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RELATIVELY ORTHOCOMPLEMENTED SKEW NEARLATTICES IN RICKART RINGS

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Abstract. A class of (right) Rickart rings, called strong, is isolated. In particular, every Rickart *-ring is strong. It is shown in the paper that every strong Rickart ring R admits a binary operation which turns R into a right normal band having an upper bound property with respect to its natural order $\leq$; such bands are known as right normal skew nearlattices. The poset $(R, \leq)$ is relatively orthocomplemented; in particular, every initial segment in it is orthomodular.

The order $\leq$ is actually a version of the so called right-star order. The one-sided star orders are well-investigated for matrices and recently have been generalized to bounded linear Hilbert space operators and to abstract Rickart *-rings. The paper demonstrates that they can successfully be treated also in Rickart rings without involution.

1. Introduction

A poset P has the *upper bound property* if every pair of its elements bounded above has the least upper bound. Hence, every section (i.e., initial segment) in P is an upper semilattice; the converse generally does not hold true. A poset possessing the upper bound property may happen to be a lower semilattice; such posets are known as *nearlattices* (though some authors use this term for dual structures). *Skew nearlattices* is a "non-commutative" generalization of nearlattices and may be considered as ordered algebras with a "skew meet" operation and (explicit or implicit) partial join operation. See the next section for more details.

General skew nearlattices were introduced by the present author in [5]. However, a particular class of them, the so called right normal skew nearlattices (rns-nearlattices, for short) were discussed already in [4]. These

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structures arose in that paper as a part of abstract descriptions of certain information systems. It was later shown in [5] that typical examples of rns-nearlattices are provided by algebras of partial functions with appropriate operations. The author's purpose in the present paper is to construct examples of another type, starting from special right Rickart rings called *strong* in the paper.

More specifically, every such a ring admits a binary operation that turns it into an rns-nearlattice. Moreover, this skew nearlattice is shown to be relatively orthocomplemented, which means, in particular, that every its section is an orthomodular lattice. Explicit descriptions of the partial join, skew meet and sectional orthocomplementation operations also are given in the paper. Its structure: Section 2 gathers some information on sectionally orthocomplemented posets and skew nearlattices; in Section 3, the class of strong Rickart rings is introduced and elementary properties of them are stated; strong Rickart rings as skew nearlattices and as relatively orthocomplemented posets is the subject of Sections 4 and 5, respectively (in the latter, also an associated orthogonality relation is discussed in short).

There is one more point, which is not developed in the paper but deserves to be mentioned. It is also disclosed in Section 4 that the natural order of the constructed skew nearlattice is in fact a version of the so called right-star partial order. What follows is some elucidative notes in this connection.

Left- and right-star orders for $m \times n$ matrices were introduced in [2] and have been intensively studied (see [18]). These orders have recently be transferred to bounded linear Hilbert space operators, thus encompassing also the infinitely dimensional case: two different but equivalent definitions are presented in [9], resp., [10]. In [8], the present author introduced a version of one-sided star orders that is generally weaker but agrees with the former ones for regular operators, i.e., those having the Moore-Penrose inverse. Since in finitely dimensional Hilbert spaces all bounded linear operators are regular, it follows that this version also is an adequate generalization of the traditional (matrix case) notion.

All these definitions are adapted to the simple fact that the sets of matrices and of Hilbert space operators have a natural structure of involution ring: involution explicitly appears in these definitions (whence the name 'star order'). In [15, 17, 7], the notion of one-sided star order has almost simultaneously been generalized to abstract involution rings (to regular *-rings, regular Rickart *-rings and Rickart *-rings, respectively). The three approaches are compared in [8]. In the latter paper, it is also shown that the order from [7] can be characterized without involving involution (see Remark 2 below for some details) and may thereby be transferred to ordinary Rickart rings. It is remarkable that the right-star order obtained in this way coincides in a strong Rickart ring with the natural order of its rns-nearlattice.

Therefore, the present paper demonstrates that such a "star-free" definition of a one-sided star order can be used to advantage in certain Rickart rings without involution.

2. Preliminaries

1. A poset $(P, \leq)$ is said to be *orthocomplemented* (an *orthoposet*, for short) if it has the least element 0, the greatest element 1, and is equipped with a unary operation $^\perp$ such that

$$x^{\perp\perp} = x, \quad \text{if } x \leq y, \text{ then } y^\perp \leq x^\perp, \quad x \wedge x^\perp = 0, \text{ (equivalently, } x \vee x^\perp = 1)$$

(then $0^\perp = 1$, $1^\perp = 0$). The operation $^\perp$ itself is called an *orthocomplementation*. In an orthoposet, the De Morgan laws are fulfilled in the following form:

$$\text{if } x \wedge y \text{ exists, then } x^\perp \vee y^\perp \text{ exists, and is equal to } (x \wedge y)^\perp,$$
$$\text{if } x \vee y \text{ exists, then } x^\perp \wedge y^\perp \text{ exists, and is equal to } (x \vee y)^\perp,$$

where $x \wedge y$ is the l.u.b. (meet) of x and y, and $x \vee y$ is the g.l.b. (join) of these elements. We write $x \perp y$ to mean that $y \leq x^\perp$; the relation $\perp$ is called the *induced orthogonality* on P. An orthoposet P is *orthomodular* if

$$\begin{gathered} x \vee y \text{ exists whenever } x \perp y, \\ \text{if } x \leq y, \text{ then } y = x \vee z \text{ for some } z \text{ with } x \perp z. \end{gathered} \tag{2.1}$$

