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FIXED POINT THEOREMS ON GENERALIZED METRIC SPACES FOR MAPPINGS IN A CLASS OF ALMOST φ-CONTRACTIONS

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Abstract. In this paper, we obtain some new fixed point theorems in generalized metric spaces for maps satisfying an implicit relation. The obtained results unify, generalize, enrich, complement and extend a multitude of related fixed point theorems from metric spaces to generalized metric spaces.

1. Introduction and preliminaries

In 2000, Branciari [7] initiated the notion of a generalized metric space (shortly gms) as a generalization of a metric space in such a way that the triangle inequality is replaced by the *Tetrahedral inequality*. Starting with the paper of Branciari, many fixed point results have been established in those interesting spaces (see [1], [8], [9], [10], [13], [14], [17], [18]).

As in the metric spaces, any generalized metric space is a topological space with a neighborhood basis given by $B = \{B(x,r) : x \in X, r > 0\}$, where $B(x,r) = \{y \in X : d(x,y) < r\}$ is the "open" ball with center x and radius r. This topology fails to provide some useful topological properties: an "open" ball in generalized metric space need not be open set, the generalized metric need not be continuous, a convergent sequence in generalized metric space need not be Cauchy, the generalized metric space need not be Hausdorff and hence, the uniqueness of limits cannot be guaranteed (see Example 1.3).

The above properties of generalized metric spaces, that do not hold for metric spaces, were first observed by Das and Dey [10], [11] and also these facts were observed independently by Samet [17], Sarma, and Rao and Rao [18]. Initially, these were considered to be true, implying incorrect

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proofs of several theorems. Thus, some of the previous results had to be reconsidered and to corrected.

Under these circumstances, not every fixed point theorem from metric spaces can be extended to gms. This extension may be done even in this case, the proof of such theorem is more complicated than in the usual settings. Because of those difficulties, some authors have taken the Hausdorffity as additional condition in their theorems, but this is not always necessary. For example, the assertion in [18] that the space needs to be Hausdorff, is superfluous. This fact was first noted by Kikina and Kikina (see [13], [14]).

The aim of this paper is to present some fixed point theorems in generalized metric spaces for self maps in a class of almost contractions defined by an implicit relation. Our results unify, generalize, enrich and complement a multitude of related fixed point theorems for metric spaces and extend them in gms.

Let us recall the notion of a generalized metric space.

Definition 1.1. [7] Let X be a set and $d : X^2 \to R^+$ be a mapping such that for all $x, y \in X$ and for all distinct points $z, w \in X$, each of them different from x and y, one has

(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, y) \leq d(x, z) + d(z, w) + d(w, y)$ (*Tetrahedral inequality*).

Then d is called a generalized metric and (X, d) is a generalized metric space (or shortly gms).

Definition 1.2. [7] Let (X, d) be a gms, let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) We say that $\{x_n\}$ is a gms convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. We denote this by $x_n \to x$.
(2) We say that $\{x_n\}$ is a gms Cauchy sequence if and only if for each $\varepsilon > 0$ there exists a natural number $n(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m > n > n(\varepsilon)$.
(3) (X, d) is called a complete gms if every gms Cauchy sequence is gms convergent in X.

The following example shows us some of the properties of metric spaces which do not hold in gms.

Example 1.3. Let $X = \left\{1 - \frac{1}{n} : n = 1, 2, \ldots \right\} \cup \{1, 2\}$. Define $d : X \times X \to R$ as follows:

$$d(x,y)=\begin{cases}0, & \text{for } x=y,\\ \frac{1}{n}, & \text{for } x\in\{1,2\} \text{ and } y=1-\frac{1}{n} \text{ or } y\in\{1,2\}\\ & \text{and } x=1-\frac{1}{n},\ x\neq y,\\ 1, & \text{otherwise.}\end{cases}$$

Then it is easy to check that (X,d) is a generalized metric space but it is not a metric space because it lacks the *triangular inequality*: $1=d(\frac{1}{2},\frac{2}{3})>d(\frac{1}{2},1)+d(1,\frac{2}{3})=\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$.

