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TRANSLATIVE PACKING OF UNIT SQUARES INTO EQUILATERAL TRIANGLES

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Abstract. Every collection of n (arbitrary-oriented) unit squares can be packed translatively into any equilateral triangle of side length $2.3755 \cdot \sqrt{n}$.

Let the coordinate system in the Euclidean plane be given. For $0 \le \alpha_i < \pi/2$, denote by $S(\alpha_i)$ a square in the plane with sides of unit length and with an angle between the x-axis and a side of $S(\alpha_i)$ equal to α_i. Furthermore, let $T(s)$ be an equilateral triangle with sides of length s and with one side parallel to the x-axis.

A collection of unit squares admits a *translative packing* into a set C if there are mutually disjoint translated copies of the members of the collection contained in C.

The question of packing of unit squares into squares or triangles (with the possibility of rotations) is a well-known problem (see [1], [3], [4] and [9]).

Some upper bounds concerning translative packing (without the possibility of rotations) of unit squares into a square are given in [6]. Covering problems are discussed in [7]. In this note, we propose the question of translative packing of squares into an equilateral triangle. Denote by t_n the smallest number t such that any collection of n unit squares $S(\alpha_1), \ldots, S(\alpha_n)$ admits a translative packing into $T(t)$. The problem is to find t_n for $n = 1, 2, 3, \ldots$.

Claim 1. $t_1 = \sqrt{2} + \sqrt{2/3} \approx 2.23$.

Proof. Let $\lambda_1 = \sqrt{2} + \sqrt{2/3}$. Obviously, $S(\pi/4)$ cannot be packed translatively into $T(\lambda_1 - \epsilon)$ for any $\epsilon > 0$ (see Fig. 1, left). The squares $S(\pi/12)$ as well as $S(5\pi/12)$ cannot be packed translatively either. Consequently, $t_1 \ge \lambda_1$.

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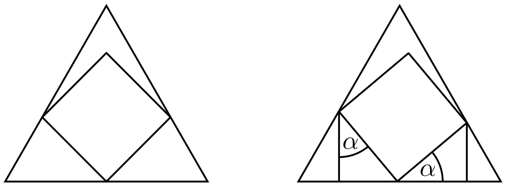

Fig. 1

Let s_α be the smallest number such that $S(\alpha)$ can be packed translatively into $T(s_\alpha)$. At least one angle between a side of $S(\alpha)$ and a side of $T(s_\alpha)$ is between $\pi/6$ and $\pi/3$. Without loss of generality, we can assume that $\frac{\pi}{6} \le \alpha \le \frac{\pi}{3}$. By Fig. 1 (right) we see that

$$s_\alpha = \left(1 + \frac{\sqrt{3}}{3}\right)(\cos\alpha + \sin\alpha).$$

It is easy to check that this value is not greater than

$$\left(1 + \frac{\sqrt{3}}{3}\right)\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right) = \left(1 + \frac{\sqrt{3}}{3}\right)\cdot\sqrt{2} = \lambda_1.$$

This implies that $t_1 \le \lambda_1$. ∎

Claim 2.

$$t_2 = \frac{2\sqrt{3}}{3}\cos[\arctan(\sqrt{3}+1)] + \left(2 + \frac{2\sqrt{3}}{3}\right)\sin[\arctan(\sqrt{3}+1)] \approx 3.359.$$

Proof. Put $\gamma = \arctan(\sqrt{3}+1)$, let $S(\alpha)$ and $S(\beta)$ be unit squares and let

$$\sigma = \frac{2\sqrt{3}}{3}\cos\gamma + \left(2 + \frac{2\sqrt{3}}{3}\right)\sin\gamma$$
$$= \sqrt{\frac{20 - 8\sqrt{3}}{39}} + 4\sqrt{\frac{11 + 6\sqrt{3}}{39}} \approx 3.359.$$

Part I of the proof. First we show that $S(\alpha)$ and $S(\beta)$ can be packed translatively into $T(\sigma)$, i.e., that $t_2 \le \sigma$. We can assume that

$$0 \le \beta \le \alpha < \pi/2.$$

We say that a square S is *packed along a side* of a triangle T provided $S \subset T$ and a vertex of S belongs to this side.

