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DECAY RATES OF THE SOLUTION TO THE CAUCHY PROBLEM OF THE TYPE III TIMOSHENKO MODEL WITHOUT ANY MECHANICAL DAMPING

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Abstract. In this paper, we study the asymptotic behavior of the solutions of the one-dimensional Cauchy problem in Timoshenko system with thermal effect. The heat conduction is given by the type III theory of Green and Naghdi. We prove that the dissipation induced by the heat conduction alone is strong enough to stabilize the system, but with slow decay rate. To show our result, we transform our system into a first order system and, applying the energy method in the Fourier space, we establish some pointwise estimates of the Fourier image of the solution. Using those pointwise estimates, we prove the decay estimates of the solution and show that those decay estimates are very slow and, in the case of nonequal wave speeds, are of regularity–loss type. This paper solves the open problem stated in [10] and shows that the stability of the solution holds without any additional mechanical damping term.

1. Introduction

The type III Green & Naghdi's model of thermoelasticity includes temperature gradient and thermal displacement gradient among the constitutive variables and proposes a heat conduction law as

$$q(x,t) = -[\kappa\nabla\theta(x,t) + \kappa^*\nabla v(x,t)], \tag{1.1}$$

where $v_t = \theta$ and v is the thermal displacement gradient, κ and κ^* are two positive constants. Equation (1.1) together with the energy balance law

$$\rho_3\theta_t + \varrho\,\mathrm{div}\,q = 0 \tag{1.2}$$

leads to the equation

$$\rho\theta_{tt} - \varrho\kappa\Delta\theta_t - \varrho\kappa^*\Delta\theta = 0, \tag{1.3}$$

which permits propagation of thermal waves at finite speed.

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The coupling of equation (1.3) with the equations of elasticity has been an active area of research in the last two decades. See for instance Zhang & Zuazua [13] and Quintanilla & Racke [5].

Concerning the coupling of (1.3) (in one-dimensional space) with Timoshenko systems, we have the recent papers of Messaoudi & Said-Houari [3, 4], in which the authors proved several stability results. More precisely, in [3], they investigated the asymptotic behavior of the problem

$$\begin{cases} \rho_1\varphi_{tt} - K\left(\varphi_x + \psi\right)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K\left(\varphi_x + \psi\right) + \beta\theta_x = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - \kappa\theta_{txx} = 0, \end{cases} \tag{1.4}$$

in $(0,1)\times(0,\infty)$ and proved an exponential decay result of the total energy corresponding to (1.4). The same problem (1.4) with an additional damping of history type of the form $\int_0^\infty g(s)\psi_{xx}(x,t-s)ds$ acting in the second equation has been analyzed in [4]. The authors of [4] proved an exponential and polynomial stability results for the equal and nonequal wave-speed propagation under conditions on the relaxation function g weaker than those in [1] and [7].

To the best of our knowledge, the Cauchy problem in Timoshenko system:

$$\varphi_{tt}\left(x,t\right) - \left(\varphi_x - \psi\right)_x\left(x,t\right) = 0, \tag{1.5a}$$

$$\psi_{tt}\left(x,t\right) - a^2\psi_{xx}\left(x,t\right) - \left(\varphi_x - \psi\right)\left(x,t\right) + \lambda\psi_t\left(x,t\right) = 0, \tag{1.5b}$$

where $(x,t)\in\mathbb{R}^+\times\mathbb{R}$, has been first studied in [2], where the authors showed some decay estimates depending on the wave speeds of the two equations in system (1.5). More precisely, they proved the following estimates

- When $a=1$,

$$\|\partial_x^k U(t)\|_{L^2} \le C\left(1+t\right)^{-1/4-k/2}\|U_0\|_{L^1} + Ce^{-ct}\|\partial_x^k U_0\|_{L^2}. \tag{1.6}$$

- When $a\neq 1$,

$$\|\partial_x^k U(t)\|_{L^2} \le C\left(1+t\right)^{-1/4-k/2}\|U_0\|_{L^1} + C\left(1+t\right)^{-\ell/2}\|\partial_x^{k+\ell} U_0\|_{L^2}, \tag{1.7}$$

where k and ℓ are non-negative integers satisfying $k+\ell\le s$, C, c are two positive constants, $U(x,t) = (\varphi_t, \varphi_x+\psi, \psi_x, \psi_t)'(x,t)$ and $U_0 = U(x,0)$.

