\colon A \to A$, $g\colon A \to A$ *commute* if $f(g(a)) = g(f(a))$ for each $a \in A$.

Definition 2.5. The *centralizer* of a monounary algebra (A, f) is the set $C(A, f)$ of those mappings $g\colon A \to A$ which commute with the mapping f.

Put $C_1(A, f) = C(A, f)$; this set we will also call the *first centralizer* of (A, f).

The *second centralizer* of (A, f) is the set $C_2(A, f) = \bigcap_{g \in C_1(A,f)} C_1(A, g)$. In other words, it is the set of all mappings which commute with all elements from $C_1(A, f)$.

From the definition of the first and second centralizer it follows that the identity mapping on A and f belong to the sets $C_1(A, f)$ and $C_2(A, f)$. Next, the second centralizer is a subset of the first centralizer.

Let us recall the definition of Green's relations [3].

Definition 2.6. Let S be a semigroup and $a, b \in S$. We write $a\mathcal{L}b$ iff $S^1a = S^1b$; $a\mathcal{R}b$ iff $aS^1 = bS^1$; $a\mathcal{J}b$ iff $S^1aS^1 = S^1bS^1$, where S^1 is the semigroup S with an identity adjoined. Next, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$. (Obviously, $\mathcal{D} \subseteq \mathcal{J}$.) The relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ are said to be *Green's relations* on S.

We will investigate Green's relations on a semigroup $(C, \circ)$, where C is the centralizer of a given monounary algebra (A, f) with the property $C_1(A, f) = C_2(A, f)$.

Since mappings from C commute mutually, then on the semigroup C the relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}, \mathcal{J}$ coincide.

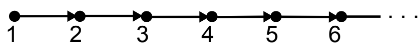
Definition 2.7. We say that $\alpha \in C$ and $\beta \in C$ are *Green equivalent* if $\alpha \circ C = \beta \circ C$, i.e., $\alpha\mathcal{R}\beta$. If $\alpha \circ C \subseteq \beta \circ C$, then we write $\alpha \leq_{\mathcal{R}} \beta$.

Let us note that $\leq_{\mathcal{R}}$ is a quasiorder on C and for all $\alpha, \beta \in C$, $\alpha\mathcal{R}\beta$ if and only if $\alpha \leq_{\mathcal{R}} \beta$ and $\beta \leq_{\mathcal{R}} \alpha$.

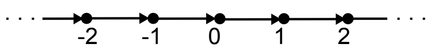
Notation 2.8. As usual, $\mathbb{Z}$ is the set of all integers. For $n \in \mathbb{N}$, we denote by $\mathbb{Z}_n = \{0_n, 1_n, \ldots, (n-1)_n\}$. The operation of successor on these sets will be denoted f. So $f(x) = x + 1$ for all $x \in \mathbb{Z}$. For every $a_n \in \mathbb{Z}_n$, $f(a_n) = (a+1)_n$ (if $0 \leq a \leq n-2$) and $f(a_n) = 0_n$ (if $a = n-1$).

We will need the following basic connected monounary algebras. (For these algebras, one symbol will denote both the support and the algebra.)

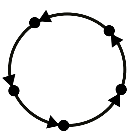
$\mathbb{N}$



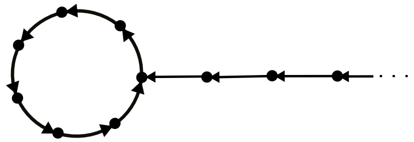
$\mathbb{Z}$



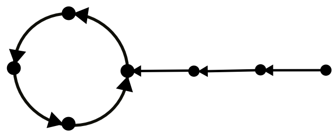
$\mathbb{Z}_n$



$$L_{n,\infty} = \mathbb{Z}_n \cup \mathbb{N} \text{ where } f(x) = \begin{cases} (a+1)_n, & \text{if } x = a_n \in \mathbb{Z}_n, \\ x-1, & \text{if } x \in \mathbb{N}\backslash\{1\}, \\ 0_n, & \text{if } x = 1. \end{cases}$$

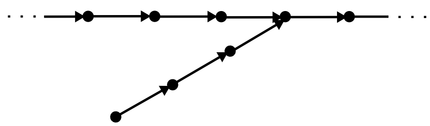


$L_{n,k}$ (for $k \in \mathbb{N}$) is a subalgebra of the algebra $L_{n,\infty}$ with the support $\mathbb{Z}_n \cup \{1, 2, \ldots, k\}$.



