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D. S. Davidova, Yu. M. Movsisyan

WEAKLY IDEMPOTENT LATTICES AND BILATTICES, NON-IDEMPOTENT PLONKA FUNCTIONS

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Abstract. In this paper, we study weakly idempotent lattices with an additional interlaced operation. We characterize interlacity of a weakly idempotent semilattice operation, using the concept of hyperidentity and prove that a weakly idempotent bilattice with an interlaced operation is epimorphic to the superproduct with negation of two equal lattices. In the last part of the paper, we introduce the concepts of a non-idempotent Plonka function and the weakly Plonka sum and extend the main result for algebras with the well known Plonka function to the algebras with the non-idempotent Plonka function. As a consequence, we characterize the hyperidentities of the variety of weakly idempotent lattices, using non-idempotent Plonka functions, weakly Plonka sums and characterization of cardinality of the sets of operations of subdirectly irreducible algebras with hyperidentities of the variety of weakly idempotent lattices. Applications of weakly idempotent bilattices in multi-valued logic is to appear.

1. Introduction

There exist various extensions of the concept of a lattice. For example, in [14], [13], weakly associative lattices were introduced and in [2], [20], [21], [23], the lattices with a third operation were studied. In [24], an algebra with a system of identities was introduced, which we call weakly idempotent lattices (also see [18], [36]).

The paper consists of Introduction and four paragraphs.

In the second paragraph, we give the definitions of a weakly idempotent semilattice, a weakly idempotent lattice, a weakly idempotent (pre-)bilattice, an interlaced operation, an interlaced weakly idempotent (pre-)bilattice and

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hyperidentities; then we prove some preliminary results. Further, we establish a connection among these concepts of these weakly idempotent structures and the corresponding quasiorders (Lemmas 2.7–2.10, Corollaries 2.12, 2.13). In the third paragraph, we prove some properties of weakly idempotent lattices. In particular, in Theorem 3.3 we characterize interlacity for the weakly idempotent semilattice operation, using the concept of hyperidentity. In paragraph four, we characterize the interlaced weakly idempotent bilattices (Theorem 4.7) and the weakly idempotent pre-bilattices (Corollary 4.8). As a corollary we also obtain a characterization of weakly idempotent distributive bilattices (Corollary 4.9). In the chapter fifth, we introduce the concepts of a non-idempotent Plonka function and a weakly Plonka sum. Here the main result for algebras with the well known Plonka function is extended to the algebras with a non-idempotent Plonka function. In the last chapter, as a corollary we characterize hyperidentities of the variety of weakly idempotent lattices and cardinality of the sets of the operations of subdirectly irreducible algebras with hyperidentities of the variety of weakly idempotent lattices.

2. Preliminary concepts and results

Definition 2.1. The algebra $(L; \wedge)$ with one binary operation is called weakly idempotent semilattice, if it satisfies the following identities:

$$a \wedge b = b \wedge a, \text{ (commutativity)} \tag{1}$$

$$(a \wedge b) \wedge c = a \wedge (b \wedge c), \text{ (associativity)} \tag{2}$$

$$a \wedge (b \wedge b) = a \wedge b. \text{ (weakly idempotency)} \tag{3}$$

The operation $\wedge$ is called product. Adding the idempotent identity $a \wedge a = a$ to it, we obtain a semilattice. The element $a \in L$ is called idempotent of the weakly idempotent semilattice $(L; \wedge)$, if $a \wedge a = a$. The set of the idempotent elements of each weakly idempotent semilattice forms a semilattice, i.e. the product of any two idempotent elements in the weakly idempotent semilattice is an idempotent element.

Definition 2.2. The algebra $(L; \wedge, \vee)$ with two binary operations is called weakly idempotent lattice if the reducts $(L; \wedge)$ and $(L; \vee)$ are weakly idempotent semilattices and the following identities are valid:

$$a \wedge (b \vee a) = a \wedge a, a \vee (b \wedge a) = a \vee a, \text{ (weakly absorption)} \tag{4}$$

$$a \wedge a = a \vee a. \text{ (equalization)} \tag{5}$$

The operation $\vee$ is called sum. The element $a \in L$ is called an idempotent of the weakly idempotent lattice $(L; \wedge, \vee)$ if $a \wedge a = a$ and $a \vee a = a$. Note that the product (sum) of two elements of a weakly idempotent lattice $(L; \wedge, \vee)$ is an idempotent element:

$(x\wedge y)\vee(x\wedge y)\overset{(5)}{=}(x\wedge y)\wedge(x\wedge y)\overset{(2)}{=}((x\wedge y)\wedge x)\wedge y\overset{(1)}{=}((y\wedge x)\wedge x)\wedge y\overset{(2)}{=}$ $(y\wedge(x\wedge x))\wedge y\overset{(3)}{=}(y\wedge x)\wedge y\overset{(1)}{=}(x\wedge y)\wedge y\overset{(2)}{=}x\wedge(y\wedge y)\overset{(3)}{=}x\wedge y.$ The other condition is proved similarly. So, the set of all idempotent elements of a weakly idempotent lattice is a lattice.

The weakly idempotent lattice $(L;\wedge,\vee)$ is called distributive, if it satisfies the following identities:

$$x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z),$$
$$x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z).$$

The weakly idempotent lattice $(L;\wedge,\vee)$ is called modular, if it satisfies the following identities:

$$(x\wedge(y\vee z))\vee(y\wedge z)=(x\vee(y\wedge z))\wedge(y\vee z),$$
$$(x\vee(y\wedge z))\wedge(y\vee z)=(x\wedge(y\vee z))\vee(y\wedge z).$$

Definition 2.3. The relation $\theta\subseteq L\times L$ is called a quasiorder if it is reflexive and transitive.

Example 2.4. The classical relation of divisibility on $\mathbb{Z}$ is a quasiorder.

Example 2.5. A cover of a set L is such a family $P=\{X_i\}_{i\in I}$ of subsets of L that $\cup_{i\in I}X_i=L$. A relation Q, defined on the set of all covers of the set L:

$$P_1QP_2\iff\forall X\in P_1\exists Y\in P_2(X\subseteq Y)$$

is a quasiorder. It is not an order, because there exist such different covers P_1 and P_2 that P_1QP_2 and P_2QP_1.

Every quasiorder generates an order as follows.

Let θ be a quasiorder on the set $L\neq\emptyset$; then $E_\theta=\theta\cap\theta^{-1}\subseteq L\times L$ is an equivalence. For any element $x\in L$ let us denote by $[x]$ the class of the relation E_θ which contains the element x. Let $\leq_\theta$ be a relation induced on the set L/E_θ from θ in the following manner: for $[a],[b]\in L/E_\theta$

$$[a]\leq_\theta[b]\in\iff a\theta b.$$

A straightforward arguments show that this definition is correct and that it is an order.

The function $K:L/E_\theta\to L$ is called a choice function, if $K([a])\in[a]$, for each $[a]\in L/E_\theta$.

Definition 2.6. The pair $(L;\theta)$ is called inf-quasiordered (a sup-quasiordered) set, if for each two classes of equivalences $[a],[b]\in L/E_\theta$ there exists: $inf([a],[b])=[a]\wedge[b]$ (dual $sup([a],[b])=[a]\vee[b]$) i.e., if $(L/E_\theta;\leq_\theta)$ is a lower (an upper) semilattice.

Lemma 2.7. *Let $(L;\theta)$ be an inf-quasiordered set and let $K : L/E_\theta \to L$ be an arbitrary choice function. If for $x, y \in L$, $x \wedge y = K(inf([x],[y])) = K([x] \wedge [y])$, then the algebra $(L;\wedge)$ is a weakly idempotent semilattice, which we call lower weakly idempotent semilattice.*

Proof. Since $(L;\theta)$ is an inf-quasiordered set, then there exists $inf([a],[b])$ for each $[a],[b] \in L/E_\theta$. Let us define the operation $\wedge$ in the following manner:

$$x \wedge y = K(inf([x],[y])).$$

Then the algebra $(L;\wedge)$ satisfies the identities: (1)–(3):

1. $a \wedge b = K(inf([a],[b])) = K(inf([b],[a])) = b \wedge a.$
2. $(a \wedge b) \wedge c = K(inf([a \wedge b],[c])) = K(inf([a] \wedge [b],[c])) =$
   $K(inf(inf([a],[b]),[c])) = (inf([a], inf([b],[c]))) =$
   $K(inf([a],[b] \wedge [c])) = K(inf([a],[b \wedge c])) = a \wedge (b \wedge c).$
3. $a \wedge (b \wedge b) = K(inf([a],[b \wedge b])) = K(inf([a],[b] \wedge [b])) =$
   $K(inf([a], inf([b],[b]))) = K(inf([a],[b]])) = a \wedge b.$ ∎

Lemma 2.8. *Let $(L;\wedge)$ be a weakly idempotent semilattice. Then the relation $a\theta b \leftrightarrow a \wedge b = a \wedge a$ is a quasiorder on the set L, the mapping $K : L/E_\theta \mapsto L$; $[a] \mapsto a \wedge a$ is a choice function and the pair (L,θ) is an inf-quasiordered set with $inf([a],[b]) = [a \wedge b]$ and $x \wedge y = K(inf([x],[y]))$.*

Proof. Let us show that the relation θ, which is defined in the following manner:

$$a\theta b \leftrightarrow a \wedge b = a \wedge a$$

is a quasiorder on the set L. Indeed, if $a\theta b$ and $b\theta c$, then $a \wedge b = a \wedge a$ and $b \wedge c = b \wedge b$ and, using the identities (1) and (3), we obtain:

$$a \wedge c = (a \wedge a) \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge (b \wedge b) = a \wedge b = a \wedge a;$$

hence $a\theta c$, i.e. θ is transitive. Reflexivity of θ is obvious.

Now, let us show that $a, b \in [x] \in L/E_\theta$ iff $a \wedge a = b \wedge b$, for any $a, b \in L$: $[a] = [b] \leftrightarrow [a] \leq_\theta [b]$ and $[b] \leq_\theta [a] \leftrightarrow a\theta b$ and $b\theta a \leftrightarrow a \wedge b = a \wedge a$ and $b \wedge a = b \wedge b \leftrightarrow a \wedge a = b \wedge b$.

It is obvious that $[a \wedge b] \leq_\theta [a]$ and $[a \wedge b] \leq_\theta [b]$. Indeed:

$$(a \wedge b) \wedge a = a \wedge b = (a \wedge b) \wedge (a \wedge b) \to (a \wedge b)\theta a \to [a \wedge b] \leq_\theta [a].$$

If $[c] \in L/E_\theta$ and $[c] \leq_\theta [a]$, $[c] \leq_\theta [b]$, then $c \wedge a = c \wedge c$ and $c \wedge b = c \wedge c$. Using the identities (1) and (2), we obtain:

$$c \wedge (a \wedge b) = (c \wedge a) \wedge b = (c \wedge c) \wedge b = c \wedge (c \wedge b) = c \wedge (c \wedge c) = c \wedge c \to c\theta(a \wedge b),$$

hence $[c] \leq_\theta [a \wedge b]$. Thus, $inf([a],[b]) = [a \wedge b]$. ∎

Lemmas 2.7 and 2.8 show that there is a one-to-one correspondence between weakly idempotent semilattices and inf-quasiordered sets.

