DE GRUYTER
OPEN

DEMONSTRATIO MATHEMATICA
Vol. XLVIII No 3 2015

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CERTAIN GENERALIZED q-OPERATORS

Communicated by H. D. Alber

Abstract. The applications of q-calculus in the approximation theory is a very interesting area of research in the recent years, several new q-operators were introduced and their behaviour were discussed by many researchers. This paper is the extension of the paper [15], in which Durrmeyer type generalization of q-Baskakov–Stancu type operators were discussed by using the concept of q-integral operators. Here, we propose to study the Stancu variant of q-Baskakov–Stancu type operators. We establish an estimate for the rate of convergence in terms of modulus of continuity and weighted approximation properties of these operators.

1. Introduction

In the year 1987, first q-analogue of classical Bernstein polynomials was given by A. Lupas [16]. In this context, Phillips [18] gave most important q-analogue of Bernstein polynomials and Gupta et al. [14] established the generating function of some q-basis functions.

Since we know that in approximation theory, the study of convergence of an operator is one of the important result, so a lot of contribution in this direction was given by [5], [4], [3] etc. For convergence of q-discrete operators Dogru–Gupta [10], [11] established some results based on q integers for Bleiman–Butzer–Hahn and Meyer–Konig–Zeller operators, respectively. Recently Aral–Gupta–Agrawal [2] published an important book mentioning applications of q-calculus in operator theory. Also in the recent year, Maheshwari–Sharma [17] studied some approximation properties on q-integral type operators. Some estimates and important facts of approximation theory can be found in the recent book of Gupta–Agrawal [13].

In the year 2003, Agrawal and Mohammad [1] gave the generalization of well known Baskakov operators having weight function of Szász basis

2010 *Mathematics Subject Classification*: 41A25, 41A36.
Key words and phrases: q-Baskakov type operators, weighted approximation, rate of convergence, q-integral operators.

DOI: 10.1515/dema-2015-0028

functions as

$$(1.1)\quad D_n(f,x) = n\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} s_{n,k-1}(t)f(t)dt + p_{n,0}(x)f(0), x \in [0,\infty),$$

where

$$p_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$$

and

$$s_{n,k}(t) = e^{-nt}\frac{(nt)^k}{k!}.$$

Wang [19] also studied some approximation properties of these operators by using iterative combination. In the year 2010, Buyukyazici–Atakut [7], [6], [8] have given the variants of several linear positive operators and established direct results. Actually Stancu variant is based on two parameters α and β. Motivated by the recent studies on Stancu type operators, we propose to study the Stancu type generalization of q-Baskakov type operators defined in (1.1).

For $0 < q < 1$ and $0 \le \alpha \le \beta$, the integral type q-Baskakov–Stancu operators are defined as

$$D_{n,\alpha,\beta}^q(f,x) = [n]_q \sum_{k=1}^{\infty} p_{n,k}^q(x)\int_0^{q(1-q^n)} q^{-k}s_{n,k-1}^q(t)$$

$$(1.2)\qquad \times f\left(\frac{[n]_q t q^{-k}+\alpha}{[n]_q+\beta}\right)d_q(t) + p_{n,0}^q f(0), \quad x \in [0,\infty),$$

where

$$p_{n,k}^q(x) = \binom{n+k-1}{k}_q q^{\frac{k(k-1)}{2}}\frac{x^k}{(1+x)_q^{n+k}}$$

and

$$s_{n,k}^q(t) = E_q(-[n]_q t)\frac{([n]_q t)^k}{[k]_q!}.$$

As a special case for $\alpha = \beta = 0$ and $q = 1$, the above operators reduce to the operators defined in (1.1) and for $\alpha = \beta = 0$, these operators become the operators studied in [15]. Aral, Gupta and Agrawal [2] have given many applications of q-calculus and important results in their book, here we use the notations of q-calculus as discussed in their book. Some notations of q-calculus, which will be very useful for finding the results of this paper, are described below.