Note also that the sequence $\{x_n\}\subset X, x_n=1-\frac{1}{n}, n\geq 1$, converges to both 1 and 2 and it is not a Cauchy sequence because $d(x_n,x_m)=d(1-\frac{1}{n},1-\frac{1}{m})=1,\forall n,m\in N, n\neq m$. The ball $B(\frac{2}{3},\frac{2}{3})=\{\frac{2}{3},1,2\}$ is not an open set because for every $r>0$, $B(1,r)\not\subset B(\frac{2}{3},\frac{2}{3})$.

The function d of Example 1.3 is not continuous in the sense presented in [7], since, although $\lim\limits_{n\to\infty}(1-\frac{1}{n})=1$, we have $\lim\limits_{n\to\infty} d(1-\frac{1}{n},\frac{1}{2})=1\neq\frac{1}{2}=d(1,\frac{1}{2})$.

The (X,d) is not a Hausdorff space because $B(1,r_1)\cap B(2,r_2)\neq\phi$, for all $r_1,r_2>0$.

The following lemma will be useful for us to prove the main theorems.

Lemma 1.4. *Let (X,d) be a generalized metric space, let $\{x_n\}$ be a sequence of distinct points ($x_n\neq x_m$ for all $n\neq m$) in X and $l\geq 0$. If*

(i) $d(x_n,x_{n+1})\leq\delta^n l, 0\leq\delta<1,\forall n\in N$ *and*
(ii) $\lim\limits_{n\to\infty} d(x_n,x_{n+2})=0$,

then $\{x_n\}$ is a Cauchy sequence.

Proof. If $m>2$ is odd, then writing $m=2k+1, k\geq 1$ (by Tetrahedral inequality) we can easily show that

$$\begin{aligned}&d(x_n,x_{n+m})\\ &\quad\leq[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+3})+\cdots+d(x_{n+m-1},x_{n+m})]\\ &\quad\leq\delta^n l+\delta^{n+1}l+\delta^{n+2}l+\cdots+\delta^{n+m-1}l=\delta^n l\frac{1-\delta^m}{1-\delta}<\delta^n\frac{l}{1-\delta}.\end{aligned}$$

Hence, $\lim\limits_{n\to\infty} d(x_n,x_{n+m})=0$.

If $m>2$ is even then writing $m=2k, k\geq 2$ and using the same arguments as before, we can get

$$\begin{aligned} d(x_n, x_{n+m}) &\leq [d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) \\ &\quad + d(x_{n+3}, x_{n+4}) + \cdots + d(x_{n+m-1}, x_{n+m})] \\ &\leq d(x_n, x_{n+2}) + \delta^{n+2} l + \delta^{n+3} l + \cdots + \delta^{n+m-1} l \\ &= d(x_n, x_{n+2}) + \delta^{n+2} l \frac{1-\delta^{m-1}}{1-\delta} \\ &< d(x_n, x_{n+2}) + \delta^{n+2} \frac{l}{1-\delta}. \end{aligned}$$

Also, $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0$. It implies that $\{x_n\}$ is a Cauchy sequence in X. ■

Let $T : X \to X$ be a mapping where X is a gms. For each $x \in X$, let

$$O(x) = \{x, Tx, T^2x, \dots\},$$

which will be called the orbit of T at x. (X, d) is called T-orbitally complete if and only if every Cauchy sequence, which is contained in $O(x)$, converges to a point in X.

Berinde, in the paper [2], introduced a class of contractive mappings initially called weak contractions, for which Berinde and Pacurar later adopted the more suggestive term of almost contractions [4]. On the other hand, in 2011, Berinde [5] obtained fixed point theorems of implicit almost contractions in metric spaces.

Definition 1.5. ([2], [4]) Let (X, d) be a metric space. A map $T : X \to X$ is called weak (almost) contraction if there exists a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta.d(x, y) + L.d(y, Tx) \text{ for all } x, y \in X.$$

2. A class of implicit relations

In this section, we consider a class of implicit relations which will give a general character to the main results of this paper.