Lemma 2.9. *Let $(L;\theta)$ be a sup-quasiordered set and let $K : L/E_\theta \mapsto L$ be an arbitrary choice function. If for each two elements $x, y \in L$:*

$$x \vee y = K(sup([x],[y])) = K([x] \vee [y]),$$

then the algebra $(L;\vee)$ is a weakly idempotent semilattice, which we call upper weakly idempotent semilattice.

Lemma 2.10. *Let $(L;\vee)$ be a weakly idempotent semilattice. Then the relation $a\theta b \leftrightarrow a \vee b = b \vee b$ is a quasiorder on the set L, the mapping $K : L/E_\theta \mapsto L$; $[a] \mapsto a \vee a$ is a choice function and the pair $(L;\theta)$ is a sup-quasiordered set with $sup([a],[b]) = [a \vee b]$ and $x \vee y = K(sup([x],[y]))$.*

Definition 2.11. The pair (L,θ) is called an $inf\,sup$-quasiordered set, if for each two classes of equivalences $[a],[b] \in L/E_\theta$, both $inf([a],[b]) = [a] \wedge [b]$ and $sup([a],[b]) = [a] \vee [b]$ exist, i.e. if $(L/E_\theta; \leq_\theta)$ is a lattice.

As a corollary of the above lemmas, we get one-to-one correspondence between weakly idempotent lattices and $inf\,sup$-quasiordered sets.

Corollary 2.12. *Let $(L;\theta)$ be an $inf\,sup$-quasiordered set and let $K : L/E_\theta \mapsto L$ be an arbitrary choice function. If for each two elements $x, y \in L$:*

$$x \wedge y = K(inf([x],[y])) = K([x] \wedge [y]),$$
$$x \vee y = K(sup([x],[y])) = K([x] \vee [y]),$$

then the algebra $(L;\wedge,\vee)$ is a weakly idempotent lattice.

Corollary 2.13. *Let $(L;\wedge,\vee)$ be a weakly idempotent lattice. Then the relation $a\theta b \leftrightarrow a \wedge b = a \wedge a \leftrightarrow a \vee b = b \vee b$ is a quasiorder on the set L, the mapping $K : L/E_\theta \mapsto L$; $[a] \mapsto a \vee a$ is a choice function and the pair (L,θ) is an $inf\,sup$-quasiordered set with $inf([a],[b]) = [a \wedge b], sup([a],[b]) = [a \vee b]$ and $x \wedge y = K(inf([x],[y])), x \vee y = K(sup([x],[y]))$.*

Example 2.14. Let's consider the relation of divisibility on the set $\mathbb{Z}\backslash\{0\}$ which is a quasiorder on $\mathbb{Z}\backslash\{0\}$. The corresponding equivalence classes are the following sets $\{x, -x\}$. Define a choice function as $K([x]) = |x|$. Then we have $x \wedge_1 y = (|x|,|y|)$, $x \vee_1 y = [|x|,|y|]$, for which $(|x|,|y|)$ and $[|x|,|y|]$ are the greatest common division (gcd) and the least common multiple (lcm) of $|x|$ and $|y|$, respectively. Thus, the algebra $(Z\backslash\{0\};\wedge_1,\vee_1)$ is a weakly idempotent lattice, which is not a lattice, since $x \wedge_1 x \neq x$ for negative x.

Example 2.15. If we define the choice function on $\mathbb{Z}\backslash\{0\}$ as follows $K([x]) = -|x|$, then we have $x \wedge_2 y = -(|x|,|y|)$, $x \vee_2 y = -[|x|,|y|]$, and the algebra $(Z\backslash\{0\};\wedge_2,\vee_2)$ also is a weakly idempotent lattice.

Definition 2.16. We say that the operation $*$ of the weakly idempotent semilattice $(L;*)$ is interlaced with the operations $\wedge$ and $\vee$ of the weakly idempotent lattice $(L;\wedge,\vee)$ if the weakly idempotent semilattice operation $*$

preserves the weakly idempotent lattice quasiorder, and the operations $\wedge, \vee$ preserve the weakly idempotent semilattice quasiorder.

Note that the basic operations of a weakly idempotent lattice are interlaced with each other and we say that a weakly idempotent lattice is interlaced.

Definition 2.17. An algebra $(L; \wedge, \vee, *, \Delta)$ with four binary operations is called a weakly idempotent pre-bilattice, if the reducts $L_1 = (L; \wedge, \vee)$ and $L_2 = (L; *, \Delta)$ are weakly idempotent lattices and the following identity is valid: $a * a = a \wedge a$. If the reducts $L_1 = (L; \wedge, \vee)$ and $L_2 = (L; *, \Delta)$ are lattices, then the algebra $(L; \wedge, \vee, *, \Delta)$ is called a pre-bilattice.

Following M. L. Ginsberg [17], we introduce the following concept of weakly idempotent bilattice.

Definition 2.18. An algebra $(L; \wedge, \vee, *, \Delta, ')$ with four binary operations and one unary operation of negation is called a weakly idempotent bilattice if $(L; \wedge, \vee, *, \Delta)$ is a weakly idempotent pre-bilattice and the following identities are valid:

$$(a \wedge b)' = a' \vee b', \ \ (a \vee b)' = a' \wedge b',$$
$$(a * b)' = a' * b', (a \Delta b)' = a' \Delta b', (a')' = a.$$

If $(L; \wedge, \vee, *, \Delta)$ is a pre-bilattice, then $(L; \wedge, \vee, *, \Delta, ')$ is called bilattice.

The weakly idempotent (pre-)bilattice $(L; \wedge, \vee, *, \Delta, ')$ ($(L; \wedge, \vee, *, \Delta)$) is called distributive (modular), if the reducts $L_1 = (L; \wedge, \vee)$ and $L_2 = (L; *, \Delta)$ are weakly idempotent distributive (modular) lattices.

The weakly idempotent (pre-)bilattices are extension of the concept of (pre-)bilattice. Bilattices were introduced by M. L. Ginsberg [17] as a general and uniform framework for a diversity of applications in artificial intelligence. In a series of papers, it was shown that these structures may serve as a foundation for many areas, such as logic programming [11], [12], artificial intelligence [17], truth theory [10] and others. For applications and characterization of (pre-)bilattices in various varieties see also [3], [5], [6], [7], [15], [22], [26]–[33], [38], [40]–[42].

Since every weakly idempotent (pre-)bilattice is structured of two weakly idempotent lattices, we have two quasiorders corresponding to every weakly idempotent (pre-)bilattice.

Definition 2.19. A weakly idempotent (pre-)bilattice is called interlaced, if each basic weakly idempotent (pre-)bilattice operation is quasiorder preserving with respect to both quasiorders.

Example 2.20. Every distributive weakly idempotent (pre-)bilattices is interlaced.

Example 2.21. Note that each operation of the weakly idempotent lattice $(L; \wedge, \vee)$ preserves the corresponding quasiorder; hence the weakly idempotent pre-bilattice $(L; \wedge, \vee, \wedge, \vee)$ is interlaced.

Let us recall that a hyperidentity is a second-order formula of the following type:

$$\forall X_1, \ldots, X_m \forall x_1, \ldots, x_n (w_1 = w_2),$$

where $X_1, \ldots, X_m$ are functional variables, and $x_1, \ldots, x_n$ are object variables in the words (terms) of w_1, w_2. Hyperidentities are usually written without quantifiers, $w_1 = w_2$. We say that the hyperidentity $w_1 = w_2$ is satisfied in the algebra $(Q; F)$ if this equality is valid, when every object variable and every functional variable in it is replaced by any element from Q and by any operation of the corresponding arity from F (supposing the possibility of such replacement) ([27]–[29], [43], [4]).

On characterization of hyperidentities of varieties of lattices, modular lattices, distributive lattices, Boolean and De Morgan algebras see in [28]–[32]. About hyperidentities in thermal (polynomial) algebras see in [8], [9], [19], [30], [34].

For example, the weakly idempotent pre-bilattice $L = (L; \wedge, \vee, *, \triangle)$ is distributive iff it satisfies the following hyperidentity:

$$X(Y(x, y), z) = Y(X(x, z), X(y, z)). \tag{6}$$

For the categorical definition of the hyperidentity in [27], the (bi)homomorphisms between two algebras $(Q; F)$ and $(Q'; F')$ are defined as the pair $(\varphi; \tilde{\psi})$ of the mappings:

$$\varphi : Q \to Q', \tilde{\psi} : F \to F', |A| = |\tilde{\psi} A|,$$

with the following condition:

$$\varphi A(a_1, \ldots, a_n) = (\tilde{\psi} A)(\varphi a_1, \ldots, \varphi a_n)$$

for any $A \in F, a_1, \ldots, a_n \in Q, |A| = n$ (about application of such morphisms in the cryptography see [1]).

Algebras with their (bi)homomorphisms $(\varphi; \tilde{\psi})$ (as morphisms) form a category. The product in this category is called superproduct of algebras and is denoted by $Q \bowtie Q'$ for the two algebras Q and Q'. For example, the superproduct of the two weakly idempotent lattices $(Q; +, \cdot)$ and $(Q'; +, \cdot)$ is the binary algebra $(Q \times Q'; (+, \cdot), (\cdot, +), (+, +), (\cdot, \cdot))$, with four binary operations, where the pairs of the operations operate component-wise, i.e.

$$(A, B)((x, y), (u, v)) = (A(x, u), B(y, v)).$$

Example 2.22. The superproduct $Q \bowtie Q'$ is an interlaced weakly idempotent pre-bilattice for any weakly idempotent lattices Q and Q' (it follows from Corollary 3.5, too).

3. Some properties of weakly idempotent lattices

Let $(L; \wedge, \vee)$ be a weakly idempotent lattice and $(L; *)$ be a weakly idempotent semilattice. Denote the quasiorder corresponding to $(L; \wedge, \vee)$ by $\leq_\wedge$ and the quasiorder corresponding to $(L; *)$ by $\leq_*$ for $(L; *)$, which are defined as follows:

$$a \leq_\wedge b \iff a \wedge b = a \wedge a \iff a \vee b = b \vee b, \tag{7}$$

$$a \leq_* b \iff a * b = a * a. \tag{8}$$

The following lemmas are proved, using the definition of weakly idempotent lattices and weakly idempotent semilattices.

Lemma 3.1. *Let $(L; \wedge, \vee)$ be a weakly idempotent lattice and $(L; *)$ be a weakly idempotent semilattice. Then:*

$$a \leq_* b \leq_* a \to a * a = b * b,$$

$$a \leq_\wedge b \leq_\wedge a \to a \wedge a = b \wedge b.$$

Lemma 3.2. *Let $(L; \wedge, \vee)$ be a weakly idempotent lattice. Then:*

$$x \wedge y \leq_\wedge x \leq_\wedge x \vee y,$$

where $x, y \in L$.