The latter condition is equivalent to

$$\text{if } x \leq y, \text{ then } y = x \vee (x \vee y^\perp)^\perp.$$

The induced orthogonality $\perp$ on an orthoposet P has the properties

$$0 \perp x, \quad \text{if } x \perp y, \text{ then } y \perp x, \quad \text{if } x \leq y \text{ and } y \perp z, \text{ then } x \perp z. \tag{2.2}$$

If P is even orthomodular, then also

$$\text{if } x \perp y, z \text{ and } y \leq x \vee z, \text{ then } y \leq z. \tag{2.3}$$

Following [6], we call a binary relation $\perp$ on an poset $(P, \leq, 0)$ an *orthogonality* (with respect to $\leq$) if it satisfies (2.2), and define a *quasi-orthomodular* poset to be a system $(P, \leq, \perp, 0)$, where $(P, \leq, 0)$ is a poset with 0 and $\perp$ is an orthogonality on P satisfying also (2.1) and (2.3).

A poset with the least element is said to be *sectionally orthocomplemented* (*orthomodular*), if every its section, i.e., initial segment $[0, x]$, is orthocomplemented (resp., orthomodular). We denote by $\frac{\perp}{p}$, the orthocomplementantion living in the section $[0, p]$ of such poset P, and by $\perp_p$, the corresponding induced orthogonality in $[0, p]$. The union of all local orthogonalities $\perp_p$ satisfies (2.2); we call it the *induced orthogonality* on P.

For example, an orthomodular poset is always sectionally orthomodular with the orthocomplement of x in $[0,p]$ given by $p \wedge x^{\perp}$. More generally, a sectionally orthocomplemented poset is said to be *relatively orthocomplemented* [6] if (i) any pair of elements $x, y \leq p$ has the join whenever $x \perp_p y$, and (ii) if $x \leq p \leq q$, then $x_p^{\perp} \leq x_q^{\perp}$. The latter condition can even be strengthened to (ii') if $x \leq p \leq q$, then $x_p^{\perp} = p \wedge x_q^{\perp}$; moreover, such a poset is, in fact, sectionally orthomodular. The subsequent proposition shows that the notions of quasi-orthomodular and of relatively orthocomplemented posets are essentially equivalent.

Proposition 2.1. [6, Theorem 5.5] *A poset P with the least element, supplied with a binary relation $\perp$, is quasi-orthomodular if and only if it is relatively o-complemented and $\perp$ is its induced orthogonality.*

2. The notation $a \mathbin{\downarrow\!\circ} b$ will mean that the elements a an b of a poset have the l.u.b. (join). Recall that a nearlattice is a lower semilattice having the upper bound property, which can be stated as $a \mathbin{\downarrow\!\circ} b$ iff $a, b \leq x$ for some x.

Let $(S, \cdot)$ be an idempotent semigroup, or *band*. The *natural order* $\leq$ on S is defined by

$$x \leq y \Leftrightarrow yx = x = xy.$$

Therefore, elements x and y of S commute if and only if the product xy is their greatest lower bound $x \wedge y$. If S has the zero element 0, then it is the least element, and if S has the unit 1, then it is the greatest element in S. S is said to be a *skew nearlattice* [5] if it has the upper bound property relatively to $\leq$. (Alternatively, a skew nearlattice may be characterized as a poset with the upper bound property which is a band for which the underlaying order relation is the natural order; cf. [4].) Therefore, nearlattices are just commutative skew nearlattices.

The band S is said to be

- *right-handed*, if $xy = x \Rightarrow yx = x$ [5],
- *right regular*, if $xyx = yx$ [14],
- *right normal*, if $xyz = yxz$ [19].

Thus, in a right-handed band,

$$x \leq y \text{ iff } xy = x.$$

Standard calculations show that a band is right-handed if and only if it is right regular. Every right normal band is right regular. A right-handed band is right normal if and only if the operation $\cdot$ commutes in every initial segment, i.e., if and only if every such a segment is a lower semilattice with $x \wedge y = xy$. Right normal bands have been studied also under the name *restrictive semigroup*, which comes from [20]. Right normal skew nearlattices were introduced in [4] and further studied in [5].

Another important relation $\sqsubseteq$ in a right-handed band is the preorder defined by

$$x \sqsubseteq y \text{ iff } yx = x.$$

In particular, $x \leq y$ iff $x \sqsubseteq y$ and $xy = yx$. The skew meet operation $\cdot$ can be restored from the relations $\leq$ and $\sqsubseteq$. The following theorem is a simplified version of [20, Theorem 1].