Definition 2.1. The set of all real functions $\varphi : R_+^6 \to R$, which are *upper semi-continuous in each coordinate variable* and satisfy at least one of the following conditions:

(a) if $\varphi(u, v, v, u, u, 0) \leq 0$ for all $u, v \geq 0$, then there exists a real constant $h \in [0, 1)$ such that $u \leq hv$,

(b) if $\varphi(u, v_1, v_2, v_3, 0, v_4) \leq 0$ for all $u, v_1, v_2, v_3, v_4 \geq 0$, then there exists a real constant $\delta \in [0, 1)$ and some $L \geq 0$ such that

$$u \leq \delta \max\{v_1, v_2, v_3, v_4\} + Lv_4,$$

(c) $\varphi(u, u, 0, 0, u, u) \leq 0 \Rightarrow u = 0$, will be denoted by Φ_6 and every such function will be called a Φ_6-function.

Some examples of Φ_6-functions are as follows:

Example 2.2. Let $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2$, where $0 \le a < 1$, then φ is a Φ_6-function and satisfies the conditions (a), (b) and (c) with constants $h = \delta = a$ and $L = 0$.

Example 2.3. Let $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - b(t_2 + t_3)$, where $0 \le b < \frac{1}{2}$, then φ is a Φ_6-function and satisfies the conditions (a), (b) and (c) with constants $h = \delta = 2b < 1$ and $L = 0$.

Example 2.4. Let $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - b(t_3 + t_4) - ct_6$, where $0 \le b < \frac{1}{2}$ and $c \ge 0$, then φ is a Φ_6-function and satisfies the conditions (a) and (b) with constants $h = \frac{b}{1-b} < 1$, $\delta = 2b$ and $L = c$, but, in general, does not satisfy the condition (c).

Example 2.5. Let $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - bt_6$, where $0 \le a < 1$ and $b \ge 0$, then φ is a Φ_6-function and satisfies the conditions (a) and (b) with constants $h = \delta = a < 1$ and $L = b$, but, in general, does not satisfy the condition (c).

Example 2.6. Let $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - bt_3$, where $0 \le a < 1$ and $b \ge 0$, then φ is a Φ_6-function and satisfies the condition (c), but, in general, does not satisfy the conditions (a) and (b) (satisfies (a) if $a + b < 1$ and (b) if $a + b < \frac{1}{2}$).

Example 2.7. Let $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \max\{t_3, t_4\} - bt_6$, where $0 \le a < 1$ and $b \ge 0$, then φ is a Φ_6-function and satisfies the conditions (a) and (b) with constants $h = \delta = a < 1$ and $L = b$, but, in general, does not satisfy the condition (c) (satisfies (c) if $b = 0$).

Example 2.8. Let $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - (at_2 + bt_3 + ct_4) + dt_6$, where a, b, c and d are nonnegative numbers such that $a + b + c < 1$, then φ is a Φ_6-function and satisfies the conditions (a) and (b) with constants $h = \frac{a+b}{1-c} < 1, \delta = a + b + c < 1$ and $L = d$, but, in general, does not satisfy the condition (c) (satisfies (c) if $d = 0$).

Example 2.9. $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a[\max\{t_2 t_3, t_3 t_4, t_4 t_5\}]^{\frac{1}{2}}$ $-b \min\{t_2, t_3, t_4, t_5, t_6\}$, where $0 \le a < 1$ and $b \ge 0$.

Example 2.10. $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \max\{t_2, \frac{t_3+t_4}{2}, t_5, t_6\} - bt_6$, where $0 \le a < 1$ and $b \ge 0$.

Example 2.11. $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \max\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\} - bt_4 t_5 t_6$, where $0 \le a < 1$ and $b \ge 0$.

Example 2.12. $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a\sqrt{t_2 t_3} - bt_6$, where $0 \le a < 1$ and $b \ge 0$.

Example 2.13. $\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1-(a_2t_2^p+a_3t_3^p+a_4t_4^p+a_5t_5^p)^{\frac{1}{p}}-Lt_6$, where $p>0, L\geq 0$ and $0\leq a_i, \sum_{i=2}^{5}a_i<1$, etc.

3. Main results

Before stating the main fixed point theorems, we introduce the definition of almost φ-contraction.

Definition 3.1. Let (X,d) be a generalized metric space and $\varphi\in\Phi_6$. A map $T:X\to X$ is called an almost φ-contraction if

(1) $$\varphi\ [d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(y,T^2x),d(y,Tx)]\leq 0,$$

for all $x,y\in X$.