Theorem 3.3. *The operation $*$ of the weakly idempotent semilattice $(L; *)$ is interlaced with the operations of the weakly idempotent lattice $(L; \wedge, \vee)$ iff the algebra $(L; \wedge, \vee, *)$ satisfies the following hyperidentity:*

$$X(Y(X(x, y), z), Y(y, z)) = X(Y(X(x, y), z), Y(X(x, y), z)). \tag{9}$$

In particular, the class of such algebras $(L; \wedge, \vee, *)$ forms a variety.

Proof. Let us show, for example, that if the operation $*$ is interlaced with the operations $\wedge, \vee$, then the following identity is valid:

$$((x \wedge y) * z) \wedge (y * z) = ((x \wedge y) * z) \wedge ((x \wedge y) * z). \tag{10}$$

From Lemma 3.2, we have: $x \wedge y \leq_\wedge y$, then $((x \wedge y) * z) \leq_\wedge y * z$. Hence, $((x \wedge y) * z) \wedge (y * z) = ((x \wedge y) * z) \wedge ((x \wedge y) * z)$. Conversely, let in $(L; \wedge, \vee, *)$ the hyperidentity (9) is valid, then
$((x \wedge x) * z) \vee (x * z) = ((x \vee x) * z) \vee (x * z) = ((x \vee x) * z) \vee ((x \vee x) * z) = ((x \vee x) * z) \wedge ((x \vee x) * z)$,
hence, $x * z \leq_\wedge (x \wedge x) * z \leq_\wedge x * z$, thus by Lemma 3.1, we get:
$((x \wedge x) * z) \wedge ((x \wedge x) * z) = (x * z) \wedge (x * z)$.

Suppose that $x \leq_\wedge y$, then
$(x*z)\wedge(y*z) = ((x*z)\wedge(x*z))\wedge(y*z) = ((x\wedge x)*z)\wedge((x\wedge x)*z)\wedge(y*z) = ((x\wedge y)*z)\wedge((x\wedge y)*z) = ((x\wedge x)*z)\wedge((x\wedge x)*z) = (x*z)\wedge(x*z)$.
So, if $x \leq_\wedge y$ then $x*z \leq_\wedge y*z$.

In the same way from $x \leq_* y$, using the hyperidentity (9), we get $x\wedge z \leq_* y\wedge z$ and $x\vee z \leq_* y\vee z$. ∎

Corollary 3.4. *Let $(L;\wedge,\vee)$ be a weakly idempotent lattice and $(L,*)$ be a weakly idempotent semilattice and the following identity $a\wedge a = a*a$ is valid. Then the operation $*$ is interlaced with the operations $\wedge,\vee$ iff the algebra $(L;\wedge,\vee,*)$ satisfies the following hyperidentity:*

$$X(Y(X(x,y),z),Y(y,z)) = Y(X(x,y),z). \tag{11}$$

Corollary 3.5. *A weakly idempotent bilattice $(L;\wedge,\vee,*,\Delta,')$ (pre-bilattice $(L;\wedge,\vee,*,\Delta)$) is interlaced iff the algebra $(L;\wedge,\vee,*,\Delta)$ satisfies the hyperidentity (11).*

In Lemmas 3.6–3.17 we assume that $(L;\wedge,\vee)$ is a weakly idempotent lattice, $(L;*)$ is a weakly idempotent semilattice and the operation $*$ is interlaced with the operations $\wedge,\vee$ and the following identity $a\wedge a = a*a$ is valid.

Lemma 3.6. *For any $x,y\in L$, the following inequalities are valid:*

$$x\wedge y \leq_\wedge x*y \leq_\wedge x\vee y,$$
$$x*y \leq_* x\wedge y, x*y \leq_* x\vee y.$$

Lemma 3.7. *For $a,x,b\in L$, we have:*

$$a \leq_\wedge x \leq_\wedge b \text{ and } a \leq_* b \to a \leq_* x \leq_* b;$$
$$a \leq_\wedge x \leq_\wedge b \text{ and } b \leq_* a \to b \leq_* x \leq_* a.$$

Proof. Let us prove the first statement. From $a \leq_* b$, we obtain that $a\wedge x \leq_* b\wedge x$ and $a\vee x \leq_* b\vee x$. Using the condition: $a \leq_\wedge x \leq_\wedge b$, we get: $a\wedge a \leq_* x\wedge x$ and $x\wedge x \leq_* b\wedge b$, and since $a \leq_* a\wedge a, b\wedge b \leq_* b$, we have: $a \leq_* x \leq_* b$. ∎

Lemma 3.8. *For $u,x,y\in L$ we have:*

$$u \leq_\wedge x \text{ and } u \leq_\wedge y \text{ and } u \leq_* x \text{ and } u \leq_* y \to x\wedge y = x*y;$$
$$x \leq_\wedge u \text{ and } y \leq_\wedge u \text{ and } u \leq_* x \text{ and } u \leq_* y \to x\vee y = x*y.$$

Proof. From $u \leq_\wedge x$ and $u \leq_\wedge y$, by Lemma 3.6, it follows that $u\wedge u \leq_\wedge x\wedge y \leq_\wedge x*y$. From the conditions $u \leq_* x$ and $u \leq_* y$ and $x*x = x\wedge x$, we have: $u\wedge u = u*u \leq_* x*y$, then by Lemma 3.7 we get $u\wedge u \leq_* x\wedge y \leq_* x*y$. On the other hand, $x*y \leq_* x\wedge y$ (see Lemma 3.6), then $x*y \leq_* x\wedge y \leq_* x*y$, hence, by Lemma 3.1: $x\wedge y = (x\wedge y)*(x\wedge y) = (x*y)*(x*y) = x*y$. ∎

Lemma 3.9. *Let $(L;*,\Delta)$ be a weakly idempotent lattice. Then* $a \leq_\wedge b \to a \leq_\wedge a\Delta b \leq_\wedge b$, *where* $a, b \in L$.

Proof. By Lemma 3.6 $a \wedge a = a * a = a * (a\Delta b) \leq_* a \vee (a\Delta b)$, then from $a \leq_\wedge b$, we get $a * a = a \wedge a = b \wedge a = b \wedge (a \wedge a) \leq_* b \wedge [a \vee (a\Delta b)]$, hence,

$$a * a \leq_* b \wedge [a \vee (a\Delta b)]. \tag{12}$$

From $b \leq_* a\Delta b$ and $a \leq_\wedge b$, we obtain $b \wedge b = a \vee b \leq_* a \vee (a\Delta b)$, hence, $b * b = b \wedge b = (b \wedge b) \wedge b \leq_* b \wedge [a \vee (a\Delta b)]$. Hence,

$$b * b \leq_* b \wedge [a \vee (a\Delta b)]. \tag{13}$$

From 12 and 13 it follows that:

$$(a * a)\Delta(b * b) \leq_* (b \wedge [a \vee (a\Delta b)])\Delta(b \wedge [a \vee (a\Delta b)]) =$$
$$(b \wedge [a \vee (a\Delta b)]) \wedge (b \wedge [a \vee (a\Delta b)]) = b \wedge [a \vee (a\Delta b)].$$

Then $a\Delta b \leq_* b \wedge [a \vee (a\Delta b)]$.

Further, from $a \leq_* a\Delta b$, it follows that:

$$a \vee (a\Delta b) \leq_* (a\Delta b) \vee (a\Delta b) = a\Delta b, a \wedge (a\Delta b) \leq_* (a\Delta b) \wedge (a\Delta b) = a\Delta b.$$

From $b \leq_* a\Delta b$, we deduce that $b \vee (a\Delta b) \leq_* a\Delta b, b \wedge (a\Delta b) \leq_* a\Delta b$. So, $a\Delta b \leq_* b \wedge [a \vee (a\Delta b)] \leq_* b \wedge (a\Delta b) \leq_* a\Delta b$. This implies $[b \wedge (a\Delta b)] * [b \wedge (a\Delta b)] = (a\Delta b) * (a\Delta b)$. Hence,

$$b \wedge (a\Delta b) = [b \wedge (a\Delta b)] \wedge [b \wedge (a\Delta b)] = [b \wedge (a\Delta b)] * [b \wedge (a\Delta b)]$$
$$= (a\Delta b) * (a\Delta b) = (a\Delta b) \wedge (a\Delta b),$$

which shows that $a\Delta b \leq_\wedge b$. The second part of the inequality is proved in the same way. ■

Let us define the relations θ_1 and θ_2 on the algebra $(L;\wedge,\vee,*)$ in the following manner:

$$a\theta_1 b \iff a * b = a \vee b, \tag{14}$$
$$a\theta_2 b \iff a * b = a \wedge b. \tag{15}$$

Lemma 3.10. *θ_1, θ_2 are congruences on $(L;\wedge,\vee)$.*

Proof. Reflexivity and symmetricity are obvious, and let us show the transitivity. Let $a\theta_1 b$ and $b\theta_1 c$, then $a \vee b = a * b$ and $b \vee c = b * c$; hence by Lemma 3.6 $a \vee b \leq_* b$ and $b \vee c \leq_* b$. So, $a \vee b \vee c \leq_* b \vee c$ and $a \vee b \vee c \leq_* a \vee b$. On the other hand, by Lemma 3.2 we have: $b \vee c \leq_\wedge a \vee b \vee c, a \vee b \leq_\wedge a \vee b \vee c$. Using Lemma 3.8 we get: $a \vee b \vee c = (a \vee b) \vee (b \vee c) = (a \vee b) * (b \vee c) = (a*b)*(b*c) = ((a*b)*b)*c = (a*b)*c$. It follows that $a \leq_\wedge a*b*c, b \leq_\wedge a*b*c$. From the inequalities $a * b * c \leq_* a, a * b * c \leq_* c$, applying Lemma 3.8, we get: $a \vee c = a * c$; hence $a\theta_1 c$.

Congruence: Let $a\theta_1 b$, then $a * b = a \vee b$, hence $a \vee b \leq_* a$ and $a \vee b \leq_* b$; then $a \vee b \vee c \leq_* a \vee c, a \vee b \vee c \leq_* b \vee c$ for any $c \in L$. On the other hand, by Lemma 3.2 we have: $a \vee c \leq_\wedge a \vee b \vee c$ and $b \vee c \leq_\wedge a \vee b \vee c$, then using Lemma 3.8 we get: $(a \vee c) \vee (b \vee c) = (a \vee c) * (b \vee c)$, so $a \vee c\theta_1 b \vee c$. ∎

Lemma 3.11. *θ_1, θ_2 are congruences on $(L; *)$.*

Proof. Let $a\theta_1 b$, i.e. $a * b = a \vee b$, then $a \leq_\wedge a * b$ and $b \leq_\wedge a * b$, hence $a*c \leq_\wedge a*b*c$ and $b*c \leq_\wedge a*b*c$. Further, by Lemma 3.2: $a*b*c \leq_* a*c$ and $a*b*c \leq_* b*c$. Then, by Lemma 3.8, we have: $(a*c)*(b*c) = (a*c) \vee (b*c)$, hence $a * c\theta_1 b * c$. Similarly, we prove that $a * c\theta_2 b * c$, if $a\theta_2 b$. ∎

Remark 3.12. $a \in [a \wedge a]_{\theta_1} = [a \vee a]_{\theta_1}$, $a \in [a \wedge a]_{\theta_2} = [a \vee a]_{\theta_2}$, for all $a \in L$.