Let us remark that we can also consider $k = 0$. Then $L_{n,0}$ is coincident with $\mathbb{Z}_n$.

$$L_{\infty,k} = \mathbb{Z} \cup \{1', 2', \ldots, k'\} \text{ where } f(x) = \begin{cases} x+1, & \text{if } x \in \mathbb{Z}, \\ (x-1)', & \text{if } x \in \{2', \ldots, k'\}, \\ 0, & \text{if } x = 1'. \end{cases}$$



Analogously as above, if no misunderstanding could occur, sometimes we will use the same symbol for an algebra and for its support. Also, the operation f restricted to a component of (A, f) will be denoted by the symbol f.

Let A be, up to isomorphism, equal to $L_{\infty,k}$, $L_{n,k}$, where $k \geq 1$, or $L_{n,\infty}$. Then there exists a unique element $x \in A$ such that $\left|f^{-1}(x)\right| = 2$ and $\left|f^{-1}(a)\right| \leq 1$ for all $a \in A$, $a \neq x$. Denote by B the subalgebra of A that is isomorphic to $\mathbb{Z}_n$ or $\mathbb{Z}$ in these algebras. For every element $y \in A\backslash B$, there exist unique elements $t(y) \in B$ and $m(y) \in \mathbb{N}$ such that $f^{m(y)}(y) = f^{m(y)}(t(y)) = x \in B$, $f^{m(y)-1}(y) \notin B$.

Put

$$p(y) = \begin{cases} y, & \text{if } y \in B, \\ t(y), & \text{if } y \in A\backslash B \end{cases}$$

and $f_k(z) = z + k$, $k \in \mathbb{Z}$, $z \in \mathbb{Z}$.

The function p is important, so we give an example for an illustration. Let $A = \mathbb{Z} \cup \{a, b, c\}$, where $f(x) = x + 1$ (if $x \in \mathbb{Z}$), $f(a) = 0$, $f(b) = a$, $f(c) = b$. Then $p(x) = x$ (if $x \in \mathbb{Z}$), $p(a) = -1$, $p(b) = -2$ and $p(c) = -3$.

Let us remark that we write $(\varphi \circ \psi)(x) = \varphi(\psi(x))$ for mappings φ, ψ. The notation implies:

Lemma 2.9.

(a) $f_r = f^r$ *for every* $r \geq 0$,

(b) $p \circ p = p$,

(c) $p \circ f = f \circ p$.

In [5], the following results were proved:
$C(\mathbb{N}) = \{f^k : k \geq 0\}$,
$C(\mathbb{Z}) = \{f^k : k \in \mathbb{Z}\}$,
$C(\mathbb{Z}_n) = \{f^k : k \geq 0\}$,
$C(L_{\infty,h}) = \{f^k : 0 \leq k < h\} \cup \{p \circ f_k : k \in \mathbb{Z}\}$,
$C(L_{n,h}) = \{f^k : 0 \leq k < h\} \cup \{p \circ f^k : k \geq 0\}$,
$C(L_{n,\infty}) = \{f^k : k \geq 0\} \cup \{p \circ f^k : k \geq 0\}$.

Throughout the paper, we will use the following two theorems which characterize all connected and non-connected monounary algebras having the property that the first and the second centralizer coincide.

Theorem 2.10. ([5], Theorem 4.1.) *Let (A, f) be a connected monounary algebra. Then $C_2(A, f) = C_1(A, f)$ if and only if (A, f) is isomorphic to one of the following algebras:*

(a) $\mathbb{Z}_n$, $n \in \mathbb{N}$,
(b) $\mathbb{N}$ *or* $\mathbb{Z}$,
(c) $L_{n,\infty}$ *or* $L_{n,k}$, $k, n \in \mathbb{N}$,
(d) $L_{\infty,k}$, $k \in \mathbb{N}$.