Proposition 2.2. *In a right-normal band,*

$$xy = \max\{z\colon\ z \sqsubseteq x \text{ and } z \leq y\}.$$

Proof. We have to show that, for all x, y, z, $xy \sqsubseteq x$, $xy \leq y$ and

$$\text{if } z \sqsubseteq x \text{ and } z \leq y, \text{ then } z \leq xy.$$

We explain only the last implication. Assume that $z \sqsubseteq x$ and $z \leq y$. Then $zxy = xzy = xz = z$, i.e., $z \leq xy$. ■

3. Strong Rickart rings

We shall deal only with associative rings.

A *(right) Rickart ring* is a ring in which the right annihilator of every element is a principal ideal generated by an idempotent. Put in another way, this means that, given any element x, we can choose an idempotent x' such that, for all elements y of the ring,

$$xy = 0 \ \text{ iff } \ x'y = y \tag{3.1}$$

(see [16, 13]). The element x'' will be called the *support* of x. A Rickart ring is always unital with $1 = 0'$.

We also assume that

$$x'' = 1 - x'. \tag{3.2}$$

This identity implies that the condition (3.1) is equivalent to

$$xy = 0 \ \text{ iff } \ x''y = 0. \tag{3.3}$$

Left Rickart rings are defined dually. We shall not deal here with these; so, in this paper, by a Rickart ring, we always mean a right Rickart ring.

Example 1. Recall that a ring is said to be *regular* if every its principal right (equivalently, left) ideal is generated by an idempotent. This is the case if and only if to every x there is an element x^- (a *generalized inverse of* x) such that $xx^-x = x$. It follows immediately from the definitions that every regular unital ring is Rickart. We may put $x' := 1 - x^-x$: then x' is idempotent and the equivalence (3.1) holds. Furthermore, choosing $(x')^- = x'$ for all elements x, we obtain (3.2); it then follows that $x'' = x^-x$.

Strictly speaking, we are treating a Rickart ring as a ring equipped with an additional operation $'$ that satisfies (3.1) and (3.2). It is well known that idempotents of any unital ring form an orthocomplemented (even orthomodular) poset E, where $e \leqslant f$ if and only if $ef = e = fe$, 0 is the least, and 1, the greatest element, while $1 - e$ serves as the orthocomplement of e in E. In a Rickart ring, let P stand for the range of the operation $'$. Clearly, $0, 1 \in P$. By (3.2), the set P is closed under orthocomplementation: if $e \in P$, then $1 - e = e' \in P$. Therefore, P inherits the structure of an orthoposet from E. Evidently, an element belongs to P iff it coincides with its support:

$$x \in P \text{ iff } x = x''$$

for every $x \in R$. By analogy with [11], let us call idempotents in P *closed*.

We say that a Rickart ring is *strong* if, for all $e, f \in P$,

$$ef \in P \text{ iff } ef = fe. \tag{3.4}$$

If this is the case, then the relation $\leq$ on P defined by

$$e \leq f \text{ iff } ef = e \text{ iff } fe = e$$

is an order and agrees with the order $\leqslant$ inherited from E. The induced orthogonality in the orthoposet P is then characterized by

$$e \perp f \text{ iff } ef = 0 \text{ iff } fe = 0;$$

of course, $e \leq f$ iff $e \perp f'$ iff $f' \perp e$. The following criterion of commutativity in P may be useful.

Theorem 3.1. *Closed idempotents e and f commute if and only if they split in the following sense: $e = g + g_1$ and $f = g + g_2$ for some mutually orthogonal closed idempotents g, g_1, g_2.*

Proof. Sufficiency of the condition is immediate. Assume that $ef = fe$ and put $g := ef$, $g_1 := ef'$ and $g_2 := e'f$; then $e = g + g_1$ and $f = g + g_2$. Moreover, e commutes with $(1 - f)$; so, g_1 and, likewise, g_2 are closed. At last, the idempotents g, g_1, g_2 are indeed mutually orthogonal. ∎

Example 2. A Rickart ring in which all idempotents are central (this is the case, for instance, if it does not have nonzero nilpotent elements; see [13] for a study of such Rickart rings) is necessarily strong, with $P = E$.

Remark 1. Recently also a notion of strongly Rickart ring has been introduced; see [1]. However, Corollary 1.9 in that paper states that such a ring is actually nothing else than a right Rickart ring without nonzero nilpotent elements. Therefore, every strongly Rickart ring is strong; the converse may not hold true.

Example 3. A Rickart *-ring [3, 7, 16] may be defined as an involution ring which is Rickart with every closed idempotent e being symmetric: $e^* = e$ (symmetric idempotents are commonly called projections). In such a ring, every projection turns out to be closed, and then $ef \in P$ iff $ef = (ef)^* = f^*e^* = fe$ for all projections e, f. Therefore, any Rickart *-ring (in particular, any *-regular involution ring, i.e., regular *-ring with proper involution [3]) is an instance of a strong Rickart ring.

A number of star-free properties of Rickart *-rings can be transferred to strong Rickart rings. The relationships stated in the subsequent proposition were first obtained for Rickart *-rings in [7, Proposition 2.4].

Proposition 3.2. *In a strong Rickart ring,*

(a) $aa' = 0$,
(b) $aa'' = a$,
(c) $(ab)'' \leq b''$, *i.e.*, $(ab)'' = b''(ab)'' = (ab)''b''$,
(d) $(ab)'' = (a''b)''$,
(e) *if* $e \in P$ *and* $e \leq a''$, *then* $(ae)'' = e$.