Case 1: $\alpha \ge \pi/3$, $\beta \le \pi/6$ and $\alpha + \beta \ge \pi/2$.

Let $T(\lambda_5)$ be a triangle with vertices $(0,0)$, $(\lambda_5, 0)$, $(\frac{1}{2}\lambda_5, \frac{\sqrt{3}}{2}\lambda_5)$, where (see Fig. 2, left)

$$\lambda_5 = \frac{\cos\beta - \sin\alpha}{\tan\alpha} + \frac{\sin\beta}{\sqrt{3}} + \left(1 + \frac{\sqrt{3}}{3}\right)\cos\beta + \frac{\cos\alpha}{\sqrt{3}} + \left(1 + \frac{\sqrt{3}}{3}\right)\sin\alpha.$$

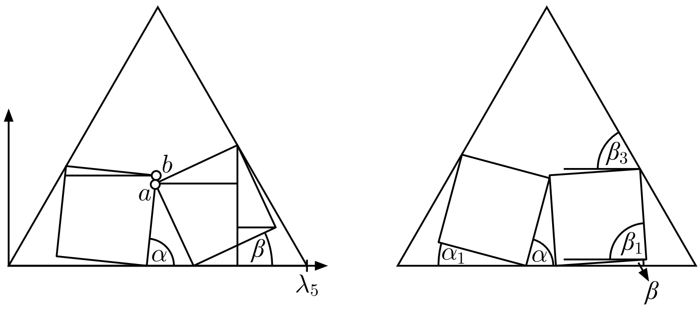

Fig. 2

Since

$$y_b - y_a = (x_b - x_a)\tan\alpha,$$

where $a(x_a, y_a)$, $b(x_b, y_b)$,

$$x_a = \lambda_5 - \frac{\sin\beta + \cos\beta}{\sqrt{3}} - \cos\beta,\ \ y_a = \cos\beta,$$

$$x_b = \frac{\sin\alpha + \cos\alpha}{\sqrt{3}} + \sin\alpha,\ \ y_b = \sin\alpha,$$

it follows that $S(\alpha)$ and $S(\beta)$ can be packed translatively into $T(\lambda_5)$. It can be shown that λ_5 is maximal at $\alpha + \beta = \pi/2$. Consequently, replacing β by $\pi/2 - \alpha$ we obtain

$$\lambda_5 \le f_1(\alpha) = \frac{2\cos\alpha}{\sqrt{3}} + 2\Big(1 + \frac{\sqrt{3}}{3}\Big)\sin\alpha.$$

By

$$f_1'(\alpha) = \frac{-2\sin\alpha}{\sqrt{3}} + 2\Big(1 + \frac{\sqrt{3}}{3}\Big)\cos\alpha,$$

we have $\lambda_5 \le f_1(\gamma) = \sigma$. This implies that $S(\alpha)$ and $S(\beta)$ can be packed translatively into $T(\sigma)$.

Case 2: $\alpha \ge \pi/3$, $\beta \le \pi/6$ and $\alpha + \beta < \pi/2$.

If $\alpha_1 = \pi/2 - \alpha$ and $\beta_1 = \pi/2 - \beta$, then $\alpha_1 \le \pi/6$, $\beta_1 \ge \pi/3$ and $\alpha_1 + \beta_1 = \pi - (\alpha + \beta) > \pi/2$ (see Fig. 2, right). By Case 1, we conclude that $S(\alpha)$ and $S(\beta)$ can be packed translatively into $T(\sigma)$.

Case 3: $\alpha \ge \pi/3$, $\beta \ge \pi/3$.