$$(1+x)_q^n := \begin{cases} (1+x)(1+qx)\dots\left(1+q^{n-1}x\right), & n = 1, 2, \dots, \\ 1, & n = 0. \end{cases}$$

$[n]_q!$ and $[n]_q$ denote the q-factorial and q-integer, respectively and are defined as

$$[n]_q! = \begin{cases} [n]_q[n-1]_q \cdots [1]_q, & n = 1, 2, \ldots, \\ 1, & n = 0, \end{cases} \qquad [n]_q = \frac{1-q^n}{1-q}.$$

q analogue of exponential function is defined as

$$E_q(z) = \prod_{j=0}^{\infty}(1+(1-q)q^j z) = \sum_{k=0}^{\infty} q^{k(k-1)/2}\frac{z^k}{[k]_q!}, |q| < 1.$$

In the present paper, we study a local approximation theorem and the rate of convergence for the operators (1.2) and also their weighted approximation properties.

2. Moments and convergence

This section deals with moments and convergence estimates.

Lemma 1. ([15]) *For the operators defined by* (1.2) *in case* $\alpha = \beta = 0$*, the following equalities hold for* $0 < q < 1$

(i) $D_n^q(1,x) = 1$,
(ii) $D_n^q(t,x) = x$,
(iii) $D_n^q(t^2,x) = x^2 + \frac{x}{[n]_q}(1+q+\frac{x}{q})$.

Lemma 2. *If* $0 < q < 1$ *and* $0 \le \alpha \le \beta$*, then for* (1.2)*, we have*

$$D_{n,\alpha,\beta}^q(1,x) = 1,$$
$$D_{n,\alpha,\beta}^q(t,x) = \frac{[n]_q x + \alpha}{[n]_q + \beta},$$
$$D_{n,\alpha,\beta}^q(t^2,x) = \frac{x^2([n]_q^2 q + [n]_q) + x([n]_q q^2 + [n]_q q + 2\alpha q[n]_q) + \alpha^2 q}{([n]_q+\beta)^2 q}.$$

Proof. From Lemma 1, it is obvious that

$$D_{n,\alpha,\beta}^q(1,x) = 1.$$

Next, we have by Lemma 1

$$D_{n,\alpha,\beta}^q(t,x) = [n]_q \sum_{k=1}^{\infty} p_{n,k}^q(x)\int_0^{q(1-q^n)} q^{-k}s_{n,k-1}(t)\left(\frac{[n]_q t q^{-k}+\alpha}{[n]_q+\beta}\right)d_q t$$
$$+ p_{n,0}^q(x)\left(\frac{\alpha}{[n]_q+\beta}\right)$$

$$= \frac{[n]_q}{[n]_q+\beta} D_n^q(t,x) + \frac{\alpha}{[n]_q+\beta} D_n^q(1,x)$$
$$= \frac{[n]_q x}{[n]_q+\beta} + \frac{\alpha}{[n]_q+\beta} = \frac{[n]_q x+\alpha}{[n]_q+\beta}.$$

Finally, we have

$$\begin{aligned}
&D_{n,\alpha,\beta}^q(t^2,x)\\
&= [n]_q \sum_{k=1}^{\infty} p_{n,k}^q(x) \int_0^{q(1-q^n)} q^{-k} s_{n,k-1}(t) \left(\frac{[n]_q t q^{-k}+\alpha}{[n]_q+\beta}\right)^2 d_q t\\
&\quad + p_{n,0}^q(x)\left(\frac{\alpha}{[n]_q+\beta}\right)^2\\
&= \left(\frac{[n]_q}{[n]_q+\beta}\right)^2 D_n^q(t^2,x) + \frac{2[n]_q\alpha}{([n]_q+\beta)^2} D_n^q(t,x) + \frac{\alpha^2}{([n]_q+\beta)^2} D_n^q(1,x)\\
&= \left(\frac{[n]_q}{[n]_q+\beta}\right)^2 \left[x^2 + \frac{x}{[n]_q}(1+q+\frac{x}{q})\right] + \frac{2[n]_q\alpha}{([n]_q+\beta)^2} x + \left(\frac{\alpha}{[n]_q+\beta}\right)^2\\
&= \frac{[n]_q^2(q[n]_q x^2 + xq + xq^2 + x^2) + 2[n]_q \alpha q [n]_q x + \alpha^2 [n]_q q}{([n]_q+\beta)^2 [n]_q q}\\
&= \frac{x^2([n]_q^2 q + [n]_q) + x([n]_q q^2 + [n]_q q + 2\alpha q [n]_q) + \alpha^2 q}{([n]_q+\beta)^2 q}. \ \blacksquare
\end{aligned}$$