Proof.
(a) By (3.1), as a' is idempotent.
(b) By (3.2) and (a), $aa'' = a(1 - a') = a$.
(c) For $(ab)'' \leq b''$ iff $(ab)''b' = 0$ iff $abb' = 0$—see (3.3) and (a).
(d) Likewise $(ab)'' \leq (a''b)''$: $(ab)''(a''b)' = 0$ iff $ab(a''b)' = 0$ iff $a''b(a''b)' = 0$, and $(a''b)'' \leq (ab)''$: $(a''b)''(ab)' = 0$ iff $a''b(ab)' = 0$ iff $ab(ab)' = 0$.
(e) If $a''e = e$, then, by (d), $(ae)'' = (a''e)'' = e'' = e$. ∎

We now turn to lattice operations in the poset of closed projections. Let, for any a, $C(a) := \{e \in P\colon\ ea = ae\}$.

Lemma 3.3. *Suppose that* e, f, ef *are closed idempotents in a strong Rickart ring. Then*

(a) $e'f, ef', (ef)' \in P$,
(b) $e \wedge f$ *exists in* P, *and* $e \wedge f = ef$,
(c) $e \vee f$ *exists in* P, *and* $e \vee f = e + f - ef$,
(d) *if* $e, f \in C(a)$ *for some* $a \in R$, *then also* $e \wedge f, e \vee f, e' \in C(a)$,
(e) *if* $e, f \leq g$ *for some* $g \in P$, *then* $e, f \in C(g)$ *and* $e \wedge f, e \vee f, g - e \leq g$.

Proof. Assume that $e, f, ef \in P$. Then also $efe, fef \in P$. Recall that $e' = 1 - e$. Of course, P is closed under the operation $'$.
(a) Evidently, $e'f = fe'$ and $ef' = f'e$.
(b) Clearly, $ef \leq e,f$, and if $g \leq e,f$ with $g \in P$, then $g = ge = gf = gef$ and $g \leq ef$. Thus ef is the greatest lower bound of e and f in P.

(c) As e and f commute, similar calculations show that $e+f-ef$ is the l.u.b. of e and f in P.

(d) Evident by virtue of (a)–(c).

(e) Evident. Observe that $g-e=ge'=g\wedge e'$. ∎

To justify the next proposition, we adjust the proof of a similar result for Baer *-semigroups in Sect. 2 of [11]. See [3, Proposition 1.3.7] for the case of Rickart *-rings.

Proposition 3.4. *The set of closed idempotents of a strong Rickart ring is an orthomodular lattice with*

$$e\wedge f=(e'f)'f=f-(e'f)'',\quad e\vee f=((ef')'f')'=f+(ef')''.$$

Proof. Assume that $e,f\in P$. Recall that P is an orthoposet with orthocomplementation $'$ and that $e'=1-e$.

By Proposition 3.2(c), $(e'f)''\le f$; so, f commutes with $(e'f)''$ and $(e'f)'$. Moreover, $(e'f)'f=(e'f)'\wedge f$ by Lemma 3.3(b).

Now, $(e'f)'f\le f$. From Proposition 3.2(a), $0=e'f(e'f)'=e'(e'f)'f$, whence $(e'f)'f\le e$ by (3.1). Thus, $(e'f)'f$ is a lower bound of e and f. Let $g\in P$ be any other such a lower bound; then $g=eg=fg$, $e'fg=e'eg=0$ and, by (3.1), $g\le(e'f)'$. Therefore, $g\le(e'f)'\wedge f=(e'f)'f$, i.e., $(e'f)'f$ is actually the greatest lower bound of e and f, as needed. Consequently, $e\wedge f=(1-(e'f)'')f=f-f(e'f)''=f-(e'f)''$; see Proposition 3.2(c).

Further, P is an orthoposet, hence $e\vee f=(e'\wedge f')'=((ef')'f')'$. As $(ef')'f'\in P$, then $e\vee f=1-(1-(ef')'')f'=f+(ef')''f'=f+(ef')''$ (Proposition 3.2(c)).

Thus, P is an ortholattice. Finally, if $e\le f$, then f commutes with e and e', consequently, $fe'\in P$, e' commutes with fe' and $(fe')'$, $(fe')'e'\in P$ (Lemma 3.3), and $f=f\vee e=e+(fe')''=e+fe'$; on the other hand, $e\perp fe'$. By Lemma 3.3(c), $e+fe'=e\vee fe'$, and P is orthomodular. ∎

Now, we can continue Proposition 3.2.

Proposition 3.5. *In a strong Rickart ring,*

(a) *for every a, the subset $\{e\in P\colon ae=0\}$ is a sublattice of P,*
(b) *if $ae=be$, $af=bf$ and $a''=e\vee f=b''$ with $e,f\in P$, then $a=b$.*

Proof. Let a,b be arbitrary elements of the ring.

(a) The subset under question is an initial segment of P: for every $e\in P$, $ae=0$ iff $a'e=e$ iff $e\le a'$.