Theorem 2.11. ([5], Theorem 5.1.) *Let (A, f) be a non-connected monounary algebra. Then $C_2(A, f) = C_1(A, f)$ if and only if there exists a component B of (A, f) such that exactly one of the following conditions is satisfied:*

(a) $B \cong \mathbb{Z}$ *and* $A \backslash B \cong L_{1,k}$ *for some* $k \in \mathbb{N}_0$,
(b) $B \cong L_{\infty,k}$, *where* $k \in \mathbb{N}$, *and* $A \backslash B \cong \mathbb{Z}_1$,
(c) $B \cong \mathbb{N}$ *and* $A \backslash B \cong \mathbb{Z}_1$,
(d) $B \cong L_{1,\infty}$ *and the system of connected components of* $A \backslash B$ *is isomorphic to* $\{\mathbb{Z}_{n_i}\}_{i \in I}$, *where* $n_i \nmid n_j$ *for* $i, j \in I$, $i \neq j$,

(e) $B \cong L_{1,m}$ *for some* $m \in \mathbb{N}$ *and the system of connected components* $A \backslash B$ *is isomorphic to* $\{L_{n_i,k_i}\}_{i \in I}$, $k_i \in \{0, \infty\}$ *for* $i \in I$, *where* $n_i \nmid n_j$ *for* $i, j \in I$, $i \neq j$,

(f) $B \cong \mathbb{Z}_1$ *and the system of connected components* $A \backslash B$ *is isomorphic to* $\{L_{n_i,k_i}\}_{i \in I}$, $k_i \in \mathbb{N}_0 \cup \{\infty\}$ *for* $i \in I$, *where* $n_i \nmid n_j$ *for* $i, j \in I$, $i \neq j$,

(g) *the system of connected components of* (A, f) *is isomorphic to* $\{L_{n_i,k_i}\}_{i \in I}$, $k_i \in \mathbb{N}_0 \cup \{\infty\}$ *for* $i \in I$, *where* $n_i \nmid n_j$ *for* $i, j \in I$, $i \neq j$.

3. The relations $\mathcal{R}$ and $\leq_{\mathcal{R}}$ for connected monounary algebras

In this section, we suppose that A is a connected monounary algebra with $C_1(A) = C_2(A)$. For α, β from the centralizer of A, it will be determined when $\alpha \mathcal{R} \beta$. Successively, the algebras of types from Theorem 2.10 will be dealt with. We will use the above description of the centralizers implicitly.

In the proofs we write only C instead of $C(A)$.

Proposition 3.1. *Let* $\alpha, \beta \in C(\mathbb{N})$. *If* $\alpha = f^r$ *and* $\beta = f^s$, *where* $r, s \geq 0$, *then* $\alpha \leq_{\mathcal{R}} \beta$ *if and only if* $r \geq s$.

Proof. Let $\alpha = f^r$, $r \geq 0$ and $\beta = f^s$, $s \geq 0$. We have $\alpha \circ C = \left\{f^{r+k} : k \geq 0\right\} = \{f^m : m \geq r\}$ and $\beta \circ C = \left\{f^{s+k} : k \geq 0\right\} = \left\{f^j : j \geq s\right\}$. So the relation $\alpha \circ C \subseteq \beta \circ C$ is valid if and only if $r \geq s$. ∎

The following corollary follows immediately from Proposition 3.1.

Corollary 3.2. *Let* $\alpha, \beta \in C(\mathbb{N})$. *Then* $\alpha \mathcal{R} \beta$ *if and only if* $\alpha = \beta$.

Proposition 3.3. *Let* $\alpha, \beta \in C(\mathbb{Z})$. *Then* $\alpha \mathcal{R} \beta$ *for all* α, β.