Lemma 3. *For $x \in [0,\infty)$ and $q \in (0,1)$, the central moments are given as*

$$D_{n,\alpha,\beta}^q(t-x,x) = \frac{\alpha-\beta x}{[n]_q+\beta},$$
$$\begin{aligned}
D_{n,\alpha,\beta}^q((t-x)^2,x) &= x^2\left[\frac{[n]_q^2 q + [n]_q}{([n]_q+\beta)^2 q} - \frac{2[n]_q}{[n]_q+\beta} + 1\right]\\
&\quad + x\left[\frac{[n]_q q^2 + [n]_q q + 2\alpha q [n]_q}{([n]_q+\beta)^2 q} - \frac{2\alpha}{[n]_q+\beta}\right] + \frac{\alpha^2}{([n]_q+\beta)^2}.
\end{aligned}$$

Proof.

$$\begin{aligned}
D_{n,\alpha,\beta}^q(t-x,x) &= D_{n,\alpha,\beta}^q(t,x) - x = \frac{[n]_q x+\alpha}{[n]_q+\beta} - x\\
&= \frac{\alpha-\beta x}{[n]_q+\beta},
\end{aligned}$$

$$\begin{aligned}
D_{n,\alpha,\beta}^q((t-x)^2,x) &= D_{n,\alpha,\beta}^q(t^2,x) - 2xD_{n,\alpha,\beta}^q(t,x) + x^2 D_{n,\alpha,\beta}^q(1,x) \\
&= \frac{x^2([n]_q^2 q + [n]_q) + x([n]_q q^2 + [n]_q q + 2\alpha q[n]_q) + \alpha^2 q}{([n]_q+\beta)^2 q} \\
&\quad - 2x\frac{[n]_q x + \alpha}{[n]_q+\beta} + x^2 \\
&= x^2\left[\frac{[n]_q^2 q + [n]_q}{([n]_q+\beta)^2 q} - \frac{2[n]_q}{[n]_q+\beta} + 1\right] \\
&\quad + x\left[\frac{[n]_q q^2 + [n]_q q + 2\alpha q[n]_q}{([n]_q+\beta)^2 q} - \frac{2\alpha}{[n]_q+\beta}\right] + \frac{\alpha^2}{([n]_q+\beta)^2}. \blacksquare
\end{aligned}$$

Definition 1. By $C_B[0,\infty)$, we mean the space of real valued continuous bounded function f on the interval $[0,\infty)$, the norm $\|.\|$ on the space $C_B[0,\infty)$ is given by

$$\|f\| = \sup_{0\le x<\infty} |f(x)|.$$

Definition 2. The Peetre's K-functional is defined by

$$K_2(f,\delta) = \inf\{\|f-g\| + \delta\|g''\| : g \in W_\infty^2\},$$

where $W_\infty^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. According to [9], there exists a positive constant $C > 0$ such that $K_2(f,\delta) \le C\omega_2(f,\delta^{1/2}), \delta > 0$, where the second order modulus of smoothness is given by

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\le\sqrt{\delta}} \sup_{0\le x<\infty} |f(x+2h) - 2f(x+h) + f(x)|.$$