Lemma 3.13. *For all $a, b \in L$, we have: $a(\theta_1 \cap \theta_2)b \iff a \wedge a = b \wedge b$.*

Proof. $a(\theta_1 \cap \theta_2)b \iff a\theta_1 b$ and $a\theta_2 b \iff a * b = a \vee b$ and $a * b = a \wedge b \iff a \vee b = a \wedge b \iff (a \vee b) \wedge a = (a \wedge b) \wedge a$ and $(a \vee b) \wedge b = (a \wedge b) \wedge b \iff a \wedge a = a \wedge b$ and $b \wedge b = a \wedge b \iff a \wedge a = b \wedge b$. ∎

Lemma 3.14. *For all $a, b \in L$, we have: $(a \wedge b)\theta_1(a * b)\theta_2(a \vee b)$.*

Proof. By Lemma 3.6, we have: $(a \wedge b) * (a * b) = a * b = (a \wedge b) \vee (a * b)$ hence, $a \wedge b\theta_1 a * b$. Similarly, we get: $(a \vee b) * (a * b) = a * b = (a \vee b) \wedge (a * b)$ hence, $a \vee b\theta_2 a * b$. Hence, we have that $a * b\theta_2 a \vee b$. ∎

Lemma 3.15. *$a \leq_\wedge b \rightarrow a\theta_1\theta_2 b$, where $a, b \in L$.*

Proof. From Lemma 3.14 it follows that $a \wedge b\theta_1 a * b$ and $a * b\theta_2 a \vee b$, then $a \wedge b\theta_1\theta_2 a \vee b$, hence $a \wedge a\theta_1\theta_2 b \wedge b$ and by Remark 3.12 we get: $a\theta_1\theta_2 b$. ∎

Lemma 3.16. *Let $(L; *, \Delta)$ be a weakly idempotent lattice, then $a \leq_\wedge b \rightarrow a\theta_2\theta_1 b$, where $a, b \in L$.*

Proof. By Lemma 3.9 from $a \leq_\wedge b$, then $a \leq_\wedge a\Delta b \leq_\wedge b$. Then $a * (a\Delta b) = a * a = a \wedge a = a \wedge (a\Delta b)$ and $b * (a\Delta b) = b * b = b \wedge b = b \vee (a\Delta b)$, hence $a\theta_2 a\Delta b$ and $b\theta_1 a\Delta b$. So, $a\theta_2\theta_1 b$. ∎

Lemma 3.17. *The algebras $(L/\theta_1; \wedge, \vee)$ and $(L/\theta_2; \wedge, \vee)$ are lattices.*

Proof. Every element of algebra $(L/\theta_1; \wedge, \vee)$ is idempotent. Indeed, $[a]_{\theta_1} \wedge [a]_{\theta_1} = [a \wedge a]_{\theta_1} = [a]_{\theta_1}$ (by Remark 3.12). ∎

Lemma 3.18. *The congruencies Θ, Φ of a weakly idempotent lattice $L = (L; \wedge, \vee)$ with the properties $a\Theta a \wedge a$, $a\Phi a \wedge a$, for all $a \in L$ commute iff for any $a, b \in L$, $a \leq_\wedge b$, the following condition is satisfied: $a\Theta\Phi b \iff a\Phi\Theta b$.*

Proof. Let us assume that congruences Θ and Φ commute for all $a \leq_\wedge \in L$.

Let $x, y, z \in L$ and $x\Theta z\Phi y$, then $x \wedge y \wedge z\Phi x \wedge z\Theta x \wedge x$, so $x \wedge y \wedge z\Phi\Theta x \wedge x$. From Lemma's condition (since $x \wedge y \wedge z \leq_\wedge x \leq_\wedge x \wedge x$) we get that

$x \wedge y \wedge z \Theta \Phi x \wedge x$, hence, there exists $t \in L$ such that $x \wedge y \wedge z \Theta t \Phi x \wedge x$, thus $(x \wedge y \wedge z) \vee y \Theta y \vee t$ and $y \vee y \Theta y \vee t$.

Furthermore, from $x \Theta z \Phi y$, we get $x \wedge y \wedge z \Theta y \wedge z \Phi y \wedge y$, hence by Lemma's condition, we have also $x \wedge y \wedge z \Phi \Theta y \wedge y$. Hence, $x \wedge y \wedge z \Theta \Phi y \vee t$, $y \wedge z \Phi \Theta y \vee t$, $t \Theta y \wedge z$. Then $t \Theta y \wedge z \Theta x \wedge y \wedge z \Theta \Phi y \vee t$, so $y \vee t \Theta \Phi t$, yield $x \wedge x \Phi \Theta y \wedge y$. Thus, $x \Phi \Theta y$. ∎

Consider the subset $[a, b] = \{x \in L | a \leq_\wedge x \leq_\wedge b\}$ of $(L; \wedge, \vee)$. It is obvious that $[a, b]$ is closed under the operations of the weakly idempotent lattice $(L; \wedge, \vee)$.

Remark 3.19. Let $L = (L; \wedge, \vee)$ be a weakly idempotent lattice and let θ be a congruence of L. If $b, c \in [a, d]$ and $a \theta d$, then $b \theta c$.

Lemma 3.20. *The reflexive binary relation θ for the weakly idempotent lattice $(L; \wedge, \vee)$, which satisfies the condition $a \theta (a \wedge a)$, is a congruence of $(L; \wedge, \vee)$ iff the following conditions are valid:*

1. $x \theta y \iff x \wedge y \theta x \vee y$;
2. $x \leq_\wedge y \leq_\wedge z, x \theta y, y \theta z \to x \theta z$;
3. $x \leq_\wedge y, x \theta y \to x \wedge t \theta y \wedge t, x \vee t \theta y \vee t$.

Proof. The necessity is clear, let us show its sufficiency.

Let us prove that θ is transitive. Let $x \theta y$ and $y \theta z$, then we have: $x \wedge y \theta x \vee y$. From condition 3, we obtain: $y \vee z = (y \vee z) \vee (y \wedge x) \theta (y \vee z) \vee (y \vee x) = x \vee y \vee z$. Similarly, we show that $x \wedge y \wedge z \theta y \wedge z$. Hence, we obtain: $x \wedge y \wedge z \theta y \wedge z \theta y \vee z \theta x \vee y \vee z$, and by Lemma 3.2, we have: $x \wedge y \wedge z \leq_\wedge y \wedge z \leq_\wedge y \vee z \leq_\wedge x \vee y \vee z$. Applying twice condition 2, we get: $x \wedge y \wedge z \theta x \vee y \vee z$. Now let us make the following designations: $a = x \wedge y \wedge z$, $b = x$, $c = z$, $d = x \vee y \vee z$, then $x \theta z$ (by Remark 3.19).

Let $x \theta y$, we will show that $x \vee t \theta y \vee t$. Indeed, from $x \theta y$, we get that $x \wedge y \theta x \vee y$; hence, $(x \wedge y) \vee t \theta x \vee y \vee t$. Take $a = (x \wedge y) \vee t$, $d = x \vee y \vee t$, $b = x \vee t$, $c = y \vee t$, then we have: $x \vee t \theta y \vee t$ (by Remark 3.19). Let us show that if $x_0 \theta y_0$ and $x_1 \theta y_1$, then $x_0 \vee x_1 \theta y_0 \vee y_1$. Since $x_0 \theta y_0$ and $x_1 \theta y_1$, then $x_0 \vee x_1 \theta x_0 \vee y_1 \theta y_0 \vee y_1$, hence $x_0 \vee x_1 \theta y_0 \vee y_1$. Similarly, we show that $x_0 \wedge x_1 \theta y_0 \vee y_1$. ∎

Lemma 3.21. *Let Θ and Φ be congruencies for a weakly idempotent lattice $(L; \wedge, \vee)$ such that $a \Theta (a \wedge a)$ and $a \Phi (a \wedge a)$ for any $a \in L$. Then the union of this congruencies can be described in the following manner:*

$$x(\Theta \cup \Phi)y \iff \text{there exists a sequence } z_0 = x \wedge y, z_1, \dots, z_{(n-1)} = x \vee y \text{ such that } z_0 = x \wedge y \leq_\wedge z_1 \leq_\wedge \dots \leq_\wedge z_{(n-1)} = x \vee y,$$

where $z_i \Theta z_{(i+1)}$ or $z_i \Phi z_{(i+1)}$, for all $i = 0, \dots, n-1$.

4. Interlaced weakly idempotent (pre-)bilattices

Theorem 4.1. *Let $(L; \wedge, \vee)$ be a weakly idempotent lattice, $(L; *)$ be a weakly idempotent semilattice, having an operation interlaced with the operations $\wedge, \vee$ and the following identity be valid: $a \wedge a = a * a$. Then there exists a pair of congruencies (θ_1, θ_2) for the weakly idempotent lattice $(L; \wedge, \vee)$, which satisfies the following conditions:*

1. $a(\theta_1 \cap \theta_2)b \iff a \wedge a = b \wedge b$;
2. $a \leq_\wedge b \to a\theta_1\theta_2 b$;
3. $X(Y(X(x,y),z), Y(y,z))\theta_i Y(X(x,y),z)$,

where $X, Y \in \{\wedge, \vee\}, x, y, z \in L$, for all $i = 1, 2$.

Conversely, let $(L; \wedge, \vee)$ be a weakly idempotent lattice and let θ_1 and θ_2 be two congruences of $(L; \wedge, \vee)$ satisfying conditions 1–3. Then there is a weakly idempotent semilattice $(L; *)$ with the operation $*$ interlaced with the operations $\wedge$ and $\vee$ and such that $a \wedge a = a * a$.

Proof. Define the relations θ_1 and θ_2 on $(L; \wedge, \vee, *)$, as above, i.e.:

$$a\theta_1 b \iff a * b = a \vee b, \quad a\theta_2 b \iff a * b = a \wedge b.$$

From Lemmas 3.10, 3.13 and 3.15, it follows that θ_1 and θ_2 are congruences for $(L; \wedge, \vee)$ satisfying conditions 1 and 2. Condition 3 is valid, since any weakly idempotent lattice is interlaced.

Conversely, let θ_1 and θ_2 be congruences satisfying the conditions of this theorem. Define the operation $*$ by the following rule: $a * b = d \wedge d \iff d\theta_1 a \wedge b$ and $d\theta_2 a \vee b$. Existence of such d follows from condition 2 and correctness of the operation $*$ holds from condition 1. Indeed, let there exist d_1 and d_2 such that $a*b = d_1 \wedge d_1$ and $a*b = d_2 \wedge d_2$ i.e., $d_1\theta_1 a \wedge b, d_1\theta_2 a \vee b$, and $d_2\theta_1 a \wedge b, d_2\theta_2 a \vee b$; hence, $d_1\theta_1 d_2$ and $d_1\theta_2 d_2$, thus by condition 1, we get $d_1 \wedge d_1 = d_2 \wedge d_2$.

