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FRACTIONAL REPRESENTATION FORMULAE UNDER INITIAL CONDITIONS AND FRACTIONAL OSTROWSKI TYPE INEQUALITIES

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Abstract. Here we present very general fractional representation formulae for a function in terms of the fractional Riemann–Liouville integrals of different orders of the function and its ordinary derivatives under initial conditions. Based on these, we derive general fractional Ostrowski type inequalities with respect to all basic norms.

1. Introduction

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$, and let $f' : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, then the following Montgomery identity holds [2]:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \int_a^b P_1(x,t)f'(t)dt, \tag{1}$$

where $P_1(x, t)$ is the Peano kernel

$$P_1(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \le t \le x, \\ \frac{t-b}{b-a}, & x < t \le b. \end{cases} \tag{2}$$

The Riemann–Liouville integral operator of order $\alpha > 0$ with anchor point $a \in \mathbb{R}$ is defined by

$$J_a^\alpha f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \tag{3}$$

$$J_a^0 f(x) := f(x), \quad x \in [a, b]. \tag{4}$$

Properties of the above operator can be found in [3].

When $\alpha = 1$, J_a^1 reduces to the classical integral.

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In [1], we proved the following fractional representation formula of Montgomery identity type.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$, and $f' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, $\alpha \geq 1$, $x \in [a,b)$. Then*

$$f(x) = (b-x)^{1-\alpha}\Gamma(\alpha) \cdot \left\{ \frac{J_a^{\alpha} f(b)}{b-a} - J_a^{\alpha-1}(P_1(x,b)f(b)) + J_a^{\alpha}(P_1(x,b)f'(b)) \right\}. \tag{5}$$

When $\alpha = 1$, the last (5) reduces to classic Montgomery identity (1).

In this article, we find higher order fractional representation for $f(x)$, similar to basic (5), and from there we derive interesting fractional Ostrowski type inequalities.

2. Main results

Next, we give higher order fractional representation of f subject to initial conditions.

Theorem 2. *Let $\alpha > 2$, $x \in [a,b)$ fixed, $f : [a,b] \to \mathbb{R}$ twice differentiable, with $f'' : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Assume $f'(x) = 0$. Then*

$$f(x) = \frac{(b-x)^{2-\alpha}}{\alpha-1}\Bigg[-(b-a)^{\alpha-2}f(a) + \Gamma(\alpha)\Bigg\{ \frac{2}{b-a}J_a^{\alpha-1}f(b) - J_a^{\alpha-2}(P_1(x,b)f(b)) + J_a^{\alpha}(P_1(x,b)f''(b)) \Bigg\}\Bigg]. \tag{6}$$

Proof. Let here $\alpha > 2$ and there exists $f'' : [a,b] \to \mathbb{R}$ that is integrable on $[a,b]$.

We have

$$\Gamma(\alpha)J_a^{\alpha}(P_1(x,b)f''(b)) = \int_a^b (b-t)^{\alpha-1}P_1(x,t)f''(t)dt \tag{7}$$

$$= \int_a^x \left(\frac{t-a}{b-a}\right)(b-t)^{\alpha-1}f''(t)dt + \int_x^b \left(\frac{t-b}{b-a}\right)(b-t)^{\alpha-1}f''(t)dt$$

$$= \int_a^x \left(\frac{t-a}{b-a}\right)(b-t)^{\alpha-1}f''(t)dt + \int_a^b \left(\frac{t-b}{b-a}\right)(b-t)^{\alpha-1}f''(t)dt - \int_a^x \left(\frac{t-b}{b-a}\right)(b-t)^{\alpha-1}f''(t)dt \tag{8}$$

$$= \int_a^x (b-t)^{\alpha-1}f''(t)dt - \frac{1}{(b-a)}\int_a^b (b-t)^{\alpha}f''(t)dt.$$

That is

$$\begin{aligned}(9)\qquad &\Gamma(\alpha)J_a^\alpha(P_1(x,b)f''(b))\\ &= \int_a^x (b-t)^{\alpha-1}f''(t)dt - \frac{1}{(b-a)}\int_a^b (b-t)^\alpha f''(t)dt =: (\xi_1).\end{aligned}$$

Next, we use integration by parts, plus the assumption $f'(x)=0$. We have

$$\begin{aligned}(10)\qquad &\int_a^x (b-t)^{\alpha-1}f''(t)dt = \int_a^x (b-t)^{\alpha-1}df'(t)\\ &= -(b-a)^{\alpha-1}f'(a) - \int_a^x f'(t)d(b-t)^{\alpha-1}\\ &= -(b-a)^{\alpha-1}f'(a) + (\alpha-1)\int_a^x (b-t)^{\alpha-2}df(t)\\ &= -(b-a)^{\alpha-1}f'(a)\\ &\quad + (\alpha-1)\left[(b-x)^{\alpha-2}f(x) - (b-a)^{\alpha-2}f(a) - \int_a^x f(t)d(b-t)^{\alpha-2}\right]\\ (11)\qquad &= -(b-a)^{\alpha-1}f'(a) + (\alpha-1)(b-x)^{\alpha-2}f(x) - (\alpha-1)(b-a)^{\alpha-2}f(a)\\ &\quad + (\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3}f(t)dt.\end{aligned}$$

That is

$$\begin{aligned}(12)\qquad &\int_a^x (b-t)^{\alpha-1}f''(t)dt = -(b-a)^{\alpha-1}f'(a) + (\alpha-1)(b-x)^{\alpha-2}f(x)\\ &- (\alpha-1)(b-a)^{\alpha-2}f(a) + (\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3}f(t)dt =: (\lambda_1).\end{aligned}$$

Next, we observe

$$\begin{aligned}(13)\qquad &-\frac{1}{(b-a)}\int_a^b (b-t)^\alpha f''(t)dt = -\frac{1}{(b-a)}\left[\int_a^b (b-t)^\alpha df'(t)\right]\\ &= -\frac{1}{(b-a)}\left[-(b-a)^\alpha f'(a) + \alpha\int_a^b (b-t)^{\alpha-1}df(t)\right]\\ &= -\frac{1}{(b-a)}\\ &\quad\cdot\left[-(b-a)^\alpha f'(a) - \alpha(b-a)^{\alpha-1}f(a) + \alpha(\alpha-1)\int_a^b (b-t)^{\alpha-2}f(t)dt\right]\\ &= (b-a)^{\alpha-1}f'(a) + \alpha(b-a)^{\alpha-2}f(a) - \frac{\alpha(\alpha-1)}{(b-a)}\int_a^b (b-t)^{\alpha-2}f(t)dt.\end{aligned}$$

That is

$$(14)\qquad -\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f''(t)dt = (b-a)^{\alpha-1} f'(a)$$
$$+\alpha(b-a)^{\alpha-2} f(a) - \frac{\alpha(\alpha-1)}{(b-a)}\int_a^b (b-t)^{\alpha-2} f(t)dt =: (\lambda_2).$$

We have that

$$(\xi_1) = (\lambda_1) + (\lambda_2).$$

Thus

$$(15)\qquad \Gamma(\alpha) J_a^{\alpha}(P_1(x,b) f''(b)) = (\alpha-1)(b-x)^{\alpha-2} f(x) + (b-a)^{\alpha-2} f(a)$$
$$+(\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3} f(t)dt - \frac{\alpha(\alpha-1)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)dt.$$