Let $T(\lambda_6)$ be a triangle with vertices $(0,0)$, $(\lambda_6, 0)$, $(\frac{1}{2}\lambda_6, \frac{\sqrt{3}}{2}\lambda_6)$, where (see Fig. 3, left)

$$\lambda_6 = \frac{\sin\alpha + \cos\alpha - \sin\beta}{\tan\alpha} + \frac{\sin\beta + \cos\beta}{\sqrt{3}} + \sin\beta + \Big(1 + \frac{\sqrt{3}}{3}\Big)\sin\alpha.$$

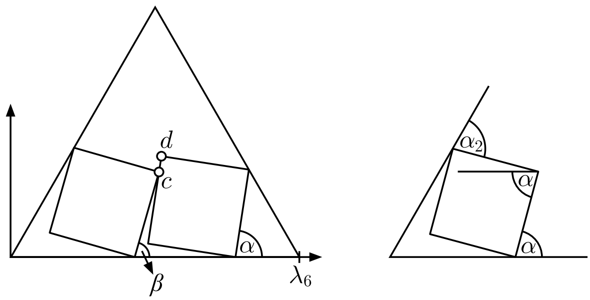

Fig. 3

Since

$$y_d - y_c = (x_d - x_c)\tan\alpha,$$

where $c(x_c, y_c),\ d(x_d, y_d)$,

$$x_c = \frac{\sin\beta + \cos\beta}{\sqrt{3}} + \sin\beta,\ \ y_c = \sin\beta,$$
$$x_d = \lambda_6 - \frac{\sin\alpha}{\sqrt{3}} - \sin\alpha,\ \ y_d = \sin\alpha + \cos\alpha,$$

it follows that $S(\alpha),\ S(\beta)$ can be packed translatively into $T(\lambda_6)$.

By $\frac{\partial\lambda_6}{\partial\alpha} \le 0$ and $\alpha \ge \beta$, replacing α by β, we obtain

$$\lambda_6 \le f_2(\beta) = \frac{\cos^2\beta}{\sin\beta} + \frac{\cos\beta}{\sqrt{3}} + \left(2 + \frac{2\sqrt{3}}{3}\right)\sin\beta.$$

It can be shown that $f_2'(\beta) < 0$ provided $\pi/3 \le \beta \le \pi/2$. This implies that

$$\lambda_6 \le f_2\left(\frac{\pi}{3}\right) = 1 + \frac{4}{3}\sqrt{3} \approx 3.309 < \sigma.$$

Case 4: $\alpha \le \pi/6,\ \beta \le \pi/6$.

If $\alpha_1 = \pi/2 - \alpha$ and $\beta_1 = \pi/2 - \beta$, then $\alpha_1 \ge \pi/3$ and $\beta_1 \ge \pi/3$. By Case 3, we conclude that $S(\alpha)$ and $S(\beta)$ can be packed translatively into $T(\sigma)$.

Case 5: $\alpha \ge \pi/3,\ \pi/6 < \beta < \pi/3$.

Observe that angles between a side of $S(\alpha)$ and $S(\beta)$ and the left-hand side of $T(\sigma)$ are equal to $\alpha_2 = \frac{\pi}{3} + \frac{\pi}{2} - \alpha = \frac{5}{6}\pi - \alpha > \frac{\pi}{3}$ and $\beta_2 = \frac{\pi}{3} - \beta < \frac{\pi}{6}$, respectively (see Fig. 3, right and Fig. 4, left). By Cases 1 and 2, we deduce that the squares can be packed translatively into $T(\sigma)$ along the left-hand side of this triangle.

Case 6: $\pi/6 < \alpha < \pi/3,\ \pi/6 < \beta < \pi/3$.

Observe that angles between a side of $S(\alpha)$ and $S(\beta)$ and the left-hand side of $T(\sigma)$ are equal to $\alpha_2 = \pi/3 - \alpha < \pi/6$ and $\beta_2 = \pi/3 - \beta < \pi/6$, respectively. By Case 4, we deduce that the squares can be packed translatively into $T(\sigma)$.

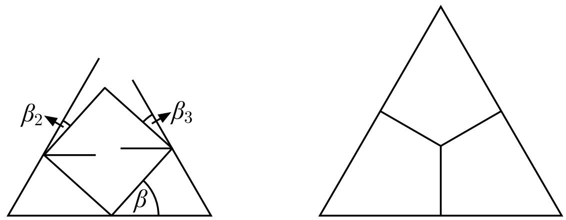

Fig. 4

Case 7: $\pi/6 < \alpha < \pi/3$, $\beta \le \pi/6$.