Obviously, the operation $*$ is commutative, and the following identities are true:
$a * (b * b) = a * b$, $a * a = a \wedge a$. Let $d_1, d_2 \in L$ be such that $d_1 \wedge d_1 = a * (b * c), d_2 \wedge d_2 = (a * b) * c$. Then $d_1\theta_1 a \wedge b \wedge c\theta_1 d_2$ and $d_1\theta_2 a \vee b \vee c\theta_2 d_2$. Consequently, $d_1(\theta_1 \cap \theta_2)d_2$, hence by condition 1, $a * (b * c) = d_1 \wedge d_1 = d_2 \wedge d_2 = (a * b) * c$. In the same way, using Corollary 3.4, we show that the operation $*$ is interlaced with the operations $\wedge, \vee$. ■

Theorem 4.2. *Let $(L; \wedge, \vee)$ be a weakly idempotent lattice, $(L; *)$ be a weakly idempotent semilattice, having the operation interlaced with the operations $\wedge$ and $\vee$ and such that $a \wedge a = a * a$, for any $a \in L$. Let (θ_1, θ_2) be a pair of congruences of $(L; \wedge, \vee)$ satisfying conditions 1–3. Then the mapping $\phi : (L; \wedge, \vee) \mapsto L/\theta_1 \times L/\theta_2$, where $\phi(x) = ([x]_{\theta_1}, [x]_{\theta_2})$, is a homomorphism from $(L; \wedge, \vee)$ onto a subdirect product of two lattices L/θ_1 and*

L/θ_2, satisfying the condition $\phi(x) = \phi(y) \iff x \wedge x = y \wedge y$. Moreover, if $(a, b), (a_1, b_1) \in \phi(L)$ and $(a, b) \leq_\wedge (a_1, b_1)$, then $(a, b_1) \in \phi(L)$.

Conversely, let ϕ be an epimorphism from a weakly idempotent lattice $(L; \wedge, \vee)$ to a subdirect product of two lattices satisfying the condition $\phi(x) = \phi(y) \iff x \wedge x = y \wedge y$ and let the subdirect product satisfy the following condition: if (a, b) and (a_1, b_1) are elements of this subdirect product and $(a, b) \leq_\wedge (a_1, b_1)$, then (a, b_1) belongs to this subdirect product. Then there exists a weakly idempotent semilattice $(L; *)$ with the operation $*$ interlaced with the operations $\wedge$, $\vee$ and such that $a * a = a \wedge a$ for any $a \in L$. Moreover, if $\phi(x) = (a, b), \phi(y) = (a_1, b_1)$, then $\phi(x * y) = (a \wedge a_1, b \vee b_1)$.

Proof. Let $(L; *)$ be a weakly idempotent semilattice satisfying Theorem's conditions, then by Theorem 4.1, there are congruencies θ_1 and θ_2 of $(L; \wedge, \vee)$, defined by the rules (14) and (15), satisfying conditions 1–3. From Lemma 3.17, it follows that the quotient algebras L/θ_1 and L/θ_2 are lattices. It is obvious, that the following set $L' = \{([x]_{\theta_1}, [x]_{\theta_2}) | x \in L\}$ is closed under the operations of quotient algebras L/θ_1 and L/θ_2 and L' is a subdirect product of L/θ_1 and L/θ_2. Indeed, for any $[x]_{\theta_i} \in L/\theta_i$ there is $([x]_{\theta_1}, [x]_{\theta_2}) \in L/\theta_1 \times L/\theta_2$ such that $e_i([x]_{\theta_1}, [x]_{\theta_2})) = [x]_{\theta_i}$, hence $L/\theta_i \subseteq e_i(L')$. The converse inclusion is obviously, so $L/\theta_i = e_i(L')$. The mapping ϕ from $(L; \wedge, \vee)$ to L', defined in the following way: $\phi(x) = ([x]_{\theta_1}, [x]_{\theta_2})$ is an epimorphism. Indeed, as the surjection is obvious, let us show that ϕ is a homomorphism:
$\phi(x \wedge y) = ([x \wedge y]_{\theta_1}, [x \wedge y]_{\theta_2}) = ([x]_{\theta_1} \wedge [y]_{\theta_1}, [x]_{\theta_2} \wedge [y]_{\theta_2}) = ([x]_{\theta_1}, [x]_{\theta_2}) \wedge ([y]_{\theta_1}, [y]_{\theta_2}) = \phi(x) \wedge \phi(y)$.

Similarly, we get that $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$.

Now let us prove that $\phi(x) = \phi(y) \iff x \wedge x = y \wedge y$.
$\varphi(x) = \varphi(y) \iff ([x]_{\theta_1}, [x]_{\theta_2}) = ([y]_{\theta_1}, [y]_{\theta_2}) \iff [x]_{\theta_1} = [y]_{\theta_1}$ and $[x]_{\theta_2} = [y]_{\theta_2} \iff x\theta_1 y$ and $x\theta_2 y \iff x(\theta_1 \cap \theta_2)y \iff x \wedge x = y \wedge y$.

Suppose that, (a, b), $(a_1, b_1) \in L'$ and $(a, b) \leq_\wedge (a_1, b_1)$. Then there exist $u, v \in L$ such that $\phi(u) = (a, b)$ and $\phi(v) = (a_1, b_1)$ and $\phi(u) \leq \phi(v)$. Then by condition 2 of Theorem 4.1, we have that there is $t \in L$ with the property $u\theta_1 t\theta_2 v$; hence, $\phi(t) = (a, b_1)$.

Conversely, let ϕ be an epimorphism between L and a subdirect product of the lattices A and B satisfying the theorem's conditions.

Let us define relations θ_1 and θ_2 on $(L; \wedge, \vee)$ as follows:

$$u\theta_1 v \iff \pi_1(\phi(u)) = \pi_1(\phi(v)),$$

$$u\theta_1 v \iff \pi_2(\phi(u)) = \pi_2(\phi(v)),$$

where π_1, π_2 are projections of a subdirect product of lattices A and B. It is obvious that θ_1 and θ_2 are congruencies of $(L; \wedge, \vee)$. Let us show that θ_1 and θ_2 satisfy conditions 1–3 of Theorem 1.

Suppose $a, b \in L$, then since ϕ is an epimorphism, there are $(a_1, a_2), (b_1, b_2)$ elements of the subdirect product of the lattices A and B such that $\phi(a) = (a_1, a_2)$ and $\phi(b) = (b_1, b_2)$. Thus:
$a(\theta_1 \cap \theta_2)b \iff a\theta_1 b$ and $a\theta_2 b \iff \pi_1(\phi(a)) = \pi_1(\phi(b))$ and $\pi_2(\phi(a)) = \pi_2(\phi(b)) \iff \pi_1(a_1, a_2) = \pi_1(b_1, b_2)$ and $\pi_2(a_1, a_2) = \pi_2(b_1, b_2) \iff a_1 = b_1$ and $a_2 = b_2 \iff (a_1, a_2) = (b_1, b_2) \iff \phi(a) = \phi(b) \iff a \wedge a = b \wedge b$.

L/θ_1 is isomorphic to A and L/θ_2 is isomorphic to B. Indeed, define a map $f : L/\theta_1 \mapsto A$ in the following way: $f([x]_{\theta_1}) = \pi_1(\phi(x))$. f is an isomorphism. Surjection and injection are obvious, let us show that f is a homomorphism:
$f([x]_{\theta_1} \wedge [y]_{\theta_1}) = f([x \wedge y]_{\theta_1}) = \pi_1(\phi(x \wedge y)) = \pi_1(\phi(x) \wedge \phi(y)) = \pi(\phi(x)) \wedge \pi(\phi(y)) = f([x]_{\theta_1}) \wedge f([y]_{\theta_1})$.
Similarly, we have that $f([x]_{\theta_1} \vee [y]_{\theta_1}) = f([x]_{\theta_1}) \vee f([y]_{\theta_1})$. In the same way, we show that L/θ_2 is isomorphic to B. Hence, $L/\theta_i($ where $i = 1, 2)$ satisfy condition 1 of Theorem 4.1.

Consider $a, b \in L$ such that $a \leq_\wedge b$, then $\phi(a) \leq_\wedge \phi(b)$; hence, by the theorem's assumption, we get that $([a]_{\theta_1}, [b]_{\theta_2})$ belong to the subdirect product, hence there exists $t \in L$ such that $\phi(t) = ([a]_{\theta_1}, [b]_{\theta_2})$, so $a\theta_1 t$, $t\theta_2 b$, yield $a\theta_1\theta_2 b$.

Thus, the pair of congruences (θ_1, θ_2) of L satisfies conditions 1–3 of Theorem 4.1, and it follows that there exists a weakly idempotent semilattice operation $*$ which is interlaced with the operations $\wedge$ and $\vee$ and satisfies the identity $a \wedge a = a * a$.

The last statement of theorem is proved with help of the relation $a \wedge b\theta_1 a * b\theta_2 a \vee b$. Take $\phi(x) = (a, b)$ and $\phi(y) = (a_1, b_1)$ hence, $a = [x]_{\theta_1}, b = [x]_{\theta_2}, a_1 = [y]_{\theta_1}, b_1 = [y]_{\theta_2}$, hence there are $t \in a, t_1 \in a_1$ such that $x\theta_1 t$, $y\theta_1 t_1$ and there are $s \in b, s_1 \in b_1$ such that $x\theta_2 s$, $y\theta_2 s_1$. Then $t \wedge t_1\theta_1 x \wedge y\theta_1 x * y$ and $s \vee s_1\theta_2 x \vee y\theta_2 x * y$ yield $[x * y]_{\theta_1} = a \wedge a_1$ and $[x * y]_{\theta_2} = b \vee b_1$. So, we get $\phi(x * y) = (a \wedge a_1, b \vee b_1)$. ∎

Theorem 4.3. *Let $(L; \wedge, \vee)$ be a weakly idempotent lattice, and θ_1, θ_2 be congruences of $(L; \wedge, \vee)$ satisfying conditions 1–3.*

a) *For the weakly idempotent semilattice $(L; *)$ from the Theorem 4.1, there exists a binary operation Δ on L(moreover, unique) such that $(L; *, \Delta)$ is a weakly idempotent lattice iff the corresponding congruences θ_1 and θ_2 commute.*

b) *For the weakly idempotent semilattice $(L; *)$ of Theorem 4.2, there exists a binary operation Δ on L such that $(L; *, \Delta)$ is a weakly idempotent lattice iff the corresponding subdirect product is a direct product.*

Proof. a) If $(L; *, \Delta)$ is a weakly idempotent lattice, then $\theta_1\theta_2 = \theta_2\theta_1$, by Lemmas 3.15, 3.16, 3.18. Conversely, let $\theta_1\theta_2 = \theta_2\theta_1$. By Theorem 4.1, there exists a weakly idempotent semilattice operation Δ on L, corresponding to the pair (θ_2, θ_1), which is interlaced with the operations $\wedge$ and $\vee$ and satisfies the identity $a\Delta a = a \wedge a$ and $a \wedge b\theta_2 a\Delta b\theta_1 a \vee b$. Hence, $(a\Delta b) * a\theta_2(a \wedge b) * a\theta_2(a \vee b) \wedge a = a \vee a$ and $(a\Delta b) * a\theta_1(a \vee b) * a\theta_1(a \vee b) \wedge a = a \wedge a$, so $a * a = (a\Delta b) * a$. Similarly, we get $a\Delta a = (a * b)\Delta a$. Hence, $(L; *, \Delta)$ is a weakly idempotent lattice.

b) Let $a, b \in L$; then by Lemmas 3.21, 3.14, we have: $a \wedge b\theta_1 a * b\theta_2 a \vee b$, and $a \wedge b \leq_\wedge a * b \leq_\wedge a \vee b$, then $\theta_1 \cup \theta_2 = L \times L$ (see Lemma 3.21). Hence, the subdirect product is a direct product iff $\theta_1\theta_2 = \theta_2\theta_1$ [16] (chapter 3). By a), this is equivalent to the following condition: $(L; *, \Delta)$ forms a weakly idempotent lattice. ∎

Theorem 4.4. *Let $(L; \wedge, \vee)$ and $(L; *, \Delta)$ be weakly idempotent lattices. If the operation $*$ is interlaced with the operations $\wedge$ and $\vee$ and satisfies the identity $a \wedge a = a * a$; then the operation Δ is interlaced with the operations $\wedge$ and $\vee$, too.*

Proof. The proof follows from Theorems 4.1 and 4.3. ∎

Lemma 4.5. *Let $(L; \wedge, \vee, *, \Delta, ')$ be a weakly idempotent bilattice. Then:*

$$a\theta_1 b \iff a'\theta_2 b',\ a\theta_2 b \iff a'\theta_1 b',$$

where θ_1 and θ_2 are relations defined by (14) and (15).