Notice that

$$(16)\qquad -\alpha(\alpha-1) = -(\alpha-1)(\alpha-2) - 2(\alpha-1).$$

We split

$$(17)\qquad -\frac{\alpha(\alpha-1)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)dt$$
$$= -\frac{(\alpha-1)(\alpha-2)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)dt - \frac{2(\alpha-1)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)dt.$$

But we see that

$$(18)\qquad -\frac{(\alpha-1)(\alpha-2)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)dt$$
$$= -\frac{(\alpha-1)(\alpha-2)}{b-a}\left[\int_a^x (b-t)^{\alpha-2} f(t)dt + \int_x^b (b-t)^{\alpha-2} f(t)dt\right]$$
$$(19)\quad = -\frac{(\alpha-1)(\alpha-2)}{b-a}\left[\int_a^x (b-t)(b-t)^{\alpha-3} f(t)dt + \int_x^b (b-t)(b-t)^{\alpha-3} f(t)dt\right]$$
$$= -(\alpha-1)(\alpha-2)$$
$$\cdot\left[\int_a^x (1-\left(\frac{t-a}{b-a}\right))(b-t)^{\alpha-3} f(t)dt - \int_x^b \left(\frac{t-b}{b-a}\right)(b-t)^{\alpha-3} f(t)dt\right]$$
$$(20) = -(\alpha-1)(\alpha-2)\left[\int_a^x (b-t)^{\alpha-3} f(t)dt - \left[\int_a^x \left(\frac{t-a}{b-a}\right)(b-t)^{\alpha-3} f(t)dt\right.\right.$$
$$\left.\left. + \int_x^b \left(\frac{t-b}{b-a}\right)(b-t)^{\alpha-3} f(t)dt\right]\right]$$

$$(21) = -(\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3} f(t)dt + (\alpha-1)(\alpha-2)\int_a^b P_1(x,t)(b-t)^{\alpha-3} f(t)dt.$$

Therefore

$$-\frac{\alpha(\alpha-1)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)dt = -\frac{2(\alpha-1)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)dt + (\alpha-1)(\alpha-2)\int_a^b P_1(x,t)(b-t)^{\alpha-3} f(t)dt - (\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3} f(t)dt. \tag{22}$$

Hence, it holds

$$\Gamma(\alpha) J_a^\alpha (P_1(x,b) f''(b)) = (\alpha-1)(b-x)^{\alpha-2} f(x) + (b-a)^{\alpha-2} f(a) - \frac{2(\alpha-1)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)dt + (\alpha-1)(\alpha-2)\int_a^b P_1(x,t)(b-t)^{\alpha-3} f(t)dt \tag{23}$$

$$(24) = (\alpha-1)(b-x)^{\alpha-2} f(x) + (b-a)^{\alpha-2} f(a) - \frac{2(\alpha-1)\Gamma(\alpha-1)}{b-a} J_a^{\alpha-1} f(b) + (\alpha-1)(\alpha-2)\Gamma(\alpha-2) J_a^{\alpha-2}(P_1(x,b) f(b))$$

$$(25) = (\alpha-1)(b-x)^{\alpha-2} f(x) + (b-a)^{\alpha-2} f(a) - \frac{2\Gamma(\alpha)}{b-a} J_a^{\alpha-1} f(b) + \Gamma(\alpha) J_a^{\alpha-2}(P_1(x,b) f(b)).$$

We have proved that

$$(\alpha-1)(b-x)^{\alpha-2} f(x) = -(b-a)^{\alpha-2} f(a) + \frac{2\Gamma(\alpha)}{b-a} J_a^{\alpha-1} f(b) - \Gamma(\alpha) J_a^{\alpha-2}(P_1(x,b) f(b)) + \Gamma(\alpha) J_a^{\alpha}(P_1(x,b) f''(b)) \tag{26}$$

$$= -(b-a)^{\alpha-2} f(a) + \Gamma(\alpha)\left\{\frac{2}{b-a} J_a^{\alpha-1} f(b) - J_a^{\alpha-2}(P_1(x,b) f(b)) + J_a^{\alpha}(P_1(x,b) f''(b))\right\}. \tag{27}$$

We have produced (6). ■

We continue with

THEOREM 3. *Let $\alpha > 3$, $x \in [a,b)$ fixed, $f : [a,b] \to \mathbb{R}$ three times differentiable, with $f''' : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Assume $f'(x) = f''(x) = 0$. Then*

$$f(x) = \frac{(b-x)^{3-\alpha}}{(\alpha-1)(\alpha-2)}\Bigg\{-2(\alpha-1)(b-a)^{\alpha-3}f(a) - (b-a)^{\alpha-2}f'(a) + \Gamma(\alpha)\Bigg\{\frac{3}{(b-a)}J_a^{\alpha-2}f(b) - J_a^{\alpha-3}(P_1(x,b)f(b)) + J_a^{\alpha}(P_1(x,b)f'''(b))\Bigg\}\Bigg\}. \tag{28}$$

Proof. Let here $\alpha > 3$ and there exists $f''' : [a,b] \to \mathbb{R}$ that is integrable on $[a,b]$. As before, we have that

$$\Gamma(\alpha)J_a^{\alpha}(P_1(x,b)f'''(b)) = \int_a^x (b-t)^{\alpha-1}f'''(t)dt - \frac{1}{(b-a)}\int_a^b (b-t)^{\alpha}f'''(t)dt =: (\xi_2). \tag{29}$$

By assumption, we have $f'(x) = f''(x) = 0$. Next, we use repeatedly integration by parts

$$\int_a^x (b-t)^{\alpha-1}f'''(t)dt = \int_a^x (b-t)^{\alpha-1}df''(t)$$

$$= -(b-a)^{\alpha-1}f''(a) + (\alpha-1)\int_a^x (b-t)^{\alpha-2}f''(t)dt$$

$$= -(b-a)^{\alpha-1}f''(a) + (\alpha-1)\int_a^x (b-t)^{\alpha-2}df'(t)$$

$$= -(b-a)^{\alpha-1}f''(a) + (\alpha-1)\left[-(b-a)^{\alpha-2}f'(a) + (\alpha-2)\int_a^x (b-t)^{\alpha-3}f'(t)dt\right]$$

$$= -(b-a)^{\alpha-1}f''(a) - (\alpha-1)(b-a)^{\alpha-2}f'(a) + (\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3}df(t) \tag{30}$$

$$= -(b-a)^{\alpha-1}f''(a) - (\alpha-1)(b-a)^{\alpha-2}f'(a) + (\alpha-1)(\alpha-2)\left[(b-x)^{\alpha-3}f(x) - (b-a)^{\alpha-3}f(a) + (\alpha-3)\int_a^x (b-t)^{\alpha-4}f(t)dt\right]$$

$$= -(b-a)^{\alpha-1}f''(a) - (\alpha-1)(b-a)^{\alpha-2}f'(a) + (\alpha-1)(\alpha-2)(b-x)^{\alpha-3}f(x) - (\alpha-1)(\alpha-2)(b-a)^{\alpha-3}f(a) + (\alpha-1)(\alpha-2)(\alpha-3)\int_a^x (b-t)^{\alpha-4}f(t)dt. \tag{31}$$