Proof. For every $\gamma = f^r$, $r \in \mathbb{Z}$, we have $\gamma \circ C = \left\{f^{r+k} : k \in \mathbb{Z}\right\} = \{f^m : m \in \mathbb{Z}\}$. Hence $\alpha \circ C = \beta \circ C$ for all α, β. ∎

Proposition 3.4. *Let* $\alpha, \beta \in C(\mathbb{Z}_n)$. *Then* $\alpha \mathcal{R} \beta$ *for all* α, β.

Proof. For every $\gamma = f^r$, $r \geq 0$, we have $\gamma \circ C = \left\{f^{r+k} : k \geq 0\right\} = \{f^m : m \geq 0\}$. Again, $\alpha \circ C = \beta \circ C$ holds for all α, β. ∎

Proposition 3.5. *Let* $\alpha, \beta \in C(L_{\infty,h})$, *where* $h \in \mathbb{N}$. *Then:*

(1) *if* $\alpha = f^r$ *and* $\beta = f^s$, $0 \leq r, s < h$, *then* $\alpha \leq_{\mathcal{R}} \beta$ *if and only if* $r \geq s$;
(2) *if* $\alpha = f^r$, $0 \leq r < h$, *and* $\beta = p \circ f_s$, $s \in \mathbb{Z}$, *then* $\beta \leq_{\mathcal{R}} \alpha$ *and* $\alpha \nleq_{\mathcal{R}} \beta$;
(3) *if* $\alpha = p \circ f_r$ *and* $\beta = p \circ f_s$, *where* $r, s \in \mathbb{Z}$, *then* $\alpha \leq_{\mathcal{R}} \beta$ *and* $\beta \leq_{\mathcal{R}} \alpha$.

Proof. Consider the mapping $\gamma \in C(L_{\infty,h})$. We will show what the set $\gamma \circ C$ looks like.

If $\gamma = f^j$, $0 \leq j < h$, then

$$\gamma \circ C = f^j \circ C = \left\{f^{k+j} : j \leq k + j < h + j\right\} \cup \{p \circ f_{k+j} : k \in \mathbb{Z}\}.$$

By the definition of the mapping p and Lemma 2.9, for $h \le k+j$ we have $f^{k+j} = p \circ f^{k+j} = p \circ f_{k+j}$. Therefore

$$\begin{aligned}\gamma\circ C &= \left\{f^{k+j} : j \le k+j < h\right\} \cup \{p\circ f_{k+j} : h \le k+j < h+j\} \cup \{p\circ f_{k+j} : k \in \mathbb{Z}\} \\ &= \left\{f^{k+j} : j \le k+j < h\right\} \cup \{p\circ f_s : s \in \mathbb{Z}\},\end{aligned}$$

because $\{p \circ f_{k+j} : k+j \ge h\}$ is a subset of $\{p \circ f_{k+j} : k \in \mathbb{Z}\}$.

If $\gamma = p \circ f_j$, $j \in \mathbb{Z}$, then

$$\begin{aligned}\gamma \circ C &= (p \circ f_j) \circ C \\ &= \left\{(p \circ f_j) \circ f^k : 0 \le k < h\right\} \cup \{(p \circ f_j) \circ (p \circ f_k) : k \in \mathbb{Z}\}.\end{aligned}$$

Similarly as above, the definition of p and Lemma 2.9 imply

$$\gamma \circ C = \{p \circ f_{k+j} : 0 \le k < h\} \cup \{p \circ f_{k+j} : k \in \mathbb{Z}\} = \{p \circ f_r : r \in \mathbb{Z}\},$$

because $\{p \circ f_{k+j} : 0 \le k < h\}$ is a subset of $\{p \circ f_{k+j} : k \in \mathbb{Z}\}$.