Lemma 4.6. *Let $(L; \wedge, \vee, *, \Delta, ')$ be a weakly idempotent bilattice and let the operation $\wedge$ be interlaced with the operations $*$ and Δ. Then lattices $(L/\theta_1; *, \Delta)$ and $(L/\theta_2; *, \Delta)$, where θ_1 and θ_2 are relations defined by (14) and (15), are isomorphic.*

Proof. It is easy to show that the mapping $h : (L/\theta_1; *, \Delta) \mapsto (L/\theta_2; *, \Delta)$; $h : [x]_{\theta_1} \mapsto [x']_{\theta_2}$, is a lattice isomorphism. ∎

Let $L = (L; +, \cdot)$ be a weakly idempotent lattice, then on the superproduct $L \bowtie L$, the operation of negation $'$ is defined in the following way:

$$(a, b)' = (b, a).$$

The obtained algebra $(L \times L; (+, \cdot), (\cdot, +), (+, +).(\cdot, \cdot), ')$ is called a superproduct with negation and is denoted by $L \widetilde{\bowtie} L$.

Theorem 4.7. *Let $L = (L; \wedge, \vee .*, \Delta, ')$ be a weakly idempotent bilattice. The operation $\wedge$ is interlaced with the operations $*, \Delta$ iff there exist a lattice A and an epimorphism φ between the weakly idempotent bilattice L and the superproduct with negation $A \widetilde{\bowtie} A$. Moreover, this epimorphism satisfies the following condition: $\varphi(x) = \varphi(y) \iff x * x = y * y$. Hence, this epimorphism is an isomorphism on the bilattice of idempotent elements of L.*

Proof. By Theorems 4.2 and 4.3 b) there exists an epimorphism $\phi : L \mapsto A \times B$ between the weakly idempotent lattice $(L; *, \Delta)$ and the subdirect product of the two lattices $A = (L/\theta_1; *, \Delta)$ and $B = (L/\theta_2; *, \Delta)$, which satisfies the condition $\phi(x) = \phi(y) \iff x * x = y * y$. The map ϕ is an epimorphism between the weakly idempotent pre-bilattice $(L; \wedge, \vee, *, \Delta)$ and the superproduct $A \bowtie B$, where

$$\phi(x \wedge y) = (a * a_1, b\Delta b_1), \phi(x \vee y) = (a\Delta a_1, b * b_1)$$

for $\phi(x) = (a, b)$, $\phi(y) = (a_1, b_1)$.

By Lemma 4.6, we know that there exists an isomorphism h between $(L/\theta_1; *, \Delta)$ and $(L/\theta_2; *, \Delta)$. Hence, $\psi : L \mapsto L/\theta_1 \bowtie L/\theta_2; \psi(x) = ([x]_{\theta_1}, [x']_{\theta_1})$ is an epimorphism between the weakly idempotent pre-bilattice $(L; \wedge, \vee, *, \Delta)$ and the algebra $L/\theta_1 \bowtie L/\theta_1$. Let us show that this map is an epimorphism between the bilattice L and the superproduct with the negation $L/\theta_1 \widetilde{\bowtie} L/\theta_1$. Thus, we need to show that $(\psi(x))' = \psi(x')$. Indeed,

$$(\psi(x))' = ([x]_{\theta_1}, [x']_{\theta_1})' = ([x']_{\theta_1}, [x]_{\theta_1}) = ([x']_{\theta_1}, [(x')']_{\theta_1}) = \psi(x'). \blacksquare$$

Corollary 4.8. *Let $L = (L; \wedge, \vee, *, \Delta)$ be a weakly idempotent pre-bilattice. The operation $\wedge$ is interlaced with the operations $*, \Delta$ iff there exist lattices A, B and an epimorphism φ between the weakly idempotent pre-bilattice L and the superproduct A and B. Moreover, this epimorphism satisfies the following condition: $\varphi(x) = \varphi(y) \iff x * x = y * y$. Hence, this epimorphism is an isomorphism on the pre-bilattice of idempotent elements of L.*

Note that if $L = (L; \wedge, \vee, *, \Delta)$ is a weakly idempotent pre-bilattice, then: $(L/\theta_i; *, \Delta) = (L/\theta_i; \wedge, \vee)$, since $(a \wedge b)\theta_1(a * b)$ and $(a \vee b)\theta_2(a\Delta b)$.

Corollary 4.9. *The weakly idempotent bilattice $L = (L; \wedge, \vee .*, \Delta, ')$ is distributive iff there exist a distributive lattice A and an epimorphism φ between the weakly idempotent bilattice L and the superproduct with the negation $A \widetilde{\bowtie} A$. Moreover, this epimorphism satisfies the following condition: $\varphi(x) = \varphi(y) \iff x * x = y * y$. Hence, this epimorphism is an isomorphism on the bilattice of the idempotent elements of L.*

A similar result is valid for interlaced weakly idempotent modular bilattices too.

5. Non-idempotent Plonka functions and weakly Plonka sums

Definition 5.1. An algebra $\mathfrak{U} = (U, \Sigma)$ is called weakly Plonka sum of its subalgebras $(U_i; \Sigma)$, where $i \in I$, if the following conditions are valid (cf. [35], [37], [39]):

i) $U_i \cap U_j = \emptyset$, for all $i, j \in I, i \neq j$;

ii) $U = \bigcup_{i \in I} U_i$;
iii) On the set of indexes I, there exists a relation "$\leq$" such that $(I; \leq)$ is an upper semilattice with the following properties;
iv) If $i \leq j$, then there exists a homomorphism $\varphi_{i,j} : (U_i, \Sigma) \mapsto (U_j, \Sigma)$, where $\varphi_{i,j} \cdot \varphi_{j,k} = \varphi_{i,k}$, for $i \leq j \leq k$ and $\varphi_{i,i}(x) = F(x, \dots, x)$ for any $F \in \Sigma$ and $x \in U_i$;
v) For all $A \in \Sigma$ and for all $x_1, \dots, x_n \in Q$, the following equality is valid:

$$A(x_1, \dots, x_n) = A(\varphi_{i_1,i_0}(x_1), \dots, \varphi_{i_n,i_0}(x_n)),$$

where the arity $|A| = n$, $x_1 \in U_{i_1}, \dots, x_n \in U_{i_n}, i_1, \dots, i_n \in I, i_0 = sup\{i_1, \dots, i_n\}$.

Let $T_{\mathfrak{U}} = \{|A| | A \in \Sigma\}$ be an arithmetic type of an algebra $\mathfrak{U}$.

From the conditions iv) and v) of Definition 5.1, it respectively follows that if the algebra $\mathfrak{U} = (U, \Sigma)$ is a weakly Plonka sum of its subalgebras, then $\mathfrak{U}$ satisfies the following hyperidentities:

$$X(\underbrace{x, \dots, x}_{n}) = Y(\underbrace{x, \dots, x}_{m}), \tag{16}$$

$$X(X(x, \dots, x), \dots, X(x, \dots, x)) = X(x, \dots, x),$$

where $m, n \in T_{\mathfrak{U}}$.

Definition 5.2. Let $\mathfrak{U} = (U, \Sigma)$ be an algebra. The binary operation $f : U \times U \mapsto U$ is called non-idempotent Plonka function of $\mathfrak{U}$ if it satisfies the following identities (cf. [35], [37], [39]):

1. $f(f(x, y), z) = f(x, f(y, z))$;
2. $f(x, x) = F_t(x, \dots, x)$, for any operation $F_t \in \Sigma$;
3. $f(x, f(y, z)) = f(x, f(z, y))$;
4. $f(F_t(x_1, \dots, x_{n(t)}), y) = F_t(f(x_1, y), \dots f(x_{n(t)}, y))$, for any operation $F_t \in \Sigma$;
5. $f(y, F_t(x_1, \dots, x_{n(t)})) = f(y, F_t(f(y, x_1), \dots f(y, x_{n(t)})))$, for any operation $F_t \in \Sigma$;
6. $f(F_t(x_1, \dots, x_{n(t)}), x_i) = F_t(x_1, \dots, x_{n(t)})$, for any $1 \leq i \leq n(t)$ and for any operation $F_t \in \Sigma$;
7. $f(F_t(x_1, \dots, x_{n(t)}), F_t(x_1, \dots, x_{n(t)})) = F_t(x_1, \dots, x_{n(t)})$, for any operation $F_t \in \Sigma$;
8. $f(x, f(x, y)) = f(x, y)$.

From conditions 2 and 7 of Definition 5.2, it follows that the algebra $\mathfrak{U} = (U; \Sigma)$, possessing a non-idempotent Plonka function, satisfies the hyperidentity (16) and the hyperidentity:

$$X(X(x_1, \dots, x_n), \dots, X(x_1, \dots, x_n)) = X(x_1, \dots, x_n).$$

To obtain a non-idempotent Plonka function different from Plonka function one should assume that no operation of the algebra $\mathfrak{U}$ is idempotent.

Theorem 5.3. *Let $\mathfrak{U} = (U; \Sigma)$ be an algebra with a non idempotent Plonka function. Then $\mathfrak{U}$ is a weakly Plonka sum of its subalgebras.*

Proof. Define, on the set U, the relation $\alpha \subseteq U \times U$ in the following way:

$$a\alpha b \leftrightarrow f(a,b) = f(a,a), f(b,a) = f(b,b),$$

where f is a non-idempotent Plonka function for $\mathfrak{U}$.