(b) Assume that the three hypotheses are satisfied. Then $(a-b)e=0=(a-b)f$ and, by (a), $(a-b)(e\vee f)=0$. Thus, $a=aa''=ba''=bb''=b$ (see Proposition 3.2(b)). ∎

4. A skew meet operation in a strong Rickart ring

Standing Assumption: In the sequel, we assume that $(R, +\cdot\cdot, 0,')$ is a strong Rickart ring and P is its lattice of closed idempotents.

We define on R a binary operation $\overset{\leftarrow}{\wedge}$ as follows:

$$x \mathbin{\overset{\leftarrow}{\wedge}} y := y(x'' \wedge y'').$$

Theorem 4.1. *The algebra $(R, \overset{\leftarrow}{\wedge})$ is a right normal band, and the idempotent mapping $\phi\colon x \mapsto x''$ is its homomorphism onto $(P, \wedge)$.*

Proof. First, due to Proposition 3.2(d),

$$(x \mathbin{\overset{\leftarrow}{\wedge}} y)'' = x'' \wedge y'', \tag{4.1}$$

and $\phi(e) = e$ for every closed idempotent. We have therefore obtained the second assertion of the theorem.

Evidently, the operation $\overset{\leftarrow}{\wedge}$ is idempotent. It is also associative:

$$\begin{aligned} x \mathbin{\overset{\leftarrow}{\wedge}} (y \mathbin{\overset{\leftarrow}{\wedge}} z) &= (y \mathbin{\overset{\leftarrow}{\wedge}} z)(x'' \wedge (y \mathbin{\overset{\leftarrow}{\wedge}} z)'') = z(y'' \wedge z'')(x'' \wedge (y'' \wedge z'')) \\ &= z(x'' \wedge (y'' \wedge z'')) = z((x'' \wedge y'') \wedge z'') = z((x \mathbin{\overset{\leftarrow}{\wedge}} y)'' \wedge z'') = (x \mathbin{\overset{\leftarrow}{\wedge}} y) \mathbin{\overset{\leftarrow}{\wedge}} z. \end{aligned}$$

Likewise, $(x \mathbin{\overset{\leftarrow}{\wedge}} y \mathbin{\overset{\leftarrow}{\wedge}} z) = z(x'' \wedge y'' \wedge z'') = z(y'' \wedge x'' \wedge z'') = (y \mathbin{\overset{\leftarrow}{\wedge}} x \mathbin{\overset{\leftarrow}{\wedge}} z)$. Thus, $(R, \overset{\leftarrow}{\wedge})$ is indeed a right normal band. ∎

The natural order of the band agrees on P with the order of closed idempotents; this allows us to use the same symbol $\leq$ for both orders. Thus, for all $x, y \in R$,

$$x \leq y \text{ iff } y(x'' \wedge y'') = x.$$

(However, $\leq$ is not an extension of the order $\leqslant$ on E.) We now list some useful properties of the relation $\leq$.

Lemma 4.2. *In $(R, \leq)$,*

(a) 0 *is the least element,*
(b) $P = [0, 1]$,
(c) *every left invertible element (in particular,* 1*) is maximal,*
(d) *for $e, f \in P$, $e \wedge f$ is the meet of e and f also in R,*
(e) *if $e, f \in P$, $e \leq f \leq x''$ and $x \leq y$, then $ye \leq yf \leq x$.*

Proof. (a) Evident.

(b) For every $a \in R$, $a \leq 1$ iff $a = a''$ iff $a \in P$.

(c) Suppose that $ya = 1$ for some $y \in R$. Then $1 = (ya)'' \leq a''$ (Proposition 3.2(c)) and, in virtue of (b), $a'' = 1$. Now, if $a \leq z$, then $a'' \leq z''$ (as the homomorphism ϕ is order-preserving) and $a = z(a'' \wedge z'') = zz'' = z$.

(d) Follows from (b): the meet of two elements in an initial segment of a poset is also their meet in the whole poset.

(e) Assume the hypotheses. As then $e \leq y''$, we get that $yf((ye)'' \wedge (yf)'') = yf(e \wedge f) = ye$ (see Proposition 3.2(d)); so, $ye \leq yf$. Likewise $yf \leq yx''$, but $yx'' = y(x'' \wedge y'') = x$ (for $x'' \leq y''$). ∎

Thus, the (partial) meet operation in R is an extension of the operation $\wedge$ in P. Again, we shall use the same symbol also for the extended operation. As every section of the band R is a lower semilattice, we thus have that

$$b(a'' \wedge b'') = a \wedge b = a(a'' \wedge b'') \tag{4.2}$$

for all a, b with $a \mathbin{\downarrow\!\!\circ} b$. Since $b = x(b'' \wedge x'')$ whenever $a, b \leq x$, it follows that $b(a'' \wedge b'') = x(b'' \wedge x'')(a'' \wedge b'') = xb''(a'' \wedge b'') = x(a'' \wedge b'')$. Thus, also

$$a \wedge b = x(a'' \wedge b''). \tag{4.3}$$

The following alternative description of the order $\leq$ will be useful:

$$x \leq y \text{ iff } yx'' = x = xy''.$$

Indeed, if the double identity holds, then $x'' = (xy'')'' \leq y''$ by Proposition 3.2(c), and $y(x'' \wedge y'') = yx'' = x$, i.e., $x \leq y$. Conversely, assume that $x \leq y$. Then, by (4.1), $yx'' = y(y(x'' \wedge y''))'' = y(x'' \wedge y'') = x$. Also, $xy'' = y(x'' \wedge y'')y'' = y(x'' \wedge y'') = x$.