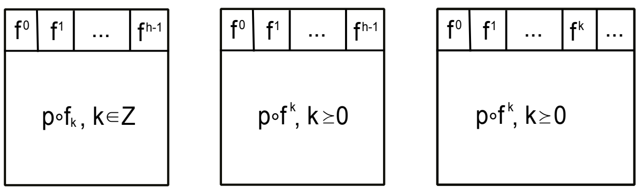
The results follow from these calculations. We have three cases for α, β. If $\alpha = f^r$, $\beta = f^s$, $0 \le r, s < h$, then $\alpha \circ C \subseteq \beta \circ C$ if and only if $r \ge s$. In case, when α is a mapping of type f^k and β is a mapping of type $p \circ f_k$, then $\alpha \circ C \supset \beta \circ C$, because $\alpha \in \alpha \circ C$, but $\alpha \notin \beta \circ C$. So $\beta \le_{\mathcal{R}} \alpha$ and $\alpha \nleq_{\mathcal{R}} \beta$. Finally, if the assumption of the third case is valid, then $\alpha \circ C = \beta \circ C$ for all α, β, i.e., $\alpha \le_{\mathcal{R}} \beta$ and $\beta \le_{\mathcal{R}} \alpha$. ∎

Remark 3.6. Proposition 3.5 is true for $L_{n,h}$ and $L_{n,\infty}$. (We need to make obvious changes regarding the range of r and s, and replace f_r with f^r and f_s with f^s.) Proofs are similar.

The following corollary follows from Proposition 3.5 and Remark 3.6.

Corollary 3.7. *Let $\alpha, \beta \in C(A)$, where $A \in \{L_{\infty,h}, L_{n,h}, L_{n,\infty} : n, h \in \mathbb{N}\}$. Then $\alpha\mathcal{R}\beta$ if and only if $\alpha = \beta$ or α and β are both of type $p \circ f_k$ (where $f_k = f^k$ if $A \in \{L_{n,h}, L_{n,\infty}\}$).*

In the figure, the $\mathcal{R}$-classes for algebras $L_{\infty,h}$, $L_{n,h}$ and $L_{n,\infty}$, $n, h \in \mathbb{N}$, are illustrated.



$L_{\infty,h}$ $L_{n,h}$ $L_{n,\infty}$

4. The relations $\mathcal{R}$ and $\leq_{\mathcal{R}}$ for non-connected monounary algebras

In what follows, let A be a non-connected monounary algebra with $C = C(A) = C_2(A)$. For $\alpha, \beta \in C$, we will show when $\alpha\mathcal{R}\beta$, i.e., when the sets $\alpha \circ C$ and $\beta \circ C$ coincide. We prove also necessary and sufficient conditions for the relation $\leq_{\mathcal{R}}$.

According to Theorem 2.11, the algebra A contains at most one one-element cycle. If A contains no one-element cycle, denote $B_0 = \emptyset$; otherwise let B_0 be the component with this cycle. Let $\{B_i\}_{i\in I}$ be the partition of $A \backslash B_0$ into components ($0 \notin I$). Let $\varphi \in C$. By [5, Lemma 5.1], $\varphi(B_0) \subseteq B_0$ and for every $i \in I$, either $\varphi(B_i) \subseteq B_i$ or $\varphi(B_i) = \{c\}$, where $\{c\}$ is the one-element cycle in B_0.

If $i \in I \cup \{0\}$, $B_i \neq \emptyset$ and $\varphi \in C$, denote $\varphi_i = \varphi \restriction B_i$.

If $B_0 = \emptyset$, then we write $\varphi = (\varphi_i)_{i\in I}$ and then

$$C(A) = \{(\varphi_i)_{i\in I} : \varphi_i \in C(B_i)\}.$$

If $B_0 \neq \emptyset$, then we write $\varphi = (\varphi_i)_{i\in I\cup\{0\}}$. For $j \in I \cup \{0\}$, denote by ε_j the mapping from B_j to B_0 defined by $\varepsilon_j(B_j) = \{c\}$, where $\{c\}$ is the one-element cycle in B_0. Then

$$C(A) = \left\{(\varphi_i)_{i\in I\cup\{0\}} : \varphi_0 \in C(B_0), \varphi_i \in C(B_i) \cup \{\varepsilon_i\} \text{ for } i \in I\right\}.$$

Next, for $\alpha \in C$ define the set $I(\alpha) = \{i \in I : \alpha_i = \varepsilon_i\}$.