The decay estimates (1.6) and (1.7) have been improved by Racke & Said-Houari [6]. In fact, by restricting the initial data U_0 to be in $H^s\left(\mathbb{R}\right)\cap L^{1,\gamma}\left(\mathbb{R}\right)$ with $\gamma\in[0,1]$, the authors derived faster decay estimates than those given in [2] and the decay has been improved by $t^{-\gamma/2}$, $\gamma\in[0,1]$. Also a global existence result for the semi-linear model has been established.

In [8], with Kasimov, we investigated the Cauchy problem of the Timoshenko system of thermoelasticity for both the Fourier and Cattaneo models.

Namely, we studied the system

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \delta\theta_x + \lambda\psi_t = 0, \\ \theta_t + q_x + \delta\psi_{tx} = 0, \\ \tau q_t + \beta q + \theta_x = 0, \end{cases} \tag{1.8}$$

where $t \in (0, \infty)$, $x \in \mathbb{R}$, and τ, δ, λ, and β are positive constants. For the Fourier model ($\tau = 0$), we proved that the solution $U = (\varphi_x - \psi, \varphi_t, a\psi_x, \psi_t)^T$ decays as in (1.6) and (1.7). For the Cattaneo model ($\tau > 0$), on the other hand, we showed that the solution $W = (\varphi_x - \psi, \varphi_t, a\psi_x, \psi_t, \theta, q)^T$ only satisfies the estimate (1.7), irrespective of the value of a. That is, the Cattaneo model has the regularity-loss property.

Recently, in [9], we proved that heat dissipation alone (i.e. $\lambda = 0$ in (1.8)) is sufficient to stabilize the system in both cases $\tau = 0$ and $\tau \neq 0$, so that additional mechanical damping is unnecessary. However, the decay rate of the L^2-norm of solutions without the mechanical damping is found to be $(1 + t)^{-1/12}$, slower than that with mechanical damping. Furthermore, in contrast to earlier results of [8, 12], we find that the Timoshenko–Fourier and the Timoshenko–Cattaneo systems have the same decay rate. The rate depends on a certain number α, (first identified in [11] in a related study in a bounded domain), which is a function of the parameters of the system.

In this paper, we consider the Cauchy problem of the Timoshenko type III model and show that the heat dissipation alone is strong enough to stabilize the solution, but with a slow decay rate. More precisely, we establish the decay rate $(1 + t)^{-1/12}$ of the L^2-norm of the solution, which is exactly the same as in the Timoshenko–Cattaneo and Timoshenko–Fourier models. This result improves a recent one in [10], where an additional mechanical damping has been considered. This paper is organized as follows: In Section 2, we state the problem and in Section 3, we prove our main result.

2. Statement of the problem

We consider the Cauchy problem

$$\varphi_{tt} - (\varphi_x - \psi)_x = 0, \tag{2.1a}$$

$$\psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \beta\theta_x = 0, \tag{2.1b}$$

$$\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - k\theta_{txx} = 0, \tag{2.1c}$$

with the initial data

$$(\varphi, \varphi_t, \psi, \psi_t, \theta, \theta_t)(x, 0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \theta_1), \tag{2.1d}$$

where $t \in (0,\infty)$ denotes the time variable and $x \in \mathbb{R}$ is the space variable, the functions φ and ψ denote the displacements of the elastic material, the function θ is the temperature difference and a, δ, γ, k and β are positive constants.

In order to exhibit the dissipative nature of system (2.1) and following Reference [13], we use the transformation

$$\tilde{\theta}(x,t) := \int_0^t \theta(x,s)ds + \chi(x) \tag{2.2}$$

with a function $\chi := \chi(x)$ satisfying

$$\delta\chi'' = \theta_1 - k\theta_0'' + \gamma\psi_1'.$$

Then we get from (2.1) (by writing, for simplicity θ instead of $\tilde{\theta}$)

$$\varphi_{tt} - (\varphi_x - \psi)_x = 0, \tag{2.3a}$$

$$\psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \beta\theta_{tx} = 0, \tag{2.3b}$$

$$\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{tx} - k\theta_{txx} = 0, \tag{2.3c}$$

with the initial data

$$(\varphi, \varphi_t, \psi, \psi_t, \theta, \theta_t)(0,x) = \left(\varphi_0, \varphi_1, \psi_0, \psi_1, \tilde{\theta}(x,0), \tilde{\theta}_t(x,0)\right). \tag{2.3d}$$