That is

$$\int_a^x (b-t)^{\alpha-1}f'''(t)dt = -(b-a)^{\alpha-1}f''(a) - (\alpha-1)(b-a)^{\alpha-2}f'(a) + (\alpha-1)(\alpha-2)(b-x)^{\alpha-3}f(x) - (\alpha-1)(\alpha-2)(b-a)^{\alpha-3}f(a) + (\alpha-1)(\alpha-2)(\alpha-3)\int_a^x (b-t)^{\alpha-4}f(t)dt =: (\omega_1). \tag{32}$$

Similarly, we find

$$
\begin{aligned}
(33)\qquad & -\frac{1}{(b-a)}\int_a^b (b-t)^\alpha f'''(t)dt = -\frac{1}{(b-a)}\int_a^b (b-t)^\alpha df''(t) \\
& \qquad = -\frac{1}{(b-a)}\left[-(b-a)^\alpha f''(a) + \alpha\int_a^b (b-t)^{\alpha-1} f''(t)dt\right] \\
& \qquad = (b-a)^{\alpha-1} f''(a) - \frac{\alpha}{(b-a)}\int_a^b (b-t)^{\alpha-1} df'(t) \\
(34)\qquad & = (b-a)^{\alpha-1} f''(a) \\
& \quad - \frac{\alpha}{(b-a)}\left[-(b-a)^{\alpha-1} f'(a) + (\alpha-1)\int_a^b (b-t)^{\alpha-2} f'(t)dt\right] \\
& = (b-a)^{\alpha-1} f''(a) + \alpha(b-a)^{\alpha-2} f'(a) - \frac{\alpha(\alpha-1)}{(b-a)}\int_a^b (b-t)^{\alpha-2} df(t) \\
(35)\qquad & = (b-a)^{\alpha-1} f''(a) + \alpha(b-a)^{\alpha-2} f'(a) \\
& \quad - \frac{\alpha(\alpha-1)}{(b-a)}\left[-(b-a)^{\alpha-2} f(a) + (\alpha-2)\int_a^b (b-t)^{\alpha-3} f(t)dt\right] \\
(36)\qquad & = (b-a)^{\alpha-1} f''(a) + \alpha(b-a)^{\alpha-2} f'(a) \\
& \quad + \alpha(\alpha-1)(b-a)^{\alpha-3} f(a) - \frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)dt.
\end{aligned}
$$

That is, we found

$$
\begin{aligned}
(37)\qquad & -\frac{1}{(b-a)}\int_a^b (b-t)^\alpha f'''(t)dt = (b-a)^{\alpha-1} f''(a) + \alpha(b-a)^{\alpha-2} f'(a) \\
& + \alpha(\alpha-1)(b-a)^{\alpha-3} f(a) - \frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)dt =: (\omega_2).
\end{aligned}
$$

Notice that

$$(\xi_2) = (\omega_1) + (\omega_2).$$

We have

$$
\begin{aligned}
(38)\qquad & \Gamma(\alpha) J_a^\alpha (P_1(x,b) f'''(b)) \\
& = (b-a)^{\alpha-2} f'(a) + (\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) \\
& \quad + 2(\alpha-1)(b-a)^{\alpha-3} f(a) + (a-1)(\alpha-2)(\alpha-3)\int_a^x (b-t)^{\alpha-4} f(t)dt \\
& \quad - \frac{\alpha(a-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)dt.
\end{aligned}
$$

We notice that

$$(39)\qquad -\alpha(\alpha-1)(\alpha-2) = -3(\alpha-1)(\alpha-2) - (\alpha-1)(\alpha-2)(\alpha-3).$$

Hence

$$(40)\qquad -\frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)dt = -\frac{3(\alpha-1)(\alpha-2)}{(b-a)} \cdot \int_a^b (b-t)^{\alpha-3} f(t)dt - \frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)dt.$$

But we see that

$$(41)\qquad -\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)dt$$

$$= -\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\left[\int_a^x (b-t)^{\alpha-3} f(t)dt + \int_x^b (b-t)^{\alpha-3} f(t)dt\right]$$

$$(42)\ = -\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)} \cdot \left[\int_a^x (b-t)(b-t)^{\alpha-4} f(t)dt + \int_x^b (b-t)(b-t)^{\alpha-4} f(t)dt\right]$$

$$= -\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)} \cdot \left[\int_a^x ((b-a)-(t-a))(b-t)^{\alpha-4} f(t)dt - \int_x^b (t-b)(b-t)^{\alpha-4} f(t)dt\right]$$

$$= -(\alpha-1)(\alpha-2)(\alpha-3)\left[\int_a^x (1-\left(\frac{t-a}{b-a}\right))(b-t)^{\alpha-4} f(t)dt - \int_x^b \left(\frac{t-b}{b-a}\right)(b-t)^{\alpha-4} f(t)dt\right]$$

$$(43)\ = -(\alpha-1)(\alpha-2)(\alpha-3)\left[\int_a^x (b-t)^{\alpha-4} f(t)dt - \int_a^b P_1(x,t)(b-t)^{\alpha-4} f(t)dt\right].$$

We derived that

$$(44)\qquad -\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)dt = -(\alpha-1)(\alpha-2)(\alpha-3)\int_a^x (b-t)^{\alpha-4} f(t)dt + (\alpha-1)(\alpha-2)(\alpha-3)\int_a^b P_1(x,t)(b-t)^{\alpha-4} f(t)dt.$$

Therefore, we obtain

$$(45)\qquad -\frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)dt$$
$$= -\frac{3(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)dt$$
$$-(\alpha-1)(\alpha-2)(\alpha-3)\int_a^x (b-t)^{\alpha-4} f(t)dt$$
$$+(\alpha-1)(\alpha-2)(\alpha-3)\int_a^b P_1(x,t)(b-t)^{\alpha-4} f(t)dt.$$

Combining (38) and (45), we find

$$(46)\qquad \Gamma(\alpha)J_a^\alpha(P_1(x,b)f'''(b))$$
$$= (b-a)^{\alpha-2} f'(a) + (\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x)$$
$$+ 2(\alpha-1)(b-a)^{\alpha-3} f(a) - \frac{3(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)dt$$
$$+(\alpha-1)(\alpha-2)(\alpha-3)\int_a^b P_1(x,t)(b-t)^{\alpha-4} f(t)dt$$

$$(47) = (b-a)^{\alpha-2} f'(a) + (\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) + 2(\alpha-1)(b-a)^{\alpha-3} f(a)$$
$$-\frac{3(\alpha-1)(\alpha-2)\Gamma(\alpha-2)}{b-a} J_a^{\alpha-2} f(b)$$
$$+(\alpha-1)(\alpha-2)(\alpha-3)\Gamma(\alpha-3)J_a^{\alpha-3}(P_1(x,b)f(b))$$

$$(48) = (b-a)^{\alpha-2} f'(a) + (\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) + 2(\alpha-1)(b-a)^{\alpha-3} f(a)$$
$$-\frac{3\Gamma(\alpha)}{(b-a)} J_a^{\alpha-2} f(b) + \Gamma(\alpha)J_a^{\alpha-3}(P_1(x,b)f(b)).$$