The angles between a side of $S(\alpha)$ and $S(\beta)$ and the right-hand side of $T(\sigma)$ are equal to $\alpha_3 = \frac{\pi}{3} - \left(\frac{\pi}{2} - \alpha\right) = \alpha - \frac{\pi}{6} < \frac{\pi}{6}$ (see $\beta_3 = \beta - \frac{\pi}{6}$ in Fig. 4, left) and $\beta_3 = \frac{\pi}{3} + \beta > \frac{\pi}{3}$ (see β_3 in Fig. 2, right), respectively. By Cases 1 and 2, we deduce that the squares can be packed translatively along the right-hand side of $T(\sigma)$.

Part II of the proof. We show that $S(\gamma)$ and $S(\delta)$, where $\gamma = \arctan(\sqrt{3}+1)$ and $\delta = \frac{\pi}{2} - \gamma = \arctan\frac{\sqrt{3}-1}{2}$, cannot be packed translatively into $T(\sigma - \epsilon)$ for any $\epsilon > 0$, i.e., that $t_2 \ge \sigma$.

It is sufficient to consider only packings along sides of $T(\sigma)$ (see Fig. 5 and Fig. 6). The angles between a side of $S(\gamma)$ and $S(\delta)$ and the left-hand side of $T(\sigma)$ are equal to $\gamma_2 = \frac{5}{6}\pi - \gamma$ and $\delta_2 = \frac{\pi}{3} - \delta$, respectively. The angles between a side of $S(\gamma)$ and $S(\delta)$ and the right-hand side of $T(\sigma)$ are equal to $\gamma_3 = \gamma - \frac{\pi}{6}$ and $\delta_3 = \frac{\pi}{3} + \delta$, respectively. Since $\gamma_3 = \delta_2$ and $\delta_3 = \gamma_2$, it suffices to consider four possibilities of packings (packings along the right-hand side of $T(\sigma)$ give the same results as packings along the left-hand side).

Let $\epsilon > 0$. By Case 1, we conclude that $S(\gamma)$ and $S(\delta)$ cannot be packed translatively along the base of $T(\sigma - \epsilon)$ so that $S(\gamma)$ is to the left of $S(\delta)$ (see Fig. 5, left).

Let

$$\lambda_7 = 2\left(\sin\delta + \cos\delta + \frac{\cos\delta}{\sqrt{3}}\right) \approx 3.6499 > \sigma.$$

By Fig. 5 (right), we see that $S(\gamma)$ and $S(\delta)$ cannot be packed translatively along the base of $T(\lambda_7 - \epsilon)$ so that $S(\gamma)$ is to the right of $S(\delta)$.

Let

$$\lambda_8 = \frac{\cos\delta_2 - \sin\gamma_2}{\tan\gamma_2} + \left(1 + \frac{\sqrt{3}}{3}\right)\sin\delta_2 + \cos\gamma_2 + \left(1 + \frac{\sqrt{3}}{3}\right)\sin\gamma_2 + \frac{\cos\gamma_2}{\sqrt{3}} \approx 3.394 > \sigma.$$

By Fig. 6 (left), we see that $S(\gamma)$ and $S(\delta)$ cannot be packed translatively along the left-hand side of $T(\lambda_8 - \epsilon)$ so that $S(\gamma)$ is lower than $S(\delta)$.