Now we can state and prove the first main theorem of this paper.

Theorem 3.2. *Let (X,d) be a gms, $\varphi\in\Phi_6$ and let $T:X\to X$ be an almost φ-contraction. If φ satisfies the conditions (a) and (b), and (X,d) is T-orbitally complete, then*

(1) *$Fix(T)=\{x\in X: Tx=x\}\neq\phi$.*
(2) *For any $x_0\in X$, the Picard iteration $\{x_n\}$ defined by $x_n=Tx_{n-1}, n=1,2,\ldots$ converges to some $\alpha\in Fix(T)$.*

Proof. Let x_0 be an arbitrary point in X and $\{x_n\}$ defined by $x_n=Tx_{n-1}, n=1,2,\ldots$ be the Picard iteration. By condition (1),

$$\varphi(d(Tx_{n-1},Tx_n),d(x_{n-1},x_n),d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n),d(x_n,T^2x_{n-1}), d(x_n,Tx_{n-1}))$$
$$=\varphi(d(x_n,x_{n+1}),d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}),d(x_n,x_{n+1}),0)\leq 0$$

and by property (a) of φ, there exists $\delta\in(0,1)$ such that

(2) $$d(x_n,x_{n+1})\leq\delta d(x_{n-1},x_n).$$

In general, we have

(3) $$d(x_n,x_{n+1})\leq\delta^n d(x_0,x_1)=\delta^n l, n\in N,$$

where $l=d(x_0,x_1)$, and so

(4) $$\lim_{n\to\infty}d(x_n,x_{n+1})=0.$$

By condition (1),

$$\varphi(d(Tx_{n-1},Tx_{n+1}),d(x_{n-1},x_{n+1}),d(x_{n-1},Tx_{n-1}),d(x_{n+1},Tx_{n+1}), d(x_{n+1},T^2x_{n-1}),d(x_{n+1},Tx_{n-1}))$$
$$=\varphi(d(x_n,x_{n+2}),d(x_{n-1},x_{n+1}),d(x_{n-1},x_n),d(x_{n+1},x_{n+2}),0,d(x_{n+1},x_n))\leq 0.$$

By property (b) of φ, there exists a real constant $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(x_n, x_{n+2}) \leq \delta \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2}), d(x_{n+1}, x_n)\} + Ld(x_{n+1}, x_n).$$

By (3), we get

$$\begin{aligned} d(x_n, x_{n+2}) &\leq \delta \max\{d(x_{n-1}, x_{n+1}), \delta^{n-1}l, \delta^{n+1}l, \delta^n l\} + L\delta^n l \\ &= \max\{\delta d(x_{n-1}, x_{n+1}), \delta^n l\} + L\delta^n l \end{aligned}$$

and hence

$$d(x_n, x_{n+2}) \leq \max\{\delta d(x_{n-1}, x_{n+1}), \delta^n l\} + L\delta^n l. \tag{5}$$

If we denote

$$y_n := \max\{d(x_{n-1}, x_{n+1}), \delta^{n-1}l\},$$

then by (5), we deduce that $\{y_n\}$ satisfies

$$y_{n+1} \leq \delta y_n + L\delta^n l, \ \forall n \in N,$$

thus, since $\lim_{n\to\infty} \delta^n = 0$, by Lemma 1.6 in [3], we obtain $y_n \to 0$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{6}$$

We divide the proof into two cases:

Case I: Suppose that $x_p = x_q$ for some $p, q \in N$, $p \neq q$. Let $p > q$.

Then $T^p x_0 = T^{p-q}T^q x_0 = T^q x_0$ i.e. $T^n\alpha = \alpha$ where $n = p - q$ and $T^q x_0 = \alpha$. Now if $n > 1$, by (3), we have

$$d(\alpha, T\alpha) = d(T^n\alpha, T^{n+1}\alpha) \leq \delta^n d(\alpha, T\alpha).$$

Since $0 < \delta < 1$, $d(\alpha, T\alpha) = 0$. So $T\alpha = \alpha$ and hence α is a fixed point of T.

Case II: Assume that $x_n \neq x_m$ for all $n \neq m$.