Consequently, we get

$$(49)\quad (\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) = -(b-a)^{\alpha-2} f'(a) - 2(\alpha-1)(b-a)^{\alpha-3} f(a)$$
$$+\frac{3\Gamma(\alpha)}{(b-a)} J_a^{\alpha-2} f(b) - \Gamma(\alpha)J_a^{\alpha-3}(P_1(x,b)f(b)) + \Gamma(\alpha)J_a^\alpha(P_1(x,b)f'''(b))$$

$$(50) = -(b-a)^{\alpha-2} f'(a) - 2(\alpha-1)(b-a)^{\alpha-3} f(a)$$
$$+\Gamma(\alpha)\left\{\frac{3}{(b-a)} J_a^{\alpha-2} f(b) - J_a^{\alpha-3}(P_1(x,b)f(b)) + J_a^\alpha(P_1(x,b)f'''(b))\right\},$$

proving the claim. ■

We continue with

Theorem 4. *Let $\alpha > 4$, $x \in [a,b)$ fixed, $f : [a,b] \to \mathbb{R}$ four times differentiable, with $f^{(4)} : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Assume $f'(x) = f''(x) = f'''(x) = 0$. Then*

$$f(x) = \frac{(b-x)^{4-\alpha}}{(\alpha-1)(\alpha-2)(\alpha-3)}\Big\{-3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4}f(\alpha) - 2(\alpha-1)(b-a)^{\alpha-3}f'(a) - (b-a)^{\alpha-2}f^{(2)}(a) + \Gamma(\alpha)\Big\{\frac{4J_a^{\alpha-3}(f(b))}{(b-a)} - J_a^{\alpha-4}(P_1(x,b)f(b)) + J_a^{\alpha}(P_1(x,b)f^{(4)}(b))\Big\}\Big\}. \tag{51}$$

Proof. Let here $\alpha > 4$ and there exists $f^{(4)} : [a,b] \to \mathbb{R}$ that is integrable on $[a,b]$. As before, we have that

$$\Gamma(\alpha)J_a^{\alpha}(P_1(x,b)f^{(4)}(b)) = \int_a^x (b-t)^{\alpha-1}f^{(4)}(t)dt - \frac{1}{(b-a)}\int_a^b (b-t)^{\alpha}f^{(4)}(t)dt =: (\xi_3).$$

By assumption, we have $f'(x) = f''(x) = f'''(x) = 0$. Next, we use repeatedly integration by parts

$$\int_a^x (b-t)^{\alpha-1}f^{(4)}(t)dt = \int_a^x (b-t)^{\alpha-1}df^{(3)}(t) \tag{52}$$

$$= -(b-a)^{\alpha-1}f^{(3)}(a) + (\alpha-1)\int_a^x (b-t)^{\alpha-2}df^{(2)}(t) = -(b-a)^{\alpha-1}f^{(3)}(a) + (\alpha-1)\Big[-(b-a)^{\alpha-2}f^{(2)}(a) + (\alpha-2)\int_a^x (b-t)^{\alpha-3}df'(t)\Big]$$

$$= -(b-a)^{\alpha-1}f^{(3)}(a) - (\alpha-1)(b-a)^{\alpha-2}f^{(2)}(a) + (\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3}df'(t)$$

$$= -(b-a)^{\alpha-1}f^{(3)}(a) - (\alpha-1)(b-a)^{\alpha-2}f^{(2)}(a) + (\alpha-1)(\alpha-2)\Big[-(b-a)^{\alpha-3}f'(a) + (\alpha-3)\int_a^x (b-t)^{\alpha-4}df(t)\Big]$$

$$= -(b-a)^{\alpha-1}f^{(3)}(a) - (\alpha-1)(b-a)^{\alpha-2}f^{(2)}(a) - (\alpha-1)(\alpha-2)(b-a)^{\alpha-3}f'(a) + (\alpha-1)(\alpha-2)(\alpha-3)\int_a^x (b-t)^{\alpha-4}df(t) \tag{53}$$

$$= -(b-a)^{\alpha-1}f^{(3)}(a) - (\alpha-1)(b-a)^{\alpha-2}f^{(2)}(a) - (\alpha-1)(\alpha-2)(b-a)^{\alpha-3}f'(a) + (\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4}f(x) - (\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-4}f(a) + (\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\int_a^x (b-t)^{\alpha-5}f(t)dt. \tag{54}$$

We find that

$$\begin{aligned}(55)\qquad \int_a^x (b-t)^{\alpha-1} f^{(4)}(t)dt &= -(b-a)^{\alpha-1} f^{(3)}(a) - (\alpha-1)(b-a)^{\alpha-2} f^{(2)}(a)\\ &-(\alpha-1)(\alpha-2)(b-a)^{\alpha-3} f'(a) + (\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4} f(x)\\ &-(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-4} f(a)\\ &+(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\int_a^x (b-t)^{\alpha-5} f(t)dt =: (\theta_1).\end{aligned}$$

Next, we observe that

$$\begin{aligned}(56)\qquad & -\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f^{(4)}(t)dt = -\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} df^{(3)}(t)\\ &= -\frac{1}{(b-a)}\left[-(b-a)^{\alpha} f^{(3)}(a) + \alpha\int_a^b (b-t)^{\alpha-1} df^{(2)}(t)\right]\\ &= (b-a)^{\alpha-1} f^{(3)}(a) - \frac{\alpha}{b-a}\int_a^b (b-t)^{\alpha-1} df^{(2)}(t) = (b-a)^{\alpha-1} f^{(3)}(a)\\ &-\frac{\alpha}{b-a}\left[-(b-a)^{\alpha-1} f^{(2)}(a) + (\alpha-1)\int_a^b (b-t)^{\alpha-2} df'(t)\right]\\ &= (b-a)^{\alpha-1} f^{(3)}(a) + \alpha(b-a)^{\alpha-2} f^{(2)}(a) - \frac{\alpha(\alpha-1)}{(b-a)}\int_a^b (b-t)^{\alpha-2} df'(t)\\ &= (b-a)^{\alpha-1} f^{(3)}(a) + \alpha(b-a)^{\alpha-2} f^{(2)}(a)\\ &-\frac{\alpha(\alpha-1)}{(b-a)}\left[-(b-a)^{\alpha-2} f'(a) + (\alpha-2)\int_a^b (b-t)^{\alpha-3} df(t)\right]\end{aligned}$$

$$\begin{aligned}(57) &= (b-a)^{\alpha-1} f^{(3)}(a) + \alpha(b-a)^{\alpha-2} f^{(2)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f'(a)\\ &-\frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} df(t)\end{aligned}$$

$$\begin{aligned}(58) &= (b-a)^{a-1} f^{(3)}(a) + \alpha(b-a)^{\alpha-2} f^{(2)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f'(a)\\ &-\frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\left[-(b-a)^{\alpha-3} f(a) + (\alpha-3)\int_a^b (b-t)^{\alpha-4} f(t)dt\right]\\ &= (b-a)^{\alpha-1} f^{(3)}(a) + \alpha(b-a)^{\alpha-2} f^{(2)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f'(a)\\ &+\alpha(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a) - \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-4} f(t)dt.\end{aligned}$$