The second identity $x = xy''$ in this description can be further modified using the following easy consequences of (3.1)–(3.3):

$$ab'' = a \text{ iff } ab' = 0 \text{ iff } a''b' = 0 \text{ iff } a''b'' = a'' \text{ iff } a'' \leq b''. \tag{4.4}$$

Remark 2. In [7, Remark 2], the following version $\preceq *$ of the so called right-star order on a Rickart *-ring was announced as an abstraction of this order in *-rings of bounded linear Hilbert space operators:

$$x \preceq * \, y \text{ iff } xx^* = yx^* \text{ and } x'' \leq y''$$

(for some reasons, it was named a left-star order in that paper). By Lemma 3.2(1) of [7], the first term of the defining conjunction here is equivalent, in Rickart *-rings, to $x = yx''$. Therefore, the natural order of the band $(R, \overleftarrow{\wedge})$ turns out to be an analogue of this right-star order in non-involutory Rickart rings; see Introduction.

The preorder $\sqsubseteq$ (see Section 2) is specified in R as follows:

$$x \sqsubseteq y \text{ iff } x = x(x'' \wedge y'').$$

It is easily seen that $a \sqsubseteq b$ if and only if any of the equations in (4.4) holds. For instance, if $a \sqsubseteq b$, then $a'' = (a(a'' \wedge b''))'' = (a''(a'' \wedge b''))'' = a'' \wedge b''$ and $a'' \leq b''$. Conversely, if $a'' \leq b''$, then $a(a'' \wedge b'') = aa'' = a$ and $a \sqsubseteq b$. Therefore,

$$x \sqsubseteq y \text{ iff } x'' \leq y'', \quad x \leq y \text{ iff } x = yx'' \text{ and } x \sqsubseteq y.$$

These characterizations of $\sqsubseteq$ and $\leq$ are specifications of the general equivalences (3) in [5].

Since P may be considered as a nearlattice, the first assertion of the subsequent theorem is a part of a general result [5, Theorem 2.3] on right normal bands (the implication (b) in Proposition 3.5 above is the necessary instance of the condition (4) in [5]). We present here an independent direct proof of the theorem.

Theorem 4.3. *The band R has the upper bound property, hence, it is a right normal skew nearlattice.*

Proof. Assume that $a, b \leq x$. Then $a = xa''$, $b = xa''$ and $a'', b'' \leq x''$; in particular $a'', b'' \in C(x'')$. Let $c := x(a'' \vee b'')$; we are going to show that c is the join of a and b. By Proposition 3.2(d), $c'' = x''(a'' \vee b'') = a'' \vee b''$. It follows that $c(a'' \wedge c'') = x(a'' \vee b'')(a'' \wedge (a'' \vee b'')) = xa'' = a$. Thus $a \leq c$, and likewise $b \leq c$. Suppose that y is any upper bound of a and b; then $a = ya''$, $b = yb''$ and $a'', b'' \leq y''$. Let $z := y(a'' \vee b'')$. By (4.1), $y(z'' \wedge y'') = y((a'' \vee b'') \wedge y'') = y(a'' \vee b'') = z$; thus, $z \leq y$. But $z = c$: as $(x-y)a'' = 0 = (x-y)b''$, Proposition 3.5(b) implies that $(x-y)(a'' \vee b'') = 0$. Therefore, $c \leq y$, i.e., c is indeed the least upper bound of a and b. ∎

In particular, it is seen from the proof that $c := 1(e \vee f) = e \vee f$, the join of two closed idempotents e, f in [0,1], is also their join in R. This means that the (partial) join operation in R is an extension of that in P, and we may use the symbol $\vee$ also for the former one: for all $a, b \leq x$,

$$a \vee b = x(a'' \vee b''). \tag{4.5}$$

It follows from Lemma 4.2(c) that join is a total operation on R if and only if $R = P$: as 1 is a maximal element of R, $1 \vee x = 1$ for every x, i.e., $x \in [0, 1] = P$.

Theorem 4.4. *The mapping ϕ is an idempotent 0-preserving homomorphism from the skew nearlattice $(R, \overline{\wedge}, \vee)$ onto the lattice $(P, \wedge, \vee)$. Moreover, the restriction of ϕ to any section $[0, x]$ is a lattice isomorphism onto $[0, x'']$.*

Proof. By virtue of Proposition 3.2(d), the equality (4.5) implies that

$$(a \vee b)'' = a'' \vee b'', \tag{4.6}$$

where $a \downarrow\!\!\circ\, b$. Due to Theorem 4.1, this observation leads us to the first assertion of the theorem. Further, denote by ϕ_x the restriction of ϕ to $[0, x]$, and consider a mapping $\psi_x \colon e \mapsto xe$ of $[0, x'']$ into R. According to Lemma 4.2(e) (with $y = x$), ψ_x is in fact an order homomorphism from $[0, x'']$ into $[0, x]$. The mappings ϕ_x and ψ_x are mutually inverse: if $a \leq x$, then