Lemma 4.1. *Let $\alpha, \beta \in C$, $\alpha \leq_{\mathcal{R}} \beta$. Then $I(\beta) \subseteq I(\alpha)$.*

Proof. Let $i \in I(\beta)$ and assume that $i \notin I(\alpha)$. Since $\alpha \leq_{\mathcal{R}} \beta$, there exists $\varphi \in C$ with $\alpha \circ \alpha = \beta \circ \varphi$. If $x \in B_i$, then $(\alpha \circ \alpha)(x) \in B_i$, while $(\beta \circ \varphi)(x) = \varepsilon_i(\varphi(x)) \in B_0$, a contradiction. ∎

Corollary 4.2. *Let $\alpha, \beta \in C$. If $\alpha\mathcal{R}\beta$, then $I(\alpha) = I(\beta)$.*

Lemma 4.3. *Let $\alpha, \beta \in C$ and $i \in I \cup \{0\} \setminus (I(\alpha) \cup I(\beta))$. If $\alpha \circ C \subseteq \beta \circ C$, then $\alpha_i \circ C(B_i) \subseteq \beta_i \circ C(B_i)$.*

Proof. Suppose that $\alpha \circ C \subseteq \beta \circ C$. Let $g \in \alpha_i \circ C(B_i)$. Then there exists $h \in C(B_i)$ such that $g = \alpha_i \circ h$. We define a mapping $\bar{h} : A \to A$ as follows:

$$\bar{h}(x) = \begin{cases} h(x), & \text{if } x \in B_i, \\ x, & \text{otherwise.} \end{cases}$$

Obviously, $\bar{h} \in C$. We have $\alpha \circ \bar{h} \in \alpha \circ C \subseteq \beta \circ C$, thus there is $\gamma \in C$ with $\alpha \circ \bar{h} = \beta \circ \gamma$. Let $b \in B_i$. Then

$$\gamma(\beta_i(b)) = \gamma(\beta(b)) = (\gamma \circ \beta)(b) = (\beta \circ \gamma)(b) = \left(\alpha \circ \bar{h}\right)(b) =$$
$$\alpha\left(\bar{h}(b)\right) = \alpha(h(b)) = \alpha_i(h(b)) \in B_i.$$

This implies $i \notin I(\gamma)$, therefore $g = \alpha_i \circ h = \gamma \circ \beta_i = \gamma_i \circ \beta_i = \beta_i \circ \gamma_i \in \beta_i \circ C(B_i)$. ■

Theorem 4.4. *Let $\alpha, \beta \in C$. Then $\alpha \leq_{\mathcal{R}} \beta$ if and only if :*

(i) $I(\beta) \subseteq I(\alpha)$; *and*
(ii) $\alpha_i \leq_{\mathcal{R}} \beta_i$ *for each* $i \in (I \cup \{0\}) \setminus I(\alpha)$.

Proof. If $\alpha, \beta \in C$, then (i) is satisfied according to Lemma 4.1. By (i), $I \cup \{0\} \setminus (I(\alpha) \cup I(\beta)) = (I \cup \{0\}) \setminus I(\alpha)$ and then Lemma 4.3 yields (ii).

Now, let (i) and (ii) hold. We are going to prove that $\alpha \circ C \subseteq \beta \circ C$. Take $\gamma \in C$. We will show that there is $\delta \in C$ such that $\alpha \circ \gamma = \beta \circ \delta$. If $i \in I \cup \{0\} \setminus I(\alpha)$ and $\gamma_i \neq \varepsilon_i$, then $\gamma_i \in C(B_i)$ and the condition (ii) implies that there exists $t_i \in C(B_i)$ such that $\alpha_i \circ \gamma_i = \beta_i \circ t_i$. Define $\delta : A \to A$ by

$$\delta_i = \begin{cases} t_i, & \text{if } i \in I \cup \{0\} \setminus I(\alpha), \gamma_i \neq \varepsilon_i, \\ \varepsilon_i, & \text{otherwise.} \end{cases}$$

Clearly, $\delta \in C$. Let us show that $(\alpha \circ \gamma)(b) = (\beta \circ \delta)(b)$ for each $b \in B_i$.