Let us show that α is an equivalence on U. Indeed, reflexivity and symmetricity immediately follow from the definition. Show transitivity: let $a\alpha b$ and $b\alpha c$, then $f(a,b) = f(a,a), f(b,a) = f(b,b), f(b,c) = f(b,b), f(c,b) = f(c,c)$. Hence:

$$f(a,c) \overset{8}{=} f(a, f(a,c)) \overset{1}{=} f(f(a,a),c) = f(f(a,b),c) \overset{1}{=} f(a, f(b,c)) =$$
$$f(a, f(b,b)) = f(a, f(b,a)) \overset{3}{=} f(a, f(a,b)) \overset{8}{=} f(a,b) = f(a,a);$$
$$f(c,a) \overset{8}{=} f(c, f(c,a)) \overset{1}{=} f(f(c,c),a) = f(f(c,b),a) \overset{1}{=} f(c, f(b,a)) =$$
$$f(c, f(b,b)) \overset{1}{=} f(f(c,b),b) = f(f(c,c),b) \overset{1}{=} f(c, f(c,b)) \overset{8}{=} f(c,b) = f(c,c).$$

Thus, $a\alpha c$. Denote the corresponding equivalence classes by $U_i, i \in I$. Hence, we obtain a partition of U: $\{U_i \subseteq U, i \in I\}$.

Let us prove that U_i are subalgebras. Indeed, let $a_1, \ldots, a_{n(t)} \in U_i, i \in I$; then for any $F_t \in \Sigma$ (the arity $|F_t| = n(t)$), we get:

$$f(F_t(a_1, \ldots, a_{n(t)}), a_1) \overset{6}{=} F_t(a_1, \ldots a_{n(t)}) \overset{7}{=} f(F_t(a_1, \ldots a_{n(t)}), F_t(a_1, \ldots, a_{n(t)}));$$

$$f(a_1, F_t(a_1, \ldots, a_{n(t)})) \overset{5}{=} f(a_1, F_t(f(a_1,a_1), \ldots, f(a_1, a_{n(t)}))) \overset{2}{=}$$
$$f(a_1, F_t(F_t(a_1, \ldots, a_1), \ldots, F_t(a_1, \ldots, a_1))) \overset{2,7}{=} f(a_1, F_t(a_1, \ldots, a_1)) \overset{2,8}{=} f(a_1, a_1),$$

i.e. $F_t(a_1, \ldots, a_{n(t)}), a_1 \in U_i$.

Note that for every $a, b \in U$:

$$f(f(a,b), f(a,b)) = f(a,b). \tag{17}$$

Indeed,

$$f(f(a,b), f(a,b)) \overset{3}{=} f(f(a,b), f(b,a)) \overset{1}{=} f(f(f(a,b),b),a) \overset{8}{=} f(f(a,b),a) \overset{1,8}{=} f(a,b).$$

Let us also note that from the identity: $f(f(a,b), f(a,b)) = f(f(a,b), f(b,a))$ it follows that

$$f(a,b)\alpha f(b,a). \tag{18}$$

Furthermore, if $a\alpha a'$ and $b\alpha b'$, then $f(a,b)\alpha f(a',b')$. Indeed:

$$f(f(a,b),f(a',b')) \overset{3}{=} f(f(a,b),f(b',a')) \overset{1,3}{=} f(f(a,f(b',b')),a') = f(f(a,f(b,b)),a') \overset{8}{=} f(f(a,b),a') \overset{1,3}{=} f(f(a,a'),b) = f(f(a,a),b) \overset{1,8}{=} f(a,b) = f(f(a,b),f(a,b)).$$

In the same way, we get that $f(f(a',b'),f(a,b)) = f(f(a',b'),f(a',b'))$.

Moreover, from the identity 8 of Definition 5.2, it immediately follows that $a\alpha f(a,a)$, for any $a \in U$.

On the set of indices I, we define the order "$\leq$" in the following manner: $i_1 \leq i_2$ iff there exist such $a \in U_{i_1}$, $b \in U_{i_2}$ that $f(b,a) = f(b,b)$. This order makes the set I into a structure of a semilattice. Indeed, reflexivity immediately follows from the definition. Let us show that "$\leq$" is antisymmetric:

Let $i_1 \leq i_2$ and $i_2 \leq i_1$, then there exist $a,a' \in U_{i_1}, b,b' \in U_{i_2}$ such that $f(b,a) = f(b,b)$ and $f(a',b') = f(a',a')$. Hence,

$$f(f(a',a'),f(b,b)) = f(f(a',a'),f(b,b')) \overset{3}{=} f(f(a',a'),f(b',b)) = f(f(a',a'),f(b',b') \overset{1}{=} f(f(a',f(a',b')),b') = f(f(a',f(a',a')),b') \overset{3}{=} f(f(a',a'),b') \overset{3}{=} f(a',a') \overset{17}{=} f(f(a',a'),f(a',a')).$$

In the similar way, we get that: $f(f(b,b),f(a',a')) = f(f(b,b),f(b,b))$. So, $f(a',a')\alpha f(b,b)$, hence $F(b,\ldots,b) = f(b,b) \in U_{i_i}$, thus $F(b,\ldots,b) \in U_{i_i} \cap U_{i_2}$, consequently, $i_1 = i_2$.

Let $i_1 \leq i_2$ and $i_2 \leq i_3$. Then there exist $a \in U_{i_1}, b,c \in U_{i_2}, d \in U_{i_3}$, such that $f(b,a) = f(b,b)$, $f(d,c) = f(d,d)$. So:

$$f(d,b) \overset{8}{=} f(d,f(d,b)) \overset{1}{=} f(f(d,d),b) = f(f(d,c),b) \overset{1}{=} f(d,f(c,b)) = f(d,f(c,c)) \overset{1}{=} f(f(d,c),c)) = f(f(d,d),c) \overset{1}{=} f(d,f(d,c)) = f(d,f(d,d)) \overset{8}{=} f(d,d),$$

hence,

$$f(d,a) = f(d,f(d,a)) = f(f(d,d),a) = f(f(d,b),a) \overset{1}{=} f(d,f(b,a)) = f(d,f(b,b)) = f(f(d,b),b) = f(f(d,d),b) \overset{1}{=} f(d,f(d,b)) = f(d,f(d,d)) \overset{8}{=} f(d,d),$$

which proves that $i_1 \leq i_3$. Thus, $(I;\leq)$ is an ordered set. To show that $(I;\leq)$ is a semilattice, let $a \in U_i, b \in U_j$ and $f(a,b) \in U_k$. Then:

$$f(f(a,b),a) = f(a,f(b,a)) = f(a,f(a,b)) = f(f(a,a),b) \overset{8}{=} f(a,b) \overset{17}{=} f(f(a,b),f(a,b)).$$

Thus, for any $i,j \in I$, there exists an upper bound $k \in I$ such that $f(a,b) \in U_k$, for some $a \in U_i, b \in U_j$.

Let us assume that for some $l \in I$, $i \leq l$ and $j \leq l$, there are $a' \in U_i, c \in U_l$ such that $f(c, a') = f(c, c)$ and there are $b' \in U_j, d \in U_l$ such that $f(d, b') = f(d, d)$. Hence, we have:

$f(c, f(a', b')) = f((c, a'), b') = f(f(c, c), b') = f(f(c, d), b') = f(c, f(d, b') = f(c, f(d, d)) = f(f(c, d), d) = f(f(c, c), d) = f(c, c)$.

Thus, $f(a', b')\alpha c$ and from the assertion $f(a, b)\alpha f(a', b')$, which is proven above, we obtain that $f(a, b)\alpha c$ which means $k \leq l$ and $k = sup\{i, j\}$.

Define the mappings $\varphi_{i_1,i_2} : U_{i_1} \mapsto U_{i_2}$, for $i_1 \leq i_2$, in the following way:

$$\varphi_{i_1,i_2}(a) = f(a, b),$$

where $a \in U_{i_1}, b \in U_{i_2}$.

First of all, let us show that $f(a, b) \in U_{i_2}$ for all $a \in U_{i_1}, b \in U_{i_2}$. Since $i_1 \leq i_2$, then there exist $c \in U_{i_1}, d \in U_{i_2}$ such that $f(d, c) = f(d, d)$. Thus, we obtain:

$$f(d, f(d, c)) \overset{3}{=} f(d, f(c, d)) \overset{8}{=} f(d, c) = f(d, d)$$

and $f(f(c, d), d) \overset{3,8}{=} (f(c, f), f(c, d))$.

This gives $b\alpha d\alpha f(c, d)\alpha f(a, b)$ and, hence, $f(a, b) \in U_{i_2}$.

The definition of the mappings φ_{i_1,i_2} is consistent, i.e. it is independent from the choosing of the element $b \in U_{i_2}$. Indeed, let $f(a, b_1), f(a, b_2)$ be arbitrary elements from U_{i_2} and $a \in U$, then:

$f(a, b_1) = f(f(a, b_1), f(a, b_1)) \overset{3}{=} f(f(a, b_1), f(b_1, a)) \overset{1}{=} f(f(a, f(b_1, b_1)), a)$
$= f(f(a, f(b_1, b_2)), a) \overset{3}{=} f(f(a, f(b_2, b_1)), a) = f(f(a, f(b_2, b_2)), a)$
$\overset{1,3}{=} f(f(a, b_2), f(a, b_2)) = f(a, b_2)$.

Thus, we have: $f(a, b_1) = f(a, b_2)$.

It is clear that the mappings φ_{i_1,i_2} are homomorphisms and $\varphi_{i,i}(x) = F_t(x, \ldots, x)$ for any $F_t \in \Sigma$.

Finally, we prove that for any $n(t)$-ary operation $F \in \Sigma$ and $x_1 \in U_{i_1}, \ldots, x_{n(t)} \in U_{i_{n(t)}}$, $F(x_1, \ldots, x_{n(t)}) = F(\varphi_{i_1,i_0}(x_1), \ldots, \varphi_{i_1,i_{n(t)}}(x_{n(t)}))$, where $i_0 = sup\{i_1, \ldots, i_n\}$. To make the proof easier let us make the designation $f = \cdot$.

We have already noticed that for $a \in U_i$ and $b \in U_j$, $a \cdot b \in U_{sup(i,j)}$. This implies that $y := x_1 \cdot \ldots \cdot x_{n(t)} \in U_{i_0}$.

By (8), for each $1 \leq i \leq n(t)$, $y \cdot x_i = x_1 \cdot \ldots \cdot x_{n(t)} \cdot x_i = x_1 \cdot \ldots \cdot x_{n(t)} = y$. Thus, by (5), we have:

$$y \cdot F(x_1, \ldots, x_{n(t)}) = y \cdot F(y \cdot x_1, \ldots, y \cdot x_{n(t)}) = y \cdot F(y, \ldots, y) = y \cdot y \cdot y = y \cdot y.$$

Since by (6), for each $1 \leq i \leq n(t)$, $F(x_1, \ldots, x_{n(t)}) \cdot x_i = F(x_1, \ldots, x_{n(t)})$, we obtain:

$$F(x_1, \ldots, x_{n(t)}) \cdot y = F(x_1, \ldots, x_{n(t)}) \overset{7}{=} F(x_1, \ldots, x_{n(t)}) \cdot F(x_1, \ldots, x_{n(t)}).$$

This means that $y\alpha F(x_1,\dots,x_{n(t)})$ and, as a consequence, $F(x_1,\dots,x_{n(t)}) \in U_{i_0}$.