Let us now introduce the new variables

$$v = \varphi_x - \psi, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t, \quad \eta = \theta_t, \quad w = \theta_x.$$

Then, the system (2.3) can be rewritten as

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - ay_x = 0, \\ y_t - az_x - v + \beta\eta_x = 0, \\ \eta_t - \delta w_x + \gamma y_x - k\eta_{xx} = 0, \\ w_t - \eta_x = 0, \end{cases} \qquad x \in \mathbb{R},\ t > 0 \tag{2.4a}$$

and the corresponding initial condition becomes

$$(v, u, z, y, \eta, w)(x,0) = (v_0, u_0, z_0, y_0, \eta_0, w_0)(x), \tag{2.4b}$$

where

$$v_0 = \phi_{0,x} - \psi_0, \quad u_0 = \phi_1, \quad z_0 = a\psi_{0,x}, \quad y_0 = \psi_1, \quad \eta_0 = \theta_0, \quad w_0 = \theta_{1,x}.$$

System (2.4) is a hyperbolic–parabolic system and can be written in the matrix form

$$\begin{cases} U_t + AU_x + LU = BU_{xx}, \\ U(x,0) = U_0, \end{cases} \tag{2.5}$$

where with $U = (v, u, z, y, \eta, w)^T$, $U_0 = (v_0, u_0, z_0, y_0, \eta_0, w_0)^T$ and A, L and B are the matrices in [10] (with $\delta = 0$).

Our main goal is to understand the interaction between the conservative hyperbolic part and the parabolic diffusive part in system (2.4). We show that the interaction between these two parts generates some dissipation that is strong enough to dissipate the hyperbolic part of the system. The key element in the proof is to construct some functionals that capture the dissipation of the hyperbolic components in the system, which can be done by using the classical energy method in the Fourier space. Our decay estimates read as follows:

Theorem 2.1. *Let s be a nonnegative integer and assume that $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Then, the solution U, of problem (2.4) satisfies the following decay estimates:*

- *when $a = 1$,*

$$\|\partial_x^k U(t)\|_{L^2} \le C\,(1+t)^{-1/12-k/6}\,\|U_0\|_{L^1} + Ce^{-ct}\|\partial_x^k U_0\|_{L^2}; \tag{2.6}$$

- *when $a \neq 1$,*

$$\|\partial_x^k U(t)\|_{L^2} \le C\,(1+t)^{-1/12-k/6}\,\|U_0\|_{L^1} + C\,(1+t)^{-\ell/2}\,\|\partial_x^{k+\ell} U_0\|_{L^2}, \tag{2.7}$$

where k and ℓ are non-negative integers satisfying $k + \ell \le s$, and C and c are two positive constants.

Remark 2.2. Theorem 2.1 together with the recent result in [9] show that in the absence of the linear frictional damping ψ_t, all three models: Timoshenko–Fourier, Timoshenko–Cattaneo and Timoshenko-type III give a very slow decay rate of the solution and this decay rate is the same in these three models.

Remark 2.3. The estimates in Theorem 2.1 can be improved by considering initial data in some L^1-weighted spaces with zero total mass or by assuming that the higher momenta of the initial data are zeros. See [9] for more details.

3. Proof

Now, we want to show some pointwise estimates of the Fourier image of the solution of (2.5). These estimates are necessary to establish the decay rates in Theorem 2.1. Indeed, taking the Fourier transform of (2.5), we get

$$\begin{cases} \hat{U}_t(\xi, t) = \Lambda(\xi)\hat{U}(\xi, t), & \xi \in \mathbb{R},\ t > 0, \\ \hat{U}(\xi, 0) = \hat{U}_0(\xi), & \xi \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $\Lambda(\xi) = -L - i\xi A - \xi^2 B$. Consequently, solving the above first order ordinary differential equation, we get

$$\hat{U}(\xi, t) = e^{\Lambda(\xi)t}\hat{U}_0(\xi). \tag{3.2}$$

Computing the term $e^{\Lambda(\xi)t}$ is a challenging problem and in many situations this cannot be done. Consequently, in order to show the asymptotic behavior of the solution, it suffices to find a function $\rho(\xi)$ such that

$$|e^{\Lambda(\xi)t}| \leq Ce^{-c\rho(\xi)t}, \tag{3.3}$$

for two positive constants C and c. Thus, the behavior of the solution depends on a critical way on the behavior of the function $\rho(\xi)$. Now, we have the following estimates:

Proposition 3.1. *Let $\hat{U} = (\hat{v}, \hat{u}, \hat{z}, \hat{y}, \hat{\eta}, \hat{w})^T$ be the solution of (3.1), then the following estimates hold:*

$$|\hat{U}(\xi,t)|^2 \leq \begin{cases} Ce^{-c\varrho_1(\xi)t}\left|\hat{U}(\xi,0)\right|^2, & \text{if } a=1,\ \varrho_1(\xi) = \dfrac{\xi^6}{(1+\xi^2)(1+\xi^2+\xi^4)}, \\ Ce^{-c\varrho_2(\xi)t}\left|\hat{U}(\xi,0)\right|^2, & \text{if } a\neq 1,\ \varrho_2(\xi) = \dfrac{\xi^6}{(1+\xi^2)^2(1+\xi^2+\xi^4)}, \end{cases} \tag{3.4}$$

where C and c are two positive constants.

Proof. Taking the Fourier transform of (2.4), we obtain

$$\hat{v}_t - i\xi\hat{u} + \hat{y} = 0, \tag{3.5a}$$

$$\hat{u}_t - i\xi\hat{v} = 0, \tag{3.5b}$$

$$\hat{z}_t - ia\xi\hat{y} = 0, \tag{3.5c}$$

$$\hat{y}_t - ia\xi\hat{z} - \hat{v} + i\xi\beta\hat{\eta} = 0, \tag{3.5d}$$

$$\hat{\eta}_t - i\delta\xi\hat{w} + i\gamma\xi\hat{y} + \xi^2 k\hat{\eta} = 0, \tag{3.5e}$$

$$\hat{w}_t - i\xi\hat{\eta} = 0, \tag{3.5f}$$

with the initial condition

$$(\hat{v}, \hat{u}, \hat{z}, \hat{y}, \hat{\eta}, \hat{w})(\xi, 0) = (\hat{v}_0, \hat{u}_0, \hat{z}_0, \hat{y}_0, \hat{\eta}_0, \hat{w}_0)(x). \tag{3.5g}$$

Let us define the energy functional associated to system (3.5)

$$\hat{\mathscr{E}}(\xi,t) := \frac{1}{2}\left(\gamma|\hat{v}|^2 + \gamma|\hat{u}|^2 + \gamma|\hat{z}|^2 + \gamma|\hat{y}|^2 + \beta|\hat{\eta}|^2 + \beta\delta|\hat{w}|^2\right)(\xi,t). \tag{3.6}$$

We multiply equation (3.5a) by $\gamma\bar{\hat{v}}$, equation (3.5b) by $\gamma\bar{\hat{u}}$, equation (3.5c) by $\gamma\bar{\hat{z}}$, equation (3.5d) by $\gamma\bar{\hat{y}}$, equation (3.5e) by $\beta\bar{\hat{\eta}}$ and equation (3.5f) by $\beta\delta\bar{\hat{w}}$, respectively, adding the resulting equalities and taking the real part, we obtain

$$\frac{d}{dt}\hat{\mathscr{E}}(\xi,t) \leq -\beta k\xi^2|\hat{\eta}|^2, \qquad \forall t \geq 0. \tag{3.7}$$

We multiply equation (3.5a) by $i\xi\bar{\hat{v}}$ and (3.5b) by $-i\xi\bar{\hat{u}}$, adding the results and taking the real part, we have

$$\left\{\mathrm{Re}(i\xi\hat{v}\bar{\hat{u}})\right\}_t + \xi^2(|\hat{u}|^2 - |\hat{v}|^2) + \mathrm{Re}(i\xi\bar{\hat{u}}\hat{y}) = 0. \tag{3.8}$$

Similarly, we multiply equation (3.5c) by $i\xi\bar{\hat{y}}$ and equation (3.5d) by $-i\xi\bar{\hat{z}}$, adding the resulting two equations and taking the real part, we obtain

$$\left\{\mathrm{Re}(i\xi\hat{y}\bar{\hat{z}})\right\}_t + a\xi^2(|\hat{z}|^2 - |\hat{y}|^2) - \mathrm{Re}\left\{i\xi\bar{\hat{z}}\hat{v} - \xi^2\beta\hat{\eta}\bar{\hat{z}}\right\} = 0. \tag{3.9}$$