That is

$$\begin{aligned}(59)\qquad -\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f^{(4)}(t)dt &= (b-a)^{\alpha-1} f^{(3)}(a) + \alpha(b-a)^{\alpha-2} f^{(2)}(a)\\ &+\alpha(\alpha-1)(b-a)^{\alpha-3} f'(a) + \alpha(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a)\\ &-\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-4} f(t)dt =: (\theta_2).\end{aligned}$$

Notice that

$$(\xi_3) = (\theta_1) + (\theta_2). \tag{60}$$

We find that

$$\begin{aligned}\Gamma(\alpha)J_a^\alpha(P_1(x,b)f^{(4)}(b)) &= (b-a)^{\alpha-2}f^{(2)}(a)+2(\alpha-1)(b-a)^{\alpha-3}f'(a)\\ &+3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4}f(a)+(\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4}f(x)\\ &+(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\int_a^x(b-t)^{\alpha-5}f(t)dt\\ &-\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b(b-t)^{\alpha-4}f(t)dt.\end{aligned} \tag{61}$$

We have

$$\begin{aligned}&-\alpha(\alpha-1)(\alpha-2)(\alpha-3)\\ &\quad=-4(\alpha-1)(\alpha-2)(\alpha-3)-(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4),\end{aligned} \tag{62}$$

and

$$\begin{aligned}&\frac{-\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b(b-t)^{\alpha-4}f(t)dt\\ &\quad=-\frac{4(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b(b-t)^{\alpha-4}f(t)dt\\ &\quad\quad-\frac{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b(b-t)^{\alpha-4}f(t)dt.\end{aligned} \tag{63}$$

But we see that

$$\begin{aligned}&-\frac{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b(b-t)^{\alpha-4}f(t)dt\\ &=-\frac{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\\ &\quad\cdot\left[\int_a^x((b-a)-(t-a))(b-t)^{\alpha-5}f(t)dt-\int_x^b(t-b)(b-t)^{\alpha-5}f(t)dt\right]\\ &=-(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\\ &\quad\cdot\left[\int_a^x(b-t)^{\alpha-5}f(t)dt-\int_a^b P_1(x,t)(b-t)^{\alpha-5}f(t)dt\right].\end{aligned} \tag{64}$$

Therefore, it holds

$$
\begin{aligned}
(65)\qquad & -\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-4} f(t)dt \\
&= -\frac{4(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-4} f(t)dt \\
&\quad -(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\int_a^x (b-t)^{\alpha-5} f(t)dt \\
&\quad +(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\int_a^b P_1(x,t)(b-t)^{\alpha-5} f(t)dt.
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
(66)\qquad & \Gamma(\alpha) J_a^{\alpha}(P_1(x,b) f^{(4)}(b)) = (b-a)^{\alpha-2} f^{(2)}(a) + 2(\alpha-1)(b-a)^{\alpha-3} f'(a) \\
& + 3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a) + (\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4} f(x) \\
& - \frac{4(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-4} f(t)dt \\
& + (\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\int_a^b P_1(x,t)(b-t)^{\alpha-5} f(t)dt \\
(67) = {} & (b-a)^{\alpha-2} f^{(2)}(a) + 2(\alpha-1)(b-a)^{\alpha-3} f'(a) + 3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a) \\
& + (\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4} f(x) \\
& - \frac{4(\alpha-1)(\alpha-2)(\alpha-3)\Gamma(\alpha-3)}{(b-a)} J_a^{\alpha-3}(f(b)) \\
& + (\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\Gamma(\alpha-4) J_a^{\alpha-4}(P_1(x,b) f(b)) \\
(68) = {} & (b-a)^{\alpha-2} f^{(2)}(a) + 2(\alpha-1)(b-a)^{\alpha-3} f'(a) + 3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a) \\
& + (\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4} f(x) \\
& - \frac{4\Gamma(\alpha)}{(b-a)} J_a^{\alpha-3}(f(b)) + \Gamma(\alpha) J_a^{\alpha-4}(P_1(x,b) f(b)).
\end{aligned}
$$

That is

$$
\begin{aligned}
(69)\qquad & \Gamma(\alpha) J_a^{\alpha}(P_1(x,b) f^{(4)}(b)) = (b-a)^{\alpha-2} f^{(2)}(a) + 2(\alpha-1)(b-a)^{\alpha-3} f'(a) \\
& + 3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a) + (\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4} f(x) \\
& + \Gamma(\alpha)\left\{-\frac{4 J_a^{\alpha-3}(f(b))}{(b-a)} + J_a^{\alpha-4}(P_1(x,b) f(b))\right\},
\end{aligned}
$$

proving the claim. ∎

We continue with

Theorem 5. *Let $\alpha > 5$, $x \in [a,b)$ fixed, $f : [a,b] \to \mathbb{R}$ five times differentiable, with $f^{(5)} : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Assume $f^{(j)}(x) = 0$, $j = 1,2,3,4$. Then*

$$(70) \qquad f(x) = \frac{(b-x)^{5-\alpha}}{\prod_{j=1}^{4}(\alpha-j)}\Bigg\{-4\prod_{j=1}^{3}(\alpha-j)(b-a)^{\alpha-5}f(a)$$

$$-3\prod_{j=1}^{2}(\alpha-j)(b-a)^{\alpha-4}f'(a)-2(\alpha-1)(b-a)^{\alpha-3}f^{(2)}(a)-(b-a)^{\alpha-2}f^{(3)}(a)$$

$$+\Gamma(\alpha)\Bigg\{\frac{5}{(b-a)}(J_a^{\alpha-4}(f(b)))-J_a^{\alpha-5}(P_1(x,b)f(b))+J_a^{\alpha}(P_1(x,b)f^{(5)}(b))\Bigg\}\Bigg\}.$$

Proof. Let here $\alpha > 5$ and there exists $f^{(5)} : [a,b] \to \mathbb{R}$ that is integrable on $[a,b]$. As before, we have that

$$(71) \qquad \Gamma(\alpha)J_a^{\alpha}(P_1(x,b)f^{(5)}(b))$$
$$= \int_a^x (b-t)^{\alpha-1}f^{(5)}(t)dt - \frac{1}{(b-a)}\int_a^b (b-t)^{\alpha}f^{(5)}(t)dt =: (\xi_4).$$