Let $x \in U_{i_0}$. Then $F(\varphi_{i_1,i_0}(x_1),\dots,\varphi_{i_{n(t)},i_0}(x_{n(t)})) = F_t(x_1\cdot x,\dots,x_{n(t)})\cdot x \overset{4}{=} F_t(x_1,\dots,x_{n(t)})\cdot x = \varphi_{i_0,i_0}(F(x_1,\dots,x_{n(t)})) = F(x_1,\dots,x_{n(t)})\cdot F(x_1,\dots,x_{n(t)}) \overset{4}{=} F(x_1,\dots,x_{n(t)})$, which finishes the proof. ■

6. Weakly idempotent quasilattices

Definition 6.1. The binary algebra $\mathfrak{U} = (U,\Sigma)$ is called weakly idempotent quasilattice if it satisfies the following hyperidentities:

$$X(x,x) = Y(x,x), \tag{19}$$

$$X(x,y) = X(y,x), \tag{20}$$

$$X(x,X(y,z)) = X(X(x,y),z), \tag{21}$$

$$X(x,X(y,y)) = X(x,y), \tag{22}$$

$$X(Y(X(x,y),z),Y(x,z)) = Y(X(x,y),z). \tag{23}$$

Example 6.2. Note that each weakly idempotent semilattice, weakly idempotent lattice and the superproduct of weakly idempotent lattices satisfy the above hyperidentities. Hence, any weakly idempotent semilattice, weakly idempotent lattice and the superproduct of weakly idempotent lattices are weakly idempotent quasilattices.

If $L = (L;\wedge,\vee)$ is a weakly idempotent lattice, then the superproduct $L \bowtie L$ satisfies all hyperidentities of the variety of weakly idempotent lattices. The reduct $(L\times L;(\wedge,\wedge),(\wedge,\vee))$ also satisfies the hyperidentities of the variety of weakly idempotent lattices, but it does not satisfy the law of weak absorption: $a\wedge(a\vee b) = a\wedge a, a\vee(a\wedge b) = a\vee a$, hence, it is not a weakly idempotent lattice.

To prove Theorem 6.3, we need the following hyperidentities, which are the consequences of the hyperidentities (19)–(23):

$$X(Y(X(Y(z,y),x),X(y,x)),Y(x,X(y,x)) = Y(X(Y(z,y),x),X(y,x)), \tag{24}$$

$$Y(X(Y(z,y),x),X(y,x)) = X(Y(X(Y(z,y),x),X(y,x)),Y(y,x)), \tag{25}$$

$$X(x,X(Y(x,y),Y(Y(x,y),z))) = X(x,Y(z,Y(x,y))), \tag{26}$$

$$X(y,Y(y,z)) = Y(y,X(y,z)), \tag{27}$$

$$Y(Y(x,X(z,Y(y,z))),y) = Y(x,Y(y,z)), \tag{28}$$

$$X(x,Y(x,X(y,Y(y,z)))) = Y(x,Y(y,z)), \tag{29}$$

$$X(x,Y(x,X(z,Y(y,z)))) = Y(x,Y(y,z)). \tag{30}$$

Theorem 6.3. *Every weakly idempotent quasilattice* $(Q; A, B)$ *with two binary operations is a weakly idempotent lattice or a weakly Plonka sum of subalgebras which are weakly idempotent lattices.*

Proof. Let $(Q; A, B)$ be a weakly idempotent quasilattice with two binary operations.

Define the mapping $f : Q \times Q \to Q$ in the following way:

$$f(x, y) = A(x, B(x, y)) = B(x, A(x, y)).$$

We show that the function f is a non-idempotent Plonka function. The consistence of f follows from the hyperidentity (27). Let us check the conditions of Definition 4.1.

1. $f(f(x, y), z) = f(A(x, B(x, y)), z) =$
   $A(A(x, B(x, y)), B(A(x, B(x, y)), z)) \overset{(23)}{=}$
   $A(A(x, B(x, y)), A(B(A(x, B(x, y)), z)), B(B(x, y), z)) \overset{(28),(20)}{=}$
   $A(A(x, B(x, y)), A(B(A(x, B(x, y)), z), B(B(z, A(x, B(x, y))), z))) \overset{(26)}{=}$
   $A(A(x, B(x, y)), B(B(z, A(x, B(x, y))), y)) \overset{(28)}{=}$
   $A(A(x, B(x, y)), B(B(x, y), z)) \overset{(21)}{=}$
   $A(x, A(B(x, y), B(B(x, y), z))) \overset{(26)}{=} A(x, B(B(x, y), z)).$
   $f(x, f(y, z)) = f(x, A(y, B(y, z))) = A(x, B(x, A(y, Y(y, z)))) \overset{(29)}{=}$
   $A(x, B(x, B(y, z)))).$
2. $f(x, x) = A(x, B(x, x)) \overset{(19)}{=} A(x, A(x, x)) \overset{(22)}{=} A(x, x).$
3. $f(x, f(y, z)) = f(x, A(y, B(y, z))) = A(x, B(x, A(y, B(y, z))));$
   $f(x, f(z, y)) = f(x, A(z, B(z, y))) = A(x, B(x, A(z, B(z, y)))).$

   From hyperidentity (28) it follows that $f(x, f(y, z)) = f(x, f(z, y))$. Further, without loss of generality, we suppose that $F_t = A$.
4. $f(A(x_1, x_2), y) = A(A(x_1, x_2), B(A(x_1, x_2), y)) \overset{(23)}{=}$
   $A(A(x_1, x_2), A(B(x_1, x_2), y), B(x_1, y)) \overset{(24)}{=} A(A(x_1, x_2), B(x_1, y)).$
   $A(f(x_1, y), f(x_2, y)) = A(A(x_1, Y(x_1, y)), A(x_2, Y(x_2, y))) \overset{(21)}{=}$
   $A(x_1, A(B(x_1, y), A(x_2, B(x_2, y)))) \overset{(19)}{=}$
   $A(x_1, A(x_2, A(B(x_1, y), B(x_2, y)))) \overset{(24)}{=}$
   $A(x_1, A(x_2, B(x_1, y))) \overset{(21)}{=} A(A(x_1, x_2), B(x_1, y)).$
5. $f(y, A(x_1, x_2)) = A(y, B(y, A(x_1, x_2))) = B(y, A(y, A(x_1, x_2))).$
   $f(y, A(f(y, x_1), f(y, x_2))) = f(y, A(A(y, B(y, x_1)), A(y, B(y, x_2)))) =$

$A(y, B(y, A(A(y, B(y, x_1)), A(y, (y, x_2))))) \overset{(27)}{=}$
$B(y, A(y, A(A(y, B(y, x_1)), A(y, B(y, x_2))))) \overset{(21),(22)}{=}$
$B(y, A(y, A(B(y, x_1), B(y, x_2)))) \overset{(30)}{=}$
$B(y, A(y, A(A(B(y, x_1), A(y, x_2)), B(B(x_1, x_2), y)))) \overset{(21)}{=}$
$B(y, A(B(y, x_1), A(y, A(B(y, x_2), B(B(x_1, x_2), y)))))) \overset{(25)}{=}$
$B(y, A(B(y, x_1), A(y, B(Y(x_1, x_2), y)))) =$
$B(y, A(y, B(B(x_1, x_2), y))) \overset{(27)}{=} A(y, B(y, B(B(x_1, x_2), y))) \overset{(22),(21)}{=}$
$A(y, A(B(x_1, x_2), y)) \overset{(28)}{=} B(y, A(A(x_1, x_2), y))$.

6. $f(A(x_1, x_2), x_i) = A(A(x_1, x_2), B(A(x_1, x_2), x_i)) \overset{(28)}{=}$
$B(A(x_1, x_2), A(A(x_1, x_2), x_i)) \overset{(21),(22)}{=} B(A(x_1, x_2), A(x_1, x_2)) =$
$A(A(x_1, x_2), A(x_1, x_2)) = A(x_1, x_2)$.

7. $f(A(x_1, x_2), A(x_1, x_2)) = A(B(x_1, x_2), B(A(x_1, x_2), A(x_1, x_2))) \overset{(19)}{=}$
$A(A(x_1, x_2), A(A(x_1, x_2), X(x_1, x_2))) \overset{(21),(22)}{=}$
$A(A(x_1, x_2), A(x_1, x_2)) \overset{(21),(22)}{=} A(x_1, x_2)$.

8. $f(x, f(x, y)) = f(x, A(x, B(x, y))) = A(x, B(x, A(x, B(x, y)))) \overset{(27)}{=}$
$A(x, B(x, B(x, A(x, Y)))) \overset{(21),(22)}{=} A(x, B(x, A(x, y))) \overset{(27)}{=}$
$B(x, A(x, A(x, y))) \overset{(21),(22)}{=} B(x, A(x, y)) \overset{(27)}{=} A(x, B(x, y)) = f(x, y)$.

Applying Theorem 5.3, we obtain that $(Q; A, B)$ is a weakly idempotent lattice or is a weakly Plonka sum of the subalgebras that are weakly idempotent lattices. The law of weak absorption for subalgebras U_i follows from the fact that for every $x, y \in U$, $f(x, y) = f(x, x)$ and $f(y, x) = f(y, y)$. ∎

Corollary 6.4. *Let $\mathfrak{U} = (U, \Sigma)$ be a subdirectly irreducible weakly idempotent quasilattice. Then the cardinality $|\Sigma| \leq 2$.*

Proof. We show that if the cardinality $|\Sigma| \geq 3$, then $\mathfrak{U}$ is not subdirectly irreducible. Let $|\Sigma| \geq 3$. Hence, there exist pairwise distinct operations $A_1, A_2, A_3 \in \Sigma$. Define a function $f_{i,j}$ in the following way:

$$f_{ij}(x, y) = A_i(x, A_j(x, y)),$$

and the relations $\widetilde{\theta}_{i,j}$ on U by the following rule: $x\widetilde{\theta}_{i,j}y \leftrightarrow f_{i,j}(x, y) = x, f_{i,j}(y, x) = y$.
Then $\theta_{i,j} = \widetilde{\theta}_{i,j} \cup \{(x, x)|x \in U\}$ are non-trivial congruences on $\mathfrak{U}$, having the trivial intersection: $\theta_{1,2} \cap \theta_{1,3} \cap \theta_{2,3} = \omega$. ∎

Corollary 6.5. *Every hyperidentity of the variety of weakly idempotent lattices is a consequence of the following hyperidentities:* (19)–(23) (see [28, 31]).

Corollary 6.6. *Every hyperidentity of the variety of weakly idempotent distributive lattices is a consequence of the hyperidentities* (19)–(22) *and the hyperidentity* (6).

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D. S. Davidova
APPLIED MATHEMATICS DEPARTMENT
EUROPEAN REGIONAL ACADEMY
10 Davit Anhaght str.
YEREVAN 0037, ARMENIA
E-mail: di.davidova@yandex.ru

Yu. M. Movsisyan
DEPARTMENT OF MATHEMATICS AND MECHANICS
YEREVAN STATE UNIVERSITY
1 Alex Manoogian str.
YEREVAN 0025, ARMENIA
E-mail: yurimovsisyan@yahoo.com

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