We now add the equations (3.8) and (3.9) we get the following

$$\begin{aligned}\frac{dF(\xi,t)}{dt} + \xi^2(|\hat{u}|^2 + a\,|\hat{z}|^2) - \xi^2(|\hat{v}|^2 + a\,|\hat{y}|^2)\\ = \mathrm{Re}\left\{i\xi\bar{\hat{z}}\hat{v} - \xi^2\beta\hat{\eta}\bar{\hat{z}}\right\} - \mathrm{Re}(i\xi\bar{\hat{u}}\hat{y}),\end{aligned} \tag{3.10}$$

where

$$F(\xi,t) := \mathrm{Re}\left\{i\xi(\hat{v}\bar{\hat{u}} + \hat{y}\bar{\hat{z}})\right\}(\xi,t).$$

The terms on the right-hand side of (3) can be estimated as follows:

$$\begin{aligned}&\mathrm{Re}\left\{i\xi\bar{\hat{z}}\hat{v} - \xi^2\beta\hat{\eta}\bar{\hat{z}}\right\} - \mathrm{Re}(i\xi\bar{\hat{u}}\hat{y})\\ &\qquad \le \epsilon\xi^2(|\hat{u}|^2 + |\hat{z}|^2) + C(\epsilon)\left\{|\hat{v}|^2 + \xi^2\,|\hat{\eta}|^2 + |\hat{y}|^2\right\},\end{aligned}$$

where ϵ is a small positive constant to be fixed later and $C(\epsilon)$ is a generic positive constant that may take different values in different places. Inserting the above estimate into (3.10), we find

$$\begin{aligned}\frac{dF(\xi,t)}{dt} + (1-\epsilon)\xi^2\,|\hat{u}|^2 + (a-\epsilon)\xi^2\,|\hat{z}|^2\\ \le C(\epsilon)(1+\xi^2)\,|\hat{v}|^2 + C(\epsilon)\left(1+\xi^2\right)|\hat{y}|^2 + C(\epsilon)\xi^2\,|\hat{\eta}|^2.\end{aligned} \tag{3.11}$$

Now, following the same method as in [10], we get (see the identity (34) in [10])

$$\frac{dH(\xi,t)}{dt} + |\hat{v}|^2 - |\hat{y}|^2 = (a^2-1)\,\mathrm{Re}(i\xi\bar{\hat{u}}\hat{y}) + \mathrm{Re}(i\xi\beta\hat{\eta}\bar{\hat{v}}), \tag{3.12}$$

where

$$H(\xi,t) := -\mathrm{Re}(\hat{v}\bar{\hat{y}} + a\hat{u}\bar{\hat{z}}). \tag{3.13}$$

Next, multiplying equation (3.5e) by $-i\xi\bar{\hat{y}}$ and equation (3.5d) by $i\xi\bar{\hat{\eta}}$, we find

$$\begin{aligned}\left\{\mathrm{Re}\left(i\xi\bar{\hat{\eta}}\hat{y}\right)\right\}_t + \gamma\xi^2\,|\hat{y}|^2 - \beta\xi^2\,|\hat{\eta}|^2\\ = -\xi^2\,\mathrm{Re}\left(a\hat{z}\bar{\hat{\eta}}\right) + \mathrm{Re}\left(i\xi\bar{\hat{\eta}}\hat{v}\right) - \xi^2\,\mathrm{Re}\left(\delta\hat{w}\bar{\hat{y}}\right) + \mathrm{Re}\left(i\xi^3k\hat{\eta}\bar{\hat{y}}\right).\end{aligned} \tag{3.14}$$

Once again, multiplying equation (3.5e) by $-i\xi\bar{\hat{w}}$ and equation (3.5f) by $i\xi\bar{\hat{\eta}}$, we get, by the same method as before,

$$\left\{\mathrm{Re}(i\xi\hat{\eta}\bar{\hat{w}})\right\}_t + \xi^2(\delta\,|\hat{w}|^2 - |\hat{\eta}|^2) = -\mathrm{Re}\left(i\xi^3k\hat{\eta}\bar{\hat{w}}\right) + \mathrm{Re}\left\{\gamma\xi^2\hat{y}\bar{\hat{w}})\right\}. \tag{3.15}$$

