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FREE ALGEBRAS OVER A POSET IN VARIETIES OF ŁUKASIEWICZ–MOISIL ALGEBRAS

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Abstract. A general construction of the free algebra over a poset in varieties finitely generated is given in [8]. In this paper, we apply this to the varieties of Łukasiewicz–Moisil algebras, giving a detailed description of the free algebra over a finite poset $(X, \leq)$, $\mathbf{Free}_n((X, \leq))$. As a consequence of this description, the cardinality of $\mathbf{Free}_n((X, \leq))$ is computed for special posets.

1. Introduction

In 1945, R. Dilworth ([5]) introduced the notion of free lattice over a poset. Later, this notion was adapted to different classes of algebras that arise from non-classical logics, these classes constitute varieties of algebras, which have an underlying order structure definable by means of certain equations $p_i(x, y) = q_i(x, y)$, $1 \leq i \leq n$, in terms of the algebra's operations and some positive integer n. Constructions of this particular free algebra have been exhibited for different kinds of algebras such as bounded distributive lattices, De Morgan algebras and Hilbert algebras (see [7, 8]).

Consider, now, the set Ω of operations of type τ and the set E of identities. We shall note $\mathbf{Alg}_{\{