If $i \in I \cup \{0\} \setminus I(\alpha)$ and $\gamma_i \neq \varepsilon_i$, then

$$(\alpha \circ \gamma)(b) = (\alpha_i \circ \gamma_i)(b) = (\beta_i \circ t_i)(b) = \beta(t_i(b)) = (\beta \circ \delta)(b).$$

Let $i \in I(\alpha)$. Then

$$\begin{aligned}(\alpha \circ \gamma)(b) &= \alpha(\gamma(b)) = \varepsilon_i(\gamma(b)) = \varepsilon_i(b) = \beta(\varepsilon_i(b)) = \beta(\delta(b)) \\ &= (\beta \circ \delta)(b).\end{aligned}$$

Finally, if $i \notin I(\alpha)$ and $\gamma_i = \varepsilon_i$, then

$$\begin{aligned}(\alpha \circ \gamma)(b) &= \alpha(\gamma(b)) = \alpha(\varepsilon_i(b)) = \varepsilon_i(b) = \beta(\varepsilon_i(b)) = \beta(\delta(b)) \\ &= (\beta \circ \delta)(b).\end{aligned}$$

This completes the proof. ■

The following corollary follows immediately from Theorem 4.4.

Corollary 4.5. *Let $\alpha, \beta \in C$. Then $\alpha \mathcal{R} \beta$ if and only if:*

(i) $I(\alpha) = I(\beta)$; *and*
(ii) $\alpha_i \mathcal{R} \beta_i$ *for each* $i \in (I \cup \{0\}) \setminus I(\alpha)$.

References

[1] J. Araújo, J. Konieczny, *General theorems on automorphisms of semigroups and their applications*, J. Austral. Math. Soc. 87(1) (2009), 1–17.
[2] J. Araújo, J. Konieczny, *Centralizers in the full transformation semigroup*, Semigroup Forum 86 (2013), 1–31.
[3] J. M. Howie, *Fundamentals of Semigroup Theory*, Oxford University Press, New York, 1995.

[4] D. Jakubíková-Studenovská, J. Pócs, *Monounary Algebras*, P. J. Šafárik Univ., Košice, 2009.

[5] D. Jakubíková-Studenovská, M. Šuličová, *Centralizers of a monounary algebra*, Asian-Eur. J. Math., Vol. 08, No. 01, 1550007 (2015), DOI: 10.1142/S1793557115500072.

[6] B. Jónsson, *Topics in universal algebra, Lecture Notes in Mathematics*, Vol. 250, Springer-Verlag, 1972.

[7] J. Konieczny, S. Lipscomb, *Centralizers in the semigroup of partial transformations*, Math. Japon. 48 (1998), 367–376.

[8] J. Konieczny, *Green's relations and regularity in centralizers of permutations*, Glasgow Math. J. 41(1) (1999), 45–57.

[9] J. Konieczny, *Second centralizers of patrial transformations*, Czechoslovak Math. J. 51(126) (2001), 873–888.

[10] J. Konieczny, *Semigroups of transformations commuting with idempotents*, Algebra Colloq. 9 (2002), 121–134.

[11] J. Konieczny, *Semigroups of transformations commuting with injective nilpotents*, Comm. Algebra 32 (2004), 1951–1969.

[12] J. Konieczny, *Centralizers in the semigroup of injective transformations on an infinite set*, Bull. Austral. Math. Soc. 82 (2010), 305–321.

[13] J. Konieczny, *Infinite injective transformations whose centralizers have simple structure*, Cent. Eur. J. Math. 9(1) (2011), 23–35.

[14] J. Kopeček, *Homomorphisms of partial unary algebras*, Czechoslovak Math. J. 26(101) (1976), 108–127.

[15] M. Novotný, *Über Abbildingen von Mengen*, Pacific J. Math. 13 (1963), 1359–1369.

[16] J. G. Pitkethly, B. A. Davey, *Dualisability: Unary Algebras and Beyond*, Springer-Verlag, 2005.

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