Computing (3.14)$+\frac{\delta}{\gamma}$(3.15), we get

$$\frac{d}{dt}K(\xi,t)+\gamma\xi^2|\hat{y}|^2-\beta\xi^2|\hat{\eta}|^2+\xi^2\left(\frac{\delta^2}{\gamma}|\hat{w}|^2-\frac{\delta}{\gamma}|\hat{\eta}|^2\right)$$
$$=-\xi^2\operatorname{Re}\left(a\hat{z}\bar{\hat{\eta}}\right)+\operatorname{Re}\left(i\xi\bar{\hat{\eta}}\hat{v}\right)+\operatorname{Re}\left(i\xi^3k\hat{\eta}\bar{\hat{y}}\right)-\operatorname{Re}\left(i\xi^3\frac{k\delta}{\gamma}\hat{\eta}\bar{\hat{w}}\right), \tag{3.16}$$

where

$$K(\xi,t):=\operatorname{Re}\left(i\xi\bar{\hat{\eta}}\hat{y}\right)+\frac{\delta}{\gamma}\operatorname{Re}(i\xi\hat{\eta}\bar{\hat{w}}). \tag{3.17}$$

Young's inequality gives for any $\epsilon_0,\epsilon_0'>0$,

$$\frac{d}{dt}K(\xi,t)+(\gamma-\epsilon_0)\xi^2|\hat{y}|^2+\left(\frac{\delta^2}{\gamma}-\epsilon_0\right)\xi^2|\hat{w}|^2$$
$$\leq C\left(\epsilon_0,\epsilon_0'\right)\left(1+\xi^2+\xi^4\right)|\hat{\eta}|^2+\epsilon_0'\frac{\xi^4}{1+\xi^2}|\hat{z}|^2+\epsilon_0'\xi^2|\hat{v}|^2, \tag{3.18}$$

where we have used the estimate

$$\left|\xi^2\operatorname{Re}\left(a\hat{z}\bar{\hat{\eta}}\right)\right|\leq\epsilon_0'\frac{\xi^4}{1+\xi^2}|\hat{z}|^2+C\left(\epsilon_0'\right)\left(1+\xi^2\right)|\hat{\eta}|^2.$$

Now we distinguish two cases:

Case 1: $a=1$.

In this case, the identity (3.12) becomes

$$\frac{dH(\xi,t)}{dt}+(1-\epsilon)|\hat{v}|^2-|\hat{y}|^2\leq C(\epsilon)\xi^2|\hat{\eta}|^2. \tag{3.19}$$

Thus, we define the functional

$$L_1(\xi,t)=N_1\left(1+\xi^2+\xi^4\right)\hat{\mathscr{E}}(\xi,t)$$
$$+\left\{\frac{\xi^4}{1+\xi^2}F(\xi,t)+\alpha_2\xi^4H(\xi,t)+\alpha_3\xi^2K(\xi,t)\right\}, \tag{3.20}$$

where N_1,α_2 and α_3 are positive constants that will be fixed later. Taking the derivative of $L_1(\xi,t)$ with respect to t and exploiting the estimates (3.7), (3.11), (3.18) and (3.19), we get

$$\frac{dL_1(\xi,t)}{dt}+(1-\epsilon)\frac{\xi^6}{1+\xi^2}|\hat{u}|^2+\left\{\alpha_2(1-\epsilon)-\epsilon_0'\alpha_3-C(\epsilon)\right\}\xi^4|\hat{v}|^2$$
$$+\left\{(a-\epsilon)-\alpha_3\epsilon_0'\right\}\frac{\xi^6}{1+\xi^2}|\hat{z}|^2+\left\{\alpha_3(\gamma-\epsilon_0)-\alpha_2-C(\epsilon)\right\}\xi^4|\hat{y}|^2$$
$$+\alpha_3\left(\frac{\delta^2}{\gamma}-\epsilon_0\right)\xi^4|\hat{w}|^2\leq\left\{C(\epsilon,\epsilon_0,\epsilon_0',\alpha_2,\alpha_3)-N_1\beta k\right\}\xi^2\left(1+\xi^2+\xi^4\right)|\hat{\eta}|^2.$$

In what follows, we choose ϵ and ϵ_0 small enough such that $\epsilon < \min(1, a)$ and $\epsilon_0 < \min\big(\gamma, \frac{\delta^2}{\gamma}\big)$. Next, we fix α_2 large enough such that $\alpha_2(1-\epsilon) > C(\epsilon)$. Once α_2 is fixed, we choose α_3 large enough such that $\alpha_3 > \alpha_2 + C(\epsilon)$. After that, we fix ϵ_0' small enough such that $\epsilon_0' < \min\left((a-\epsilon)/\alpha_3, (\alpha_2(1-\epsilon) - C(\epsilon))/\alpha_3\right)$. Finally, we pick N_1 large enough such that $N_1\beta k > C(\epsilon, \epsilon_0, \epsilon_0', \alpha_2, \alpha_3)$. Consequently, the above estimate takes the form

$$\frac{dL_1(\xi,t)}{dt} + \lambda_1 P_1(\xi,t) \le 0, \qquad \forall t \ge 0, \tag{3.21}$$

for some positive constant λ_1 and

$$\begin{aligned} P_1(\xi,t) &= \frac{\xi^6}{1+\xi^2}\left(|\hat{u}|^2 + |\hat{z}|^2\right) + \xi^4\left(|\hat{y}|^2 + |\hat{v}|^2 + |\hat{w}|^2\right) \\ &\quad + \xi^2\left(1+\xi^2+\xi^4\right)|\hat{\eta}|^2 \\ &\ge C\frac{\xi^6}{1+\xi^2}\hat{\mathscr{E}}(\xi,t). \end{aligned} \tag{3.22}$$

On the other hand, it is not difficult to see that for N_1 large enough, there exist two positive constants β_3 and β_4 such that

$$\begin{aligned} \beta_3\left(1+\xi^2+\xi^4\right)\hat{\mathscr{E}}(\xi,t) &\le L_1(\xi,t) \\ &\le \beta_4\left(1+\xi^2+\xi^4\right)\hat{\mathscr{E}}(\xi,t), \qquad \forall t \ge 0. \end{aligned} \tag{3.23}$$

This last inequality together with (3.21) and (3.22) yield

$$\hat{\mathscr{E}}(\xi,t) \le Ce^{-\varrho_1(\xi)t}\hat{\mathscr{E}}(\xi,0), \qquad \forall t \ge 0, \tag{3.24}$$

where $\varrho_1(\xi)$ is defined in (3.4).

Case 2: $a \ne 1$**.**

In this case, the identity (3.12) becomes

$$\begin{aligned} \frac{dH(\xi,t)}{dt} + (1-\epsilon)|\hat{v}|^2 &\le C(\epsilon_1)\left(1+\xi^2\right)|\hat{y}|^2 + \epsilon_1\frac{\xi^2}{1+\xi^2}|\hat{u}|^2 \\ &\quad + C(\epsilon)\,\xi^2|\hat{\eta}|^2. \end{aligned} \tag{3.25}$$

Also, the estimate (3.18) becomes

$$\begin{aligned} &\frac{d}{dt}K(\xi,t) + (\gamma-\epsilon_0)\,\xi^2|\hat{y}|^2 + \left(\frac{\delta^2}{\gamma} - \epsilon_0\right)\xi^2|\hat{w}|^2 \\ &\quad \le C(\epsilon_0,\epsilon_0')\left(1+\xi^2+\xi^4\right)|\hat{\eta}|^2 + \epsilon_0'\frac{\xi^4}{(1+\xi^2)^2}|\hat{z}|^2 + \epsilon_0'\frac{\xi^2}{1+\xi^2}|\hat{v}|^2, \end{aligned} \tag{3.26}$$

where we have used the estimates

$$\begin{aligned}\left|\xi^2 \operatorname{Re}\left(a\hat{z}\bar{\hat{\eta}}\right)\right| &\leq \epsilon_0' \frac{\xi^4}{(1+\xi^2)^2}\left|\hat{z}\right|^2 + C\left(\epsilon_0'\right)\left(1+\xi^2\right)^2\left|\hat{\eta}\right|^2 \\ &\leq \epsilon_0' \frac{\xi^4}{(1+\xi^2)^2}\left|\hat{z}\right|^2 + C\left(\epsilon_0'\right)\left(1+\xi^2+\xi^4\right)\left|\hat{\eta}\right|^2\end{aligned}$$

and

$$\operatorname{Re}\left(i\xi\bar{\hat{\eta}}\hat{v}\right) \leq \epsilon_0' \frac{\xi^2}{1+\xi^2}|\hat{v}|^2 + C(\epsilon_0')(1+\xi^2)|\hat{\eta}|^2.$$

Now, we define the functional

$$\begin{aligned}L_2\left(\xi,t\right) = N_2\left(1+\xi^2+\xi^4\right)\hat{\mathscr{E}}(\xi,t) + \frac{\xi^4}{(1+\xi^2)^2}F(\xi,t) \\ + \tilde{\alpha}_2 \frac{\xi^4}{1+\xi^2}H(\xi,t) + \tilde{\alpha}_3\xi^2 K\left(\xi,t\right).\end{aligned} \tag{3.27}$$

Consequently, taking the derivative of $L_2\left(\xi,t\right)$ with respect to t and using (3.7), (3.11), (3.26) and (3.25), we get

$$\begin{aligned}&\frac{d}{dt}L_2\left(\xi,t\right) + \Big\{(1-\epsilon) - \tilde{\alpha}_2\epsilon_1\Big\}\frac{\xi^6}{(1+\xi^2)^2}\left|\hat{u}\right|^2 \\ &\quad + \Big\{\left(1-\epsilon\right)\tilde{\alpha}_2 - C(\epsilon) - \epsilon_0'\tilde{\alpha}_3\Big\}\frac{\xi^4}{1+\xi^2}\left|\hat{v}\right|^2 \\ &\quad + \Big\{(a-\epsilon) - \epsilon_0'\tilde{\alpha}_3\Big\}\frac{\xi^6}{(1+\xi^2)^2}\left|\hat{z}\right|^2 + \left(\frac{\delta^2}{\gamma} - \epsilon_0\right)\tilde{\alpha}_3\xi^4\left|\hat{w}\right|^2 \\ &\quad + \Big\{\tilde{\alpha}_3\left(\gamma-\epsilon_0\right) - C\left(\epsilon_1\right)\tilde{\alpha}_2 - C\left(\epsilon\right)\Big\}\xi^4\left|\hat{y}\right|^2 \\ &\qquad \leq \Big\{C(\epsilon,\epsilon_0,\epsilon_0',\epsilon_1,\tilde{\alpha}_2,\tilde{\alpha}_3) - N_2\Big\}\xi^2(1+\xi^2+\xi^4)|\hat{\eta}|^2.\end{aligned} \tag{3.28}$$

We choose the constants as in the first case, in particular $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ like α_2 and α_3, respectively. Once these constants are fixed. We pick ϵ_1 small enough such that $\epsilon_1 < (1-\epsilon)/\tilde{\alpha}_2$. Then, we choose N_2 large enough such that $N_2 > C(\epsilon,\epsilon_0,\epsilon_0',\epsilon_1,\tilde{\alpha}_2,\tilde{\alpha}_3)$. Thus, we obtain from (3.28)

$$\frac{dL_2\left(\xi,t\right)}{dt} + \lambda_2 P_2\left(\xi,t\right) \leq 0, \qquad \forall t \geq 0, \tag{3.29}$$

for some positive constant λ_2 and

$$P_2(\xi,t) = \frac{\xi^6}{(1+\xi^2)^2}\left(|\hat{u}|^2 + |\hat{z}|^2\right) + \xi^4\left(|\hat{y}|^2 + |\hat{w}|^2\right) + \frac{\xi^4}{1+\xi^2}|\hat{v}|^2 + \xi^2\left(1+\xi^2+\xi^4\right)|\hat{\eta}|^2 \geq C\frac{\xi^6}{(1+\xi^2)^2}\hat{\mathscr{E}}(\xi,t).$$

It is also not hard to see that if N_2 is large enough, then a similar estimate like (3.23) holds for L_2. Consequently, as above we conclude

$$\hat{\mathscr{E}}(\xi,t) \leq Ce^{-\varrho_2(\xi)t}\hat{\mathscr{E}}(\xi,0), \qquad \forall t \geq 0,$$

where $\varrho_2(\xi)$ is given in (3.4). This finishes the proof of Proposition 3.1. ■

3.1. Proof of Theorem 2.1. It is clear that the functions $\varrho_i(\xi)$, $i = 1, 2$ satisfy the following estimates

$$\varrho_1(\xi) \geq \begin{cases} c_1|\xi|^6, & \text{for } |\xi| \leq 1, \\ c_2, & \text{for } |\xi| \geq 1, \end{cases}$$

and

$$\varrho_2(\xi) \geq \begin{cases} c_3|\xi|^6, & \text{for } |\xi| \leq 1, \\ c_4|\xi|^{-2}, & \text{for } |\xi| \geq 1, \end{cases}$$

where $c_i > 0$, $i = 1, \ldots, 4$. Consequently, the proof can be finished exactly as in [9]. We omit the details. ■

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