($\phi_x(a) \leq x''$ and) $xa'' = a$, and if $e \leq x''$, then $(xe)'' = e$ by Proposition 3.2(e). Therefore, the mappings are bijective, and ϕ_x is a lattice isomorphism. ∎

Observe that $u \overleftarrow{\barwedge} 1 = u''$; it follows that $a'' = b''$ if and only if $a \overleftarrow{\barwedge} x = b \overleftarrow{\barwedge} x$ for all $x \in R$. Therefore, a and b generate the same principal right ideal in $(R, \overleftarrow{\barwedge})$ if and only if $\phi(a) = \phi(b)$, i.e., the kernel congruence of ϕ is the Green's equivalence $\mathcal{R}$ of R. Evidently, the left Green's equivalence $\mathcal{L}$ is the equality relation; therefore, $\mathcal{D} = \mathcal{R}$. (See Section 2 in [12] on Green's equivalences in semigroups.) Actually, $\mathcal{R}$ is even a congruence of the skew nearlattice R, and, by [5, Proposition 2.2], no image of R that is a nearlattice includes P as a proper sublattice.

The homomorphism ϕ is full in the following strong sense: to every pair of elements a, b there are elements a_1, b_1 such that $a_1 \mathcal{R} a$, $b_1 \mathcal{R} b$, $a_1 \mathrel{\overset{\downarrow}{\circ}} b_1$ and, consequently, $\phi(a) \vee \phi(b) = \phi(a_1 \vee b_1)$. In virtue of Proposition 3.2(c) and Lemma 4.2(e), one may put here $a_1 := xa''$ and $b_1 := xb''$, with any x such that $x'' \geq a'', b''$. Indeed, then $a_1'' = (x''a_1)'' = a''$ and similarly $b_1'' = b''$. Also, $a_1, b_1 \leq x$: for instance, $x((xa'')'' \wedge x'') = x(a'' \wedge x'') = xa''$. Therefore, $a_1 \mathrel{\overset{\downarrow}{\circ}} a_2$.

Notice also that the mapping ψ_x is a lattice isomorphism $[0, x''] \to [0, x]$. In addition, the following observation is an immediate consequence of the above theorem.

Corollary 4.5. *If $x'' = y''$, then the lattices $[0, x]$ and $[0, y]$ are isomorphic.*

We end the section with a characterization of some special joins and meets in R. Let as say that two elements x and y *ϕ-commute*, if their supports x'' and y'' commute.

Lemma 4.6. *Suppose that $a \mathrel{\overset{\downarrow}{\circ}} b$ and elements a and b ϕ-commute. Then*

(a) $ab'' = a \wedge b = ba''$,
(b) $a + ba' = a \vee b = b + ab'$,
(c) $a \vee b = a + b - (a \wedge b)$,
(d) $a \vee b = ab' + (a \wedge b) + ba'$.

Proof. The identities (a) and (b) follow from (4.3) and (4.5), respectively, by Lemma 3.3(b,c), while (c) and (d) are consequences of these identities; see (3.2). ∎

5. Sectional orthocomplementations and orthogonality in R

The underlying ring structure of the skew nearlattice R allows us to introduce certain orthocomplementations in every its section.

Theorem 5.1. *R is a relatively orthocomplemented poset with the sectional orthocomplementation in every section $[0, x]$ given by*

$$a^{\perp}_{x} := x - a = xa'.$$

Proof. Choose $x \in R$ and, for every $a \leq x$, let $a^{\perp}_{x} = x - a = x - xa'' = xa'$. As $(x - a)x'' = xx'' - ax'' = x - a$ and, by Proposition 3.2(c), $x(xa')'' = xa'(xa')'' = xa'$, we conclude that $a^{\perp}_{x} \leq x$, i.e., the section $[0, x]$ is closed under the operation $^{\perp}_{x}$ defined in this way. We next check that this operation is an orthocomplementation on $[0, x]$.

Evidently, $0^{\perp}_{x} = x$, $x^{\perp}_{x} = 0$ and $a^{\perp}_{x}{}^{\perp}_{x} = a$. Further, $^{\perp}_{x}$ is antitone. Suppose that $a \leq b \leq x$, then $a'' \leq b'' \leq x''$, so that $b' \leq a'$ and x'' commutes with a'', b'' and a', b'. Consequently, $xa'(xb')'' = xa'x''b' = xx''a'b' = xb'$ and likewise $xb'(xa')'' = xb'$—see Proposition 3.2(d,b). Therefore, $xb' \leq xa'$, i.e., $b^{\perp}_{x} \leq a^{\perp}_{x}$. At last, if $a \leq x$, then $a \wedge a^{\perp}_{x} = 0$, for 0 is the single lower bound of a and xa': suppose that $y \leq a$ and $y \leq xa'$; then $y = ya''$ and $y = xa'y'' = xa'(ya'')'' = xa'a''(ya'')'' = 0$ (again, Proposition 3.2(c,b)).