Also for $f \in C_B[0,\infty)$, the usual modulus of continuity is given by

$$\omega(f,\delta) = \sup_{0<h\le\delta} \sup_{0\le x<\infty} |f(x+h) - f(x)|.$$

Our first main result is in terms of modulus of continuity, which is given as

Theorem 1. *Let $f \in C_B[0,\infty)$ and $0 < q < 1$, then for all $x \in [0,\infty)$ and $n \in N$, there exists an absolute constant $C > 0$ such that*

$$|D_{n,\alpha,\beta}^q(f,x) - f(x)| \le C\omega_2(f,\delta_n(x)) + \omega\left(f, \frac{\alpha-\beta x}{[n]_q+\beta}\right),$$

where $\delta_n^2(x) = \left[D_{n,\alpha,\beta}^q\left((t-x)^2,x\right) + \left(\frac{\alpha-\beta x}{[n]_q+\beta}\right)^2\right]^{1/2}$.

Proof. We introduce the auxiliary operators $\overline{D}_{n,\alpha,\beta}^q$ defined by

$$\overline{D}_{n,\alpha,\beta}^q(f,x) = D_{n,\alpha,\beta}^q(f,x) - f\left(x + \frac{\alpha-\beta x}{[n]_q+\beta}\right) + f(x), \tag{2.1}$$

$x \in [0, \infty)$. The operators $\overline{D}^q_{n,\alpha,\beta}(f, x)$ are linear and hence follows the linearity condition

$$\overline{D}^q_{n,\alpha,\beta}(t - x, x) = 0. \tag{2.2}$$

Let $h \in W^2$, from Taylor's expansion

$$h(t) = h(x) + h'(x)(t - x) + \int_x^t (t - u)\ h''(u)\ du, \quad t \in [0, \infty)$$

and by (2.2), we get

$$\overline{D}^q_{n,\alpha,\beta}(h, x) \ = \ h(x) + \overline{D}^q_n \left(\int_x^t (t - u)\ h''(u)\ du, x \right).$$

Hence, by (2.1) we can get

$$\begin{aligned}
&|\overline{D}^q_{n,\alpha,\beta}(h, x) - h(x)| \\
&\leq \left| D^q_{n,\alpha,\beta} \left(\int_x^t (t - u) h''(u)\, du, x \right) \right| + \left| \int_x^{\frac{(x[n]_q+\alpha)}{[n]_q+\beta}} \left(\frac{x[n]_q + \alpha}{[n]_q + \beta} - u \right) h''(u)\, du \right| \\
&\leq D^q_{n,\alpha,\beta} \left(\left| \int_x^t |t - u||h''(u)|\, du \right|, x \right) + \int_x^{\frac{x[n]_q+\alpha}{[n]_q+\beta}} \left| \frac{x[n]_q + \alpha}{[n]_q + \beta} - u \right| |h''(u)|\, du \\
&\leq \left[D^q_{n,\alpha,\beta} \left((t - x)^2, x \right) + \left(\frac{\alpha - \beta x}{[n]_q + \beta} \right)^2 \right] \|h''\| = \delta^2_n(x)\|h''\|.
\end{aligned} \tag{2.3}$$

And by (2.1), we can get

$$|\overline{D}^q_{n,\alpha,\beta}(f, x)| \leq |D^q_{n,\alpha,\beta}(f, x)| + 2\|f\| \leq 3\|f\|. \tag{2.4}$$

Now (2.1), (2.3) and (2.4) imply

$$\begin{aligned}
|D^q_{n,\alpha,\beta}(f, x) - f(x)| &\leq |\overline{D}^q_n(f - h, x) - (f - h)(x)| \\
&\quad + |\overline{D}^q_{n,\alpha,\beta}(h, x) - h(x)| + \left| f \left(x + \frac{\alpha - \beta x}{[n]_q + \beta} \right) - f(x) \right| \\
&\leq 4\|f - h\| + \delta^2_n(x)\|h''\| \\
&\quad + \left| f \left(x + \frac{\alpha - \beta x}{[n]_q + \beta} \right) - f(x) \right|.
\end{aligned}$$