By assumption, we have $f^{(j)}(x) = 0$, $j = 1,2,3,4$. Next, we use repeatedly integration by parts

$$(72) \qquad \int_a^x (b-t)^{\alpha-1}f^{(5)}(t)dt = \int_a^x (b-t)^{\alpha-1}df^{(4)}(t)$$
$$= -(b-a)^{\alpha-1}f^{(4)}(a)+(\alpha-1)\int_a^x (b-t)^{\alpha-2}df^{(3)}(t) = -(b-a)^{\alpha-1}f^{(4)}(a)$$
$$+(\alpha-1)\Bigg\{-(b-a)^{\alpha-2}f^{(3)}(a)+(\alpha-2)\int_a^x (b-t)^{\alpha-3}df^{(2)}(t)\Bigg\}$$
$$= -(b-a)^{\alpha-1}f^{(4)}(a)-(\alpha-1)(b-a)^{\alpha-2}f^{(3)}(a)$$
$$+(\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3}df^{(2)}(t)$$
$$= -(b-a)^{\alpha-1}f^{(4)}(a)-(\alpha-1)(b-a)^{\alpha-2}f^{(3)}(a)$$
$$+(\alpha-1)(\alpha-2)\Bigg\{-(b-a)^{\alpha-3}f^{(2)}(a)+(\alpha-3)\int_a^x (b-t)^{\alpha-4}df'(t)\Bigg\}$$
$$(73) = -(b-a)^{\alpha-1}f^{(4)}(a)-(\alpha-1)(b-a)^{\alpha-2}f^{(3)}(a)$$
$$-(\alpha-1)(\alpha-2)(b-a)^{\alpha-3}f^{(2)}(a)+(\alpha-1)(\alpha-2)(\alpha-3)\int_a^x (b-t)^{\alpha-4}df'(t)$$

$$= -(b-a)^{\alpha-1}f^{(4)}(a)-(\alpha-1)(b-a)^{\alpha-2}f^{(3)}(a)$$
$$-(\alpha-1)(\alpha-2)(b-a)^{\alpha-3}f^{(2)}(a)$$
$$+(\alpha-1)(\alpha-2)(\alpha-3)\left\{-(b-a)^{\alpha-4}f'(a)+(\alpha-4)\int_a^x(b-t)^{\alpha-5}df(t)\right\}$$

$$(74) = -(b-a)^{\alpha-1}f^{(4)}(a)-(\alpha-1)(b-a)^{\alpha-2}f^{(3)}(a)$$
$$-(\alpha-1)(\alpha-2)(b-a)^{\alpha-3}f^{(2)}(a)-(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-4}f'(a)$$
$$+(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\int_a^x(b-t)^{\alpha-5}df(t)$$
$$= -(b-a)^{\alpha-1}f^{(4)}(a)-(\alpha-1)(b-a)^{\alpha-2}f^{(3)}(a)$$
$$-(\alpha-1)(\alpha-2)(b-a)^{\alpha-3}f^{(2)}(a)$$
$$-(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-4}f'(a)+(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)$$
$$\cdot\left\{(b-x)^{\alpha-5}f(x)-(b-a)^{\alpha-5}f(a)+(\alpha-5)\int_a^x(b-t)^{\alpha-6}f(t)dt\right\}.$$

That is

$$(75)\qquad \int_a^x(b-t)^{\alpha-1}f^{(5)}(t)dt = -(b-a)^{\alpha-1}f^{(4)}(a)-(\alpha-1)(b-a)^{\alpha-2}f^{(3)}(a)$$
$$-(\alpha-1)(\alpha-2)(b-a)^{\alpha-3}f^{(2)}(a)-(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-4}f'(a)$$
$$+(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(b-x)^{\alpha-5}f(x)$$
$$-(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(b-a)^{\alpha-5}f(a)$$
$$+(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)\int_a^x(b-t)^{\alpha-6}f(t)dt =: (\eta_1).$$

Next, we observe that

$$(76)\qquad -\frac{1}{(b-a)}\int_a^b(b-t)^{\alpha}f^{(5)}(t)dt = -\frac{1}{(b-a)}\int_a^b(b-t)^{\alpha}df^{(4)}(t)$$
$$= -\frac{1}{(b-a)}\left\{-(b-a)^{\alpha}f^{(4)}(a)+\alpha\int_a^b(b-t)^{\alpha-1}df^{(3)}(t)\right\}$$
$$= (b-a)^{\alpha-1}f^{(4)}(a)-\frac{\alpha}{(b-a)}\int_a^b(b-t)^{\alpha-1}df^{(3)}(t)$$
$$= (b-a)^{\alpha-1}f^{(4)}(a)$$
$$-\frac{\alpha}{(b-a)}\left\{-(b-a)^{\alpha-1}f^{(3)}(a)+(\alpha-1)\int_a^b(b-t)^{\alpha-2}df^{(2)}(t)\right\}$$
$$= (b-a)^{\alpha-1}f^{(4)}(a)+\alpha(b-a)^{\alpha-2}f^{(3)}(a)-\frac{\alpha(\alpha-1)}{(b-a)}\int_a^b(b-t)^{\alpha-2}df^{(2)}(t)$$

$$= (b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a)$$
$$- \frac{\alpha(\alpha-1)}{(b-a)} \left\{ -(b-a)^{\alpha-2} f^{(2)}(a) + (\alpha-2) \int_a^b (b-t)^{\alpha-3} df'(t) \right\}$$
$$= (b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a)$$
$$- \frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)} \int_a^b (b-t)^{\alpha-3} df'(t)$$
$$= (b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a)$$
$$- \frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)} \left\{ -(b-a)^{\alpha-3} f'(a) + (\alpha-3) \int_a^b (b-t)^{\alpha-4} df(t) \right\}$$

$$(77) = (b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a)$$
$$+ \alpha(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a) - \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)} \int_a^b (b-t)^{\alpha-4} df(t)$$

$$(78) = (b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a)$$
$$+ \alpha(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a) - \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}$$
$$\cdot \left\{ -(b-a)^{\alpha-4} f(a) + (\alpha-4) \int_a^b (b-t)^{\alpha-5} f(t) dt \right\}.$$

We proved that

$$(79) \qquad - \frac{1}{(b-a)} \int_a^b (b-t)^{\alpha} f^{(5)}(t) dt = (b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a)$$
$$+ \alpha(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) + \alpha(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a)$$
$$+ \alpha(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-5} f(a)$$
$$- \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)} \int_a^b (b-t)^{\alpha-5} f(t) dt =: (\eta_2).$$

We have

$$(\xi_4) = (\eta_1) + (\eta_2).$$

Therefore, it holds

$$(80) \qquad \Gamma(\alpha) J_a^{\alpha}(P_1(x,b) f^{(5)}(b)) = (b-a)^{\alpha-2} f^{(3)}(a)$$
$$+ 2(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) + 3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a)$$
$$+ (\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(b-x)^{\alpha-5} f(x)$$
$$+ 4(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-5} f(a)$$

$$+ (\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)\int_a^x (b-t)^{\alpha-6} f(t)dt$$
$$- \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)dt.$$

We see that

$$\begin{aligned}(81)\qquad & -\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)dt \\ &= -\frac{5(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)dt \\ &\quad - \frac{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)dt.\end{aligned}$$

We have

$$\begin{aligned}(82)\qquad & -\frac{\prod_{j=1}^5(\alpha-j)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)dt \\ &= -\frac{\prod_{j=1}^5(\alpha-j)}{(b-a)} \\ &\quad \cdot \left[\int_a^x ((b-a)-(t-a))(b-t)^{\alpha-6} f(t)dt - \int_x^b (t-b)(b-t)^{\alpha-6} f(t)dt\right] \\ &= -\prod_{j=1}^5(\alpha-j)\left[\int_a^x (b-t)^{\alpha-6} f(t)dt - \int_a^b P_1(x,t)(b-t)^{\alpha-6} f(t)dt\right].\end{aligned}$$