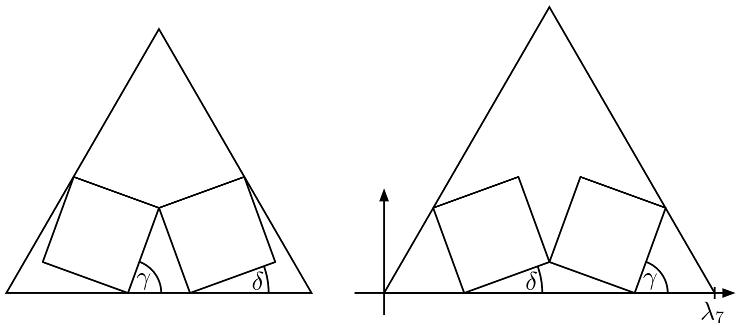

Fig. 5

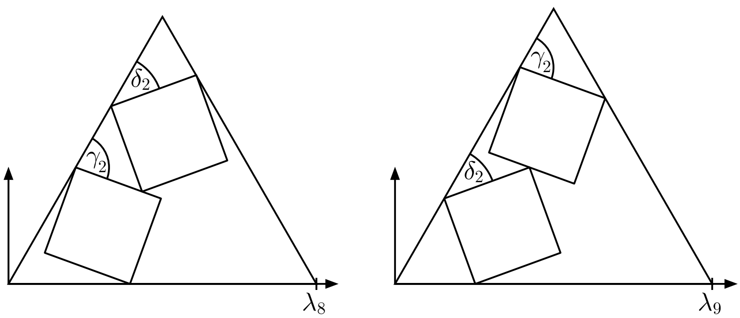

Fig. 6

Finally, let

$$\lambda_9 = \frac{\cos\gamma_2 - \sin\delta_2}{\tan\gamma_2} + \sin\delta_2 + \left(1 + \frac{\sqrt{3}}{3}\right)\cos\delta_2$$
$$+ \cos\gamma_2 + \left(1 + \frac{\sqrt{3}}{3}\right)\sin\gamma_2 \approx 3.495 > \sigma.$$

By Fig. 6 (right), we see that $S(\gamma)$ and $S(\delta)$ cannot be packed translatively along the left-hand side of $T(\lambda_9 - \epsilon)$ so that $S(\gamma)$ is higher than $S(\delta)$. ■

Results presented in Claims 3, 4 and 5 will be applied in the proof of the Theorem.

Claim 3.

$$t_3 \le 2\left(1 + \frac{\sqrt{3}}{3}\right)\cos\left(\arctan\frac{3-\sqrt{3}}{2}\right) + 2\sin\left(\arctan\frac{3-\sqrt{3}}{2}\right) \approx 3.73526.$$

Proof. Put $\vartheta = \arctan\frac{3-\sqrt{3}}{2}$, and let Q be a deltoid with vertices

$$(0,0),\ (x_p,0),\ (3x_p/2,(1+\sqrt{3}/2)x_p),\ (0,y_q),$$

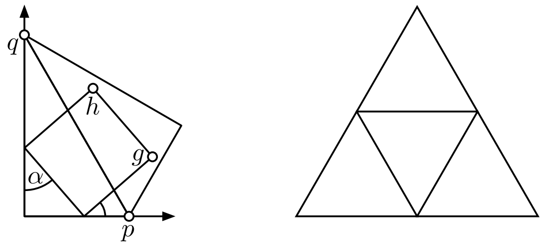

Fig. 7

where $y_q = \left(1+\sqrt{3}/3\right)\cos\vartheta + \sin\vartheta \ \approx 1.8676$ and $x_p = \sqrt{3}y_q/3 \approx 1.078$ (see Fig. 7, left).

Since any triangle $T(2y_q)$ can be divided into three deltoids congruent to Q (see Fig. 4, right), it follows that to prove Claim 3 it suffices to show that any unit square can be packed translatively into Q.

Let $S(\alpha)$ be a unit square. There is no loss of generality in assuming that $\pi/6 \le \alpha \le \pi/2$ (the straight line going through the points p and q is the axis of symmetry of Q).

A simple calculation shows that

$$y_h \le -\frac{\sqrt{3}}{3}x_h + y_q,$$

where $x_h = \cos\alpha,\ y_h = \sin\alpha + \cos\alpha$. Moreover,

$$y_g > \sqrt{3}(x_g - x_p),$$

where $x_g = \cos\alpha + \sin\alpha$, $y_g = \sin\alpha$. This implies that $S(\alpha)$ can be packed translatively into Q. ■

Obviously, $T(t)$ can be divided into four equilateral triangles of side lengths $\frac{1}{2}t$ (see Fig. 7, right).

Claim 4. $t_4 \le 2t_1 = 2(\sqrt{2} + \sqrt{2/3}) \approx 4.46$.

Claim 5. $t_7 \le 2\sqrt{2} + \frac{4}{3}\sqrt{6} \approx 6.0944$.