Then, from (3), (6) and Lemma 1.4 it follows that $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) *is T-orbitally complete*, there exists an $\alpha \in X$ such that

$$\lim_{n\to\infty} x_n = \alpha. \tag{7}$$

The limit α is unique: Assume that $\alpha' \neq \alpha$ and $\alpha' = \lim_{n\to\infty} x_n$.

Since $x_n \neq x_m$ for all $n \neq m$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \neq \alpha$ and $x_{n_k} \neq \alpha'$ for all $k \in N$. Without loss of generality, assume that $\{x_n\}$ is this subsequence. Then by *tetrahedral inequality* of Definition 1.1, we obtain

$$d(\alpha, \alpha') \leq d(\alpha, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, \alpha').$$

Letting n tend to infinity, we get $d(\alpha, \alpha') = 0$ and so $\alpha = \alpha'$.

Let us prove now that α is a fixed point of T. In contrary, if $\alpha \neq T\alpha$, then there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \neq T\alpha$ and $x_{n_k} \neq \alpha$ for all $k \in N$. Then, by *tetrahedral inequality* of Definition 1.1, we obtain

$$d(\alpha, T\alpha) \leq d(\alpha, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) + d(x_{n_k}, T\alpha).$$

Then if $k \to \infty$, we get

$$d(\alpha, T\alpha) \leq \overline{\lim_{k\to\infty}}\, d(x_{n_k}, T\alpha). \tag{8}$$

From (1),

$$\varphi(d(Tx_{n-1}, T\alpha), d(x_{n-1}, \alpha), d(x_{n-1}, Tx_{n-1}), d(\alpha, T\alpha), d(\alpha, T^2x_{n-1}), d(\alpha, Tx_{n-1}))$$
$$= \varphi(d(x_n, T\alpha), d(x_{n-1}, \alpha), d(x_{n-1}, x_n), d(\alpha, T\alpha), d(\alpha, x_{n+1}), d(\alpha, x_n)) \leq 0.$$

Letting n tend to infinity, we get

$$\varphi(\overline{\lim_{n\to\infty}}\, d(x_n, T\alpha), 0, 0, d(\alpha, T\alpha), 0, 0) \leq 0.$$

And by property (b) of φ, we have

$$\overline{\lim_{n\to\infty}}\, d(x_n, T\alpha) \leq \delta d(\alpha, T\alpha). \tag{9}$$

From (8) and (9)

$$d(\alpha, T\alpha) \leq \overline{\lim_{k\to\infty}}\, d(x_{n_k}, T\alpha) \leq \overline{\lim_{n\to\infty}}\, d(x_n, T\alpha) \leq \delta d(\alpha, T\alpha).$$

Since $0 < \delta < 1$, we have $d(\alpha, T\alpha) = 0$. So α is a fixed point of T. This completes the proof of the theorem. ∎

Theorem 3.3. *Let (X, d) be a gms, $\varphi \in \Phi_6$ and let $T : X \to X$ be an almost φ-contraction. If φ satisfies the condition (c) and $Fix(T) \neq \phi$ then T has a unique fixed point.*

Proof. Let α be a fixed point of T: $\alpha \in Fix(T)$. Assume that $\alpha' \neq \alpha$ is also a fixed point of T. From (1),

$$\varphi(d(T\alpha, T\alpha'), d(\alpha, \alpha'), d(\alpha, T\alpha), d(\alpha', T\alpha'), d(\alpha', T^2\alpha), d(\alpha', T\alpha))$$
$$= \varphi(d(\alpha, \alpha'), d(\alpha, \alpha'), 0, 0, d(\alpha', \alpha), d(\alpha', \alpha)) \leq 0.$$

By property (c) of φ, we have

$$d(\alpha, \alpha') = 0.$$

Thus, the proof follows. ∎

Theorem 3.4. *Let (X, d) be a gms, $\varphi \in \Phi_6$ and let $T : X \to X$ be an almost φ-contraction. If (X, d) is T-orbitally complete and φ satisfies the conditions (a), (b) and (c), then*

(1) *T has a unique fixed point $\alpha \in X$, i.e. $Fix(T) = \{\alpha\}$.*
(2) *For any $x_0 \in X$, the Picard iteration $(x_n) = (Tx_{n-1}), n \in N$ converges to α.*