Therefore, it holds

$$\begin{aligned}(83)\qquad & -\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)dt \\ &= -\frac{5(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)dt \\ &\quad -\prod_{j=1}^5(\alpha-j)\int_a^x (b-t)^{\alpha-6} f(t)dt + \prod_{j=1}^5(\alpha-j)\int_a^b P_1(x,t)(b-t)^{\alpha-6} f(t)dt.\end{aligned}$$

Consequently, we get

$$\begin{aligned}(84)\qquad & \Gamma(\alpha) J_a^\alpha (P_1(x,b) f^{(5)}(b)) = (b-a)^{\alpha-2} f^{(3)}(a) \\ &\quad + 2(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) + 3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a) \\ &\quad + (\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(b-x)^{\alpha-5} f(x) \\ &\quad + 4(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-5} f(a)\end{aligned}$$

$$- \frac{5(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)} \int_a^b (b-t)^{\alpha-5} f(t)dt$$
$$+ \prod_{j=1}^{5}(\alpha-j) \int_a^b P_1(x,t)(b-t)^{\alpha-6} f(t)dt.$$

So that

$$\Gamma(\alpha) J_a^\alpha (P_1(x,b) f^{(5)}(b)) = (b-a)^{\alpha-2} f^{(3)}(a)$$
$$+ 2(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) + 3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a)$$
$$+ (\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(b-x)^{\alpha-5} f(x)$$
$$+ 4(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-5} f(a)$$
$$- \frac{5(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\Gamma(\alpha-4)}{(b-a)} (J_a^{\alpha-4}(f(b)))$$
$$+ \prod_{j=1}^{5}(\alpha-j)\Gamma(\alpha-5) J_a^{\alpha-5}(P_1(x,b) f(b)). \tag{85}$$

And finally, we derive

$$\prod_{j=1}^{4}(\alpha-j)(b-x)^{\alpha-5} f(x) = -4(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-5} f(a)$$
$$-3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a) - 2(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) - (b-a)^{\alpha-2} f^{(3)}(a)$$
$$+\Gamma(\alpha)\left\{ \frac{5}{(b-a)} (J_a^{\alpha-4}(f(b))) - J_a^{\alpha-5}(P_1(x,b) f(b)) + J_a^\alpha (P_1(x,b) f^{(5)}(b)) \right\}, \tag{86}$$

proving the claim. ∎

In general, the following fractional representation formula holds

Theorem 6. *Let $\alpha > n$, $n \in \mathbb{N}$, $x \in [a,b)$ fixed, $f : [a,b] \to \mathbb{R}$ n-times differentiable, with $f^{(n)} : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Assume $f^{(j)}(x) = 0$, $j = 1, \ldots, n-1$. Then*

$$f(x) = \frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)} \left\{ -(n-1) \prod_{j=1}^{n-2}(\alpha-j)(b-a)^{\alpha-n} f(a) \right. \tag{87}$$
$$- (n-2) \prod_{j=1}^{n-3}(\alpha-j)(b-a)^{\alpha-n+1} f'(a) - (n-3) \prod_{j=1}^{n-4}(\alpha-j)(b-a)^{\alpha-n+2} f^{(2)}(a)$$
$$- (n-4) \prod_{j=1}^{n-5}(\alpha-j)(b-a)^{\alpha-n+3} f^{(3)}(a) - \ldots$$

$$-(b-a)^{\alpha-2}f^{(n-2)}(a)+\Gamma(\alpha)\left\{\frac{n}{b-a}(J_a^{\alpha-n+1}(f(b)))-J_a^{\alpha-n}(P_1(x,b)f(b)) + J_a^{\alpha}(P_1(x,b)f^{(n)}(b))\right\}\Bigg\}.$$

Above we assume that $\prod_{j=1}^{0}(\alpha-j)=1$, *and* $\prod_{j=1}^{k}(\alpha-j)=0$ *if* $k\in\{-1,-2,\dots\}$. *Also set* $f^{(-1)}(a):=0$.

Proof. Based on Theorems 1–5. ∎

Theorems 1–5 are special cases of Theorem 6.

We give applications of Theorem 6 for $n=6,7$.

Theorem 7. *Let* $\alpha>6$, $x\in[a,b)$ *fixed,* $f:[a,b]\to\mathbb{R}$ *six times differentiable, with* $f^{(6)}:[a,b]\to\mathbb{R}$ *integrable on* $[a,b]$. *Assume* $f^{(j)}(x)=0$, $j=1,\dots,5$.

$$f(x)=\frac{(b-x)^{6-\alpha}}{\prod_{j=1}^{5}(\alpha-j)}\Bigg\{-5\prod_{j=1}^{4}(\alpha-j)(b-a)^{\alpha-6}f(a) \tag{88}$$

$$-4\prod_{j=1}^{3}(\alpha-j)(b-a)^{\alpha-5}f'(a)-3\prod_{j=1}^{2}(\alpha-j)(b-a)^{\alpha-4}f^{(2)}(a)$$

$$-2(\alpha-1)(b-a)^{\alpha-3}f^{(3)}(a)-(b-a)^{\alpha-2}f^{(4)}(a)$$

$$+\Gamma(\alpha)\left\{\frac{6}{b-a}(J_a^{\alpha-5}(f(b)))-J_a^{\alpha-6}(P_1(x,b)f(b))+J_a^{\alpha}(P_1(x,b)f^{(6)}(b))\right\}\Bigg\}.$$

Theorem 8. *Let* $\alpha>7$, $x\in[a,b)$ *fixed,* $f:[a,b]\to\mathbb{R}$ *seven times differentiable, with* $f^{(7)}:[a,b]\to\mathbb{R}$ *integrable on* $[a,b]$. *Assume* $f^{(j)}(x)=0$, $j=1,\dots,6$. *Then*

$$f(x)=\frac{(b-x)^{7-\alpha}}{\prod_{j=1}^{6}(\alpha-j)}\Bigg\{-6\prod_{j=1}^{5}(\alpha-j)(b-a)^{\alpha-7}f(a) \tag{89}$$

$$-5\prod_{j=1}^{4}(\alpha-j)(b-a)^{\alpha-6}f'(a)-4\prod_{j=1}^{3}(\alpha-j)(b-a)^{\alpha-5}f''(a)$$

$$-3\prod_{j=1}^{2}(\alpha-j)(b-a)^{\alpha-4}f^{(3)}(a)-2(\alpha-1)(b-a)^{\alpha-3}f^{(4)}(a)$$

$$-(b-a)^{\alpha-2}f^{(5)}(a)$$

$$+\Gamma(\alpha)\left\{\frac{7}{b-a}(J_a^{\alpha-6}(f(b)))-J_a^{\alpha-7}(P_1(x,b)f(b))+J_a^{\alpha}(P_1(x,b)f^{(7)}(b))\right\}\Bigg\}.$$

We make

Remark 9. We rewrite (87) as follows:

$$(90)\qquad E_n(f,\alpha,x) := f(x) + \frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\Bigg\{(n-1)\prod_{j=1}^{n-2}(\alpha-j)(b-a)^{\alpha-n}f(a)$$

$$+(n-2)\prod_{j=1}^{n-3}(\alpha-j)(b-a)^{\alpha-n+1}f'(a)+(n-3)\prod_{j=1}^{n-4}(\alpha-j)(b-a)^{\alpha-n+2}f^{(2)}(a)$$

$$+(n-4)\prod_{j=1}^{n-5}(\alpha-j)(b-a)^{\alpha-n+3}f^{(3)}(a)+\cdots+(b-a)^{\alpha-2}f^{(n-2)}(a)$$

$$+\Gamma(\alpha)\Bigg\{-\frac{n}{b-a}(J_a^{\alpha-n+1}(f(b)))+J_a^{\alpha-n}(P_1(x,b)f(b))\Bigg\}\Bigg\}$$

$$= \frac{(b-x)^{n-\alpha}\Gamma(\alpha)}{\prod_{j=1}^{n-1}(\alpha-j)}J_a^{\alpha}(P_1(x,b)f^{(n)}(b))$$

$$(91)\qquad = \frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\int_a^b (b-t)^{\alpha-1}P_1(x,t)f^{(n)}(t)dt.$$

We upper bound $E_n(f,\alpha,x)$, that is we upper bound the right hand side of (91).