Proof. Melissen (see [8]) found the densest packing of seven equal circles into an equilateral triangle. We apply a part of its arrangement. We pack five circles of radius $r = \sqrt{2}/2$ as in Fig. 8 (left). The length of the segment uv equals $r(2+2\sqrt{3}) = \sqrt{2}+\sqrt{6}$. Since any unit square can be packed translatively into any circle of radius $\sqrt{2}/2$, by Claim 1 we deduce that

$$t_7 \le \sqrt{2} + \sqrt{6} + t_1 = 2\sqrt{2} + \frac{4}{3}\sqrt{6}. \ ■$$

Theorem. *Let n be a positive integer. Then*

$$t_n \le \left(\sqrt{\frac{10-4\sqrt{3}}{39}} + 4\sqrt{\frac{11+6\sqrt{3}}{78}}\right)\sqrt{n}.$$

Furthermore, the equality holds only for $n = 2$.

Proof. Put

$$\xi = \sqrt{\frac{10-4\sqrt{3}}{39}} + 4\sqrt{\frac{11+6\sqrt{3}}{78}} \approx 2.3754.$$

By Claims 1–4 we know that

$$t_n < \xi\sqrt{n} \quad \text{for} \quad n \in \{1, 3, 4\},$$

and that

$$t_2 = \xi\sqrt{2}.$$

Now assume that $5 \le n \le 15$. Denote by μ_n, the smallest number s such that n circles of unit radius can be packed into $T(s)$. The problem of minimizing the side of an equilateral triangle into which n congruent circles can be packed is a well-known question. The values of μ_n are known, among others, for $n < 16$ (see [8]). We know that:

$$\mu_5 \le \mu_6 < 7.47,\ \mu_8 \le \mu_9 \le \mu_{10} < 9.47,$$

$$\mu_{11} \le \mu_{12} < 10.93,\ \mu_{13} \le \mu_{14} \le \mu_{15} < 11.47.$$

Each unit square is contained in a circle of radius $\sqrt{2}/2$. Consequently, n unit squares can be packed translatively into $T(\sqrt{2}\mu_n/2)$. It is easy to verify that

$$t_n \le \frac{\sqrt{2}}{2}\mu_n < 2.37\sqrt{n}$$

for $n \in \{5, \dots, 15\} \setminus \{7\}$.

Unfortunately, $\mu_7 = 2 + 4\sqrt{3}$ and $\sqrt{2}\mu_7/2 > \xi\sqrt{7}$. The inequality $t_7 < 2.37\sqrt{7}$ follows from Claim 5.

Obviously, $T(t)$ can be divided into four equilateral triangles of side lengths $\frac{1}{2}t$ (see Fig. 7, right). Consequently,

$$t_{16} \le 2t_4 < 2.37\sqrt{16}.$$

Finally assume that $n \ge 17$.

There exists an integer m such that

$$\frac{(m-1)m}{2} < n \le \frac{m(m+1)}{2}.$$

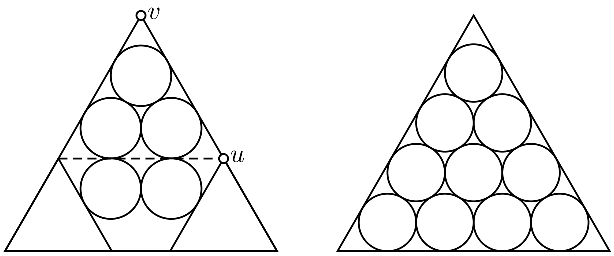

Fig. 8

By Fig. 8 (right), we deduce that $\frac{m(m+1)}{2}$ circles of radius $\sqrt{2}/2$ can be packed into $T(\sqrt{2}(m-1)+\sqrt{6})$. Since

$$\frac{(m-1)m}{2}+1\le n,$$

it follows that

$$m\le \frac{1+\sqrt{8n-7}}{2}.$$

Thus

$$t_n\le \sqrt{2}\left(\frac{1+\sqrt{8n-7}}{2}-1\right)+\sqrt{6}.$$

It is easy to check that the value on the right side of this inequality is smaller than $\xi\sqrt{n}$ provided $n\ge 17$. ■

Remark. Arguing as in the proof of Theorem 2 of [6] or Theorem 7 of [5], we can show that

$$\lim_{n\to\infty}\frac{\sqrt{3}t_n^2}{4n}=1.$$

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