Proof. The conditions of Theorem 3.2 hold, and so $Fix(T) \neq \phi$. By Theorem 3.3, $Fix(T) = \{\alpha\}$ and by Theorem 3.2, for any $x_0 \in X$, the Picard iteration $(x_n) = (Tx_{n-1}), n \in N$ converges to α. This completes the proof of the theorem. ■

Example 3.5. Let $X = \{\frac{n-1}{n} : n \in N\} \cup \{1\}$. Define $d : X \times X \to R$ as follows:

$$d(x,y) = \begin{cases} 0, & \text{for } x = y, \\ \frac{1}{n}, & \text{for } \{x,y\} = \{\frac{n-1}{n}, 1\},\ x \neq y, \\ 1, & \text{for } x, y \in X - \{1\},\ x \neq y. \end{cases}$$

(X, d) is a generalized metric space but (X, d) is not a standard metric space because it lacks the triangular property:

$$1 = d\left(\frac{1}{2}, \frac{2}{3}\right) > d\left(\frac{1}{2}, 1\right) + d\left(1, \frac{2}{3}\right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

Let $T : X \to X$ be a mapping such that $T(0) = 0$ and $T(x) = \frac{1}{2}$, for $x \in X - \{0\}$. The generalized metric space (X, d) is T-orbitally complete.

We verify the conditions of Theorem 3.2 in case

$$\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \delta t_2 - Lt_6.$$

The inequality (1) takes the form

$$d(Tx, Ty) - \delta d(x, y) - L.d(y, Tx) \leq 0$$

or

$$(1') \qquad d(Tx, Ty) \leq \delta d(x, y) + L.d(y, Tx).$$

If $x = y$ or $x, y \in X - \{0\}$, the right hand side of the inequality (1') is zero and consequently, it is satisfied for any $\delta \in (0, 1)$ and $L \geq 0$.

If $x = 0$ and $y \neq 0$, inequality (1') takes the form

$$1 = d(T0, Ty) \leq \delta d(0, y) + L.d(y, 0) = \delta + L,$$

which is satisfied for $\delta = \frac{1}{2}$ and $L \geq \frac{1}{2}$.

If $x \neq 0$ and $y = 0$, inequality (1') takes the form

$$1 = d(Tx, T0) \leq \delta d(x, 0) + L.d(0, \tfrac{1}{2}) = \delta + L.$$

For $\delta = \frac{1}{2}$ and $L \geq \frac{1}{2}$, the conditions of Theorem 3.2 are satisfied. The mapping T have two fixed points: $Fix(T) = \{0, \frac{1}{2}\} \neq \phi$ and for any $x_0 \in X$, the Picard iteration $\{x_n\}$ defined by $x_n = Tx_{n-1}, n = 1, 2, \ldots$ converges to some $\alpha \in Fix(T) = \{0, \frac{1}{2}\}$.

Example 3.6. Let (X, d) be the generalized metric space from the above example. Let $T : X \to X$ be a mapping such that $T(\frac{2}{3}) = \frac{1}{2}$ and $T(x) = 1$ for $x \in X - \{\frac{2}{3}\}$. The generalized metric space (X, d) is T-orbitally complete.

We verify the conditions of Theorem 3.4 in case

$$\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - h \max\{t_2, t_3, t_4, t_6\}, \quad 0 < h < 1.$$

The φ is a Φ_6-function and satisfies the conditions (a), (b) and (c).

The inequality (1) takes the form

$$(1'') \qquad d(Tx, Ty) \le h \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\}$$

and is satisfied for all $x, y \in X$.

If $x = y$ or $x, y \in X - \{\frac{2}{3}\}$, the left side of the above inequality is zero and consequently, it is true for any $h \in [0, 1)$.

If $x = \frac{2}{3}$ and $y \neq \frac{2}{3}$, the inequality $(1'')$ takes the form

$$\tfrac{1}{2} = d(T\tfrac{2}{3}, Ty) \le h \max\{d(\tfrac{2}{3}, y), d(\tfrac{2}{3}, T\tfrac{2}{3}), d(y, Ty), d(y, T\tfrac{2}{3})\} = h$$

since $d(\frac{2}{3}, T\frac{2}{3}) = d(\frac{2}{3}, \frac{1}{2}) = 1$ and consequently, it is true for any $h \in [\frac{1}{2}, 1)$.