Now taking the infimum on the right hand side over all $h \in W^2$, we get

$$|D^q_{n,\alpha,\beta}(f, x) - f(x)| \leq CK_2 \left(f, \delta^2_n(x) \right) + \omega \left(f, \frac{\alpha - \beta x}{[n]_q + \beta} \right).$$

Using the property of K-functional

$$|D^q_{n,\alpha,\beta}(f, x) - f(x)| \leq C\omega_2 \left(f, \delta_n(x) \right) + \omega \left(f, \frac{\alpha - \beta x}{[n]_q + \beta)} \right).$$

This completes the proof of the theorem. ■

Definition 3. By $H_{x^2}[0,\infty)$ we denote the set of all functions f defined on the positive real axis and satisfying the condition $|f(x)| \leq m_f(1+x^2)$, where m_f is a constant depending only on f. By $C_{x^2}[0,\infty)$, we mean the subspace of all continuous functions belonging to $H_{x^2}[0,\infty)$. Also, suppose that $C^*_{x^2}[0,\infty)$ denote the subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\lim_{x\to\infty} \frac{f(x)}{1+x^2}$ is finite. The norm on the class $C^*_{x^2}[0,\infty)$ is defined as

$$\|f\|_{x^2} = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

The modulus of continuity of f on the closed interval $[0,a]$, $a>0$ is given as

$$\omega_a(f,\delta) = \sup_{|t-x|\leq\delta} \sup_{x,t\in[0,a]} |f(t)-f(x)|.$$

Observe that for every function $f \in C_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero.

The following estimate is in terms of weighted approximation.

Theorem 2. *Suppose that $q=q_n$ satisfies $0<q_n<1$ and let $q_n\to 1$ as $n\to\infty$ for each $f \in C^*_{x^2}[0,\infty)$, we have*

$$\lim_{n\to\infty} \|D^{q_n}_{n,\alpha,\beta}(f,x)-f(x)\|_{x^2} = 0.$$

Proof. If we use the theorem in [12], we observe that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \|D^{q_n}_{n,\alpha,\beta}(t^\nu,x)-x^\nu\|_{x^2} = 0, \qquad \nu=0,1,2. \tag{2.5}$$

Since $D^{q_n}_{n,\alpha,\beta}(1,x)=1$, (2.5) holds for $\nu=0$.

$$\begin{aligned}\|D^{q_n}_{n,\alpha,\beta}(t,x)-x\|_{x^2} &= \sup_{x\in[0,\infty)} \frac{\alpha-\beta x}{[n]_q+\beta}.\frac{1}{1+x^2}\\ &\leq \frac{-\beta}{[n]_q+\beta}\sup_{x\in[0,\infty)}\frac{x}{1+x^2} + \frac{\alpha}{[n]_q+\beta}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}.\end{aligned}$$

Hence

$$\lim_{n\to\infty} \|D^{q_n}_{n,\alpha,\beta}(t,x)-x\|_{x^2} = 0,$$

$$\begin{aligned}\|D^{q_n}_{n,\alpha,\beta}(t^2,x)-x^2\|_{x^2} \le & \left(\frac{[n]_q^2 q+[n]_q}{([n]_q+\beta)^2 q}-1\right)\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}\\ &+\left(\frac{[n]_q q^2+2\alpha q[n]_q+[n]_q q}{([n]_q+\beta)^2 q}\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2}\\ &+\left(\frac{\alpha^2 q}{([n]_q+\beta)^2 q}\right)\sup_{x\in[0,\infty)}\frac{1}{1+x^2},\end{aligned}$$

which leads to

$$\lim_{n\to\infty}\|D^{q_n}_{n,\alpha,\beta}(t^2,x)-x^2\|_{x^2}=0.$$

This completes the proof of theorem. ∎

Acknowledgement. Authors are thankful to referees for their valuable suggestions for overall improvement of the paper.

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Received January 7, 2014; revised version July 10, 2014.