Since R has the upper bound property, $a \perp_{x} b$ implies that $a \mathrel{\downarrow\!\!\circ} b$. Finally, if $a \leq p \leq q$, then $pa' \leq qa'$ and $a^{\perp}_{p} \leq a^{\perp}_{q}$; this can be proved similarly to antitonicity of $^{\perp}_{x}$. So, R is indeed relatively orthocomplemented. ∎

The relatively orthocomplemented skew nearlattice R turns out to be locally (sectionwise) imbeddable in P.

Corollary 5.2. *Every section $[0, x]$ of R is an orthomodular lattice, which is isomorphic to $[0, x'']$.*

Proof. The first assertion follows immediately from the definition of relatively orthocomplemented poset, while the second one is a consequence of Theorem 4.4. It is easily seen that the lattice isomorphisms ϕ_x from its proof preserve also sectional orthocomplements: if $a \leq x$, then $a'' \leq x''$, $a'', a' \in C(x'')$ and, by Proposition 3.2(d), $\phi(a^{\perp}_{x}) = (xa')'' = x''a' = (\phi_x(a))^{\perp}_{x''}$. ∎

Evidently, $e \perp f$ iff $e \perp_1 f$ provided $e, f \in P$. This means that the induced orthogonality on the relative orthoposet R is an extension of the orthogonality of closed idempotents. Let us denote the induced orthogonality on the sectional orthoposet R by $\perp$:

$$a \perp b \text{ iff there is } x \in R \text{ such that } a, b \leq x \text{ and } a \perp_x b.$$

Then the poset R is quasi-orthomodular with respect to $\perp$ (Proposition 2.1). In the rest of the section, we derive some further properties of this orthogonality relation. (It differs from the standard ring orthogonality defined by $x \perp y$ iff $xy = 0 = yx$, which will not be referred to in this paper.)

Theorem 5.3. *For all $a, b \in R$, $a \perp b$ if and only if $a ⫰ b$ and any of the following conditions is fulfilled:*

(a) $a + b = a \vee b$,
(b) $a \leq a + b$,
(c) $b \leq a + b$,
(d) $a'' \perp b''$,
(e) $a''b'' = b''a''$ *and* $a \wedge b = 0$.

Proof. Notice that, in virtue of (3.3),

$$a'' \perp b'' \text{ iff } ab'' = 0 \text{ iff } ba'' = 0.$$

We shall prove that, in any section $[0, x]$, all conditions (a)–(e) are equivalent to

(f) $a \perp_x b$.

(a)→(b) Evident.

(b)→(c) Suppose that $a \leq a+b$. Then $(a+b)a'' = a$ and, further, $ba'' = 0$. Thus, $ab'' = 0$ (see the beginning of the proof) and $(a + b)b'' = b$. Also, $0 = b(1-a')$, i.e., $b = ba' = (a+b)a'$. It also follows from the supposition that $a'' \leq (a + b)''$; hence $(a + b)''$ commutes with a'' and a' (Lemma 3.3). Now, $b'' = ((a + b)a')'' = ((a + b)''a')'' = (a'(a + b)'')'' \leq (a + b)''$ (see Proposition 3.2). Thus, $(a + b)a'' = a$ and $a'' \leq (a + b)''$; consequently, $b \leq a + b$.

(c)→(d) If $a \leq a + b$, then, in particular, $a = (a + b)a'' = a + ba''$, whence $ba'' = 0$ and $a'' \perp b''$.

(d)→(e) If $a'' \perp b''$, then the first identity in (e) is evident, and 0 is the single lower bound of a and b, since for every c with $c \leq a, b$, $c'' \leq a'' \wedge b'' = a''b'' = 0$ and $c = 0$.

Now assume that $a, b \leq x$. Then (e) implies (a) in virtue of Lemma 4.6(c). Therefore, the conditions (a)–(e) are equivalent. We finally show that (d) and (f) also are equivalent under this assumption.

Notice that $a = xa''$, $b = xb''$, $a'', b'' \leq x''$, and recall that $a \perp_x b$ if and only if $b \leq x - a = xa'$, i.e., if and only if $(x - a)b'' = b = b(xa')''$. But $(x - a)b'' = b$ iff $xb'' - ab'' = b$ iff $b - ab'' = b$ iff $ab'' = 0$ iff $a'' \perp b''$. On the other hand, $b(xa')'' = b$ iff $b(x''a')'' = b$ iff $bx''a' = b$ iff $ba' = b$ iff $ba'' = 0$ iff $a'' \perp b''$ (we have $a'' \leq x''$, from where $a'' \in C(x'')$ and $a' \in C(x'')$; see also Proposition 3.2(d) and (3.2). ∎

It follows immediately that, for $a \leq x$,

$$a^{\perp}_x = \max(b \leq x \colon\ a'' \perp b'') = \max(b \colon\ b \leq x \wedge (a + b)).$$

Also, the following corollary to the theorem is easily seen. For (b), use (a), the identity (4.6) and Lemma 3.3(c). For (c), recall the implication (b)→(c) in the previous proof.

COROLLARY 5.4. *For all a, b,*

(a) *if $a \perp b$, then $a \wedge b = 0$ and $a \vee b = a + b$,*
(b) *if $a \perp b$, then $(a + b)'' = a'' + b''$,*
(c) *$a \perp b$ if and only if $a \leq a + b$.*

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