If $x \neq \frac{2}{3}$ and $y = \frac{2}{3}$, inequality $(1'')$ takes the form

$$\tfrac{1}{2} = d(Tx, T\tfrac{2}{3}) \le h \max\{d(x, \tfrac{2}{3}), d(x, Tx), d(\tfrac{2}{3}, T\tfrac{2}{3}), d(\tfrac{2}{3}, Ty)\} = h$$

and is true for any $h \in [\frac{1}{2}, 1)$.

If we take an arbitrary $h \in [\frac{1}{2}, 1)$, all conditions of Theorem 3.4 are satisfied. The mapping T has unique fixed point: $Fix(T) = \{1\}$ and, for any $x \in X$, the Picard iteration $\{x_n\}$ defined by $x_n = T^n x, n = 1, 2, \ldots,$ converges to 1.

Remark 3.7. We notice that in the ordinary metric space, the inequality $(1'')$ is not satisfied for $x = \frac{2}{3}$ and $y = \frac{1}{2}$:

$$\begin{aligned} \tfrac{1}{2} &= d(T\tfrac{2}{3}, T\tfrac{1}{2}) \\ &\le h \max\{d(\tfrac{2}{3}, \tfrac{1}{2}), d(\tfrac{2}{3}, T\tfrac{2}{3}), d(\tfrac{1}{2}, T\tfrac{1}{2}), d(\tfrac{1}{2}, T\tfrac{2}{3})\} \\ &= h \max\{\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{2}, 0\} = \tfrac{1}{2}h \end{aligned}$$

and the Theorem 3.4 can not be applied for the mapping T in the ordinary metric space.

This example shows that the Theorem 3.4 provides a larger class of mappings than that of metric spaces.

4. Corollaries

For different expressions of φ in the Theorems 3.2, 3.3 and 3.4, we get different Theorems:

1) If $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - bt_6$, ($\varphi$ is the function defined in Example 2.5), then by Theorem 3.2, we obtain a fixed point Theorem that extends the well-known Berinde weak (almost) contraction principle (Theorem 1 in [2]) in a generalized metric space.

2) If $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - bt_3$, ($\varphi$ is the function defined in Example 2.6), then by Theorem 3.3, we obtain a fixed point Theorem that extends the well-known Berinde weak (almost) contraction principle (Theorem 2 in [2]) in a generalized metric space.

3) If $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - ct_2$, ($\varphi$ is the function defined in Example 2.2), then by Theorem 3.4, we obtain a fixed point Theorem that extends the well-known Banach contraction principle in a generalized metric space.

4) If $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - b(t_3 + t_4)$, ($\varphi$ is the function defined in Example 2.4 for $c = 0$), then by Theorem 3.4, we obtain a fixed point Theorem that extends the well-known Kannan contraction principle [12] in a generalized metric space.

5) If $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\{t_2, t_3\}$, ($\varphi$ is the function defined in Example 2.7 for $b = 0$), then by Theorem 3.4, we obtain a fixed point Theorem that extends the well-known Bianchini contraction principle [6] in a generalized metric space.

6) If $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - (at_2 + bt_3 + ct_4)$, ($\varphi$ is the function defined in Example 2.8 for $d = 0$), then by Theorem 3.4, we obtain a fixed point Theorem that extends the well-known Reich contraction principle [16] in a generalized metric space.

Remark 4.1. For φ as in the Examples 2.9–2.13, other corollaries can be obtained.

5. Conclusions

In this paper, we obtain three theorems for the almost contractions defined by an implicit relation: Theorem 3.2 (sufficient condition for existence of fixed point) and Theorems 3.3 and 3.4 (sufficient conditions for existence of unique fixed point). The results are extensions and generalizations, from metric spaces to generalized metric spaces, of many well-known fixed point theorems: of Berinde [2], Banach, Kanan [12], Bianchini [6], Reich [16], and many others in Rhoades's classification [15].

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