Consequently, we produce fractional Ostrowski type inequalities motivated by [1] done there for $n = 1$.

Theorem 10. *Let $\alpha > n$, $n \in \mathbb{N}$, $x \in [a,b)$ fixed, $f : [a,b] \to \mathbb{R}$ n-times differentiable, with $f^{(n)} : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Assume $f^{(j)}(x) = 0$, $j = 1,\dots,n-1$, and $\|f^{(n)}\|_\infty < \infty$. Then*

$$(92)\qquad |E_n(f,\alpha,x)| \le \frac{\|f^{(n)}\|_\infty}{\prod_{j=1}^{n-1}(\alpha-j)}\left[\frac{(b-x)^{n-\alpha}(b-a)^{\alpha}}{\alpha(\alpha+1)} - \frac{(b-x)^n}{\alpha} + \frac{2(b-x)^{n+1}}{(b-a)(\alpha+1)}\right],$$

where $E_n(f,\alpha,x)$ as in (90).

Proof. We have that

$$(93)\qquad |E_n(f,\alpha,x)| \le \frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\int_a^b (b-t)^{\alpha-1}|P_1(x,t)||f^{(n)}(t)|dt$$

$$\le \frac{(b-x)^{n-\alpha}\|f^{(n)}\|_\infty}{(b-a)(\prod_{j=1}^{n-1}(\alpha-j))}\left[\int_a^x (b-t)^{\alpha-1}(t-a)dt + \int_x^b (b-t)^{\alpha}dt\right]$$

$$= \frac{(b-x)^{n-\alpha}\|f^{(n)}\|_\infty}{(b-a)(\prod_{j=1}^{n-1}(\alpha-j))}\left[\int_a^x (b-t)^{\alpha-1}((b-a)-(b-t))dt + \frac{(b-x)^{\alpha+1}}{\alpha+1}\right]$$

$$= \frac{\|f^{(n)}\|_\infty (b-x)^{n-\alpha}}{(b-a)(\prod_{j=1}^{n-1}(\alpha-j))}\left[(b-a)\int_a^x (b-t)^{\alpha-1}dt - \int_a^x (b-t)^{\alpha}dt + \frac{(b-x)^{\alpha+1}}{\alpha+1}\right]$$

$$= \frac{\|f^{(n)}\|_\infty (b-x)^{n-\alpha}}{(b-a)(\prod_{j=1}^{n-1}(\alpha-j))}$$

$$\cdot\left[(b-a)\left(\frac{(b-a)^\alpha}{\alpha} - \frac{(b-x)^\alpha}{\alpha}\right) - \frac{(b-a)^{\alpha+1}}{\alpha+1} + \frac{2(b-x)^{\alpha+1}}{\alpha+1}\right]$$

$$(94) = \frac{(b-x)^{n-\alpha}\|f^{(n)}\|_\infty}{\prod_{j=1}^{n-1}(\alpha-j)}\left[\frac{(b-a)^\alpha}{\alpha(\alpha+1)} - \frac{(b-x)^\alpha}{\alpha} + \frac{2(b-x)^{\alpha+1}}{(b-a)(\alpha+1)}\right]$$

$$(95) = \frac{\|f^{(n)}\|_\infty}{\prod_{j=1}^{n-1}(\alpha-j)}\left[\frac{(b-x)^{n-\alpha}(b-a)^\alpha}{\alpha(\alpha+1)} - \frac{(b-x)^n}{\alpha} + \frac{2(b-x)^{n+1}}{(b-a)(\alpha+1)}\right]. \blacksquare$$

Theorem 11. *Let all be as in the Theorem 6. Then*

$$(96) \qquad |E_n(f,\alpha,x)| \leq \left(\frac{(b-x)^{n-\alpha}(b-a)^{\alpha-2}}{2\prod_{j=1}^{n-1}(\alpha-j)}\right)(b-a+|a+b-2x|)\|f^{(n)}\|_{L_1([a,b])}.$$

Proof. We have that

$$(97) \qquad |E_n(f,\alpha,x)| \leq \frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\int_a^b (b-t)^{\alpha-1}|P_1(x,t)||f^{(n)}(t)|dt$$

$$\leq \left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\right)(b-a)^{\alpha-2}\max\{x-a,b-x\}\|f^{(n)}\|_{L_1([a,b])}$$

$$(98) = \left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\right)(b-a)^{\alpha-2}\left(\frac{b-a+|a+b-2x|}{2}\right)\|f^{(n)}\|_{L_1([a,b])}. \blacksquare$$

Theorem 12. *Let $p,q,r>1$ such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$. Let all be as in the Theorem 6, but now $f^{(n)} \in L_r([a,b])$. Then*

$$(99) \qquad |E_n(f,\alpha,x)| \leq \left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\right)\frac{(b-a)^{\alpha-2+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}$$

$$\cdot\left\{\frac{(b-x)^{q+1}+(x-a)^{q+1}}{(q+1)}\right\}^{\frac{1}{q}}\|f^{(n)}\|_{L_r([a,b])}.$$

Proof. We have

$$|E_n(f,\alpha,x)| \overset{(97)}{\leq} \left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\right)\left(\int_a^b (b-t)^{(\alpha-1)p}dt\right)^{\frac{1}{p}} \cdot \left(\int_a^b |P_1(x,t)|^q dt\right)^{\frac{1}{q}} \|f^{(n)}\|_{L_r([a,b])} \tag{100}$$

$$= \left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\right)\frac{(b-a)^{(\alpha-2)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot \left(\int_a^x (t-a)^q dt + \int_x^b (b-t)^q dt\right)^{\frac{1}{q}} \|f^{(n)}\|_{L_r([a,b])}$$

$$= \left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\right)\frac{(b-a)^{(\alpha-2)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left(\frac{(x-a)^{q+1}}{(q+1)} + \frac{(b-t)^{q+1}}{(q+1)}\right)^{\frac{1}{q}} \|f^{(n)}\|_{L_r([a,b])}$$

$$= \left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\right)\frac{(b-a)^{\alpha-2+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot \left\{\frac{(b-x)^{q+1}+(x-a)^{q+1}}{(q+1)}\right\}^{\frac{1}{q}} \|f^{(n)}\|_{L_r([a,b])}, \tag{101}$$

proving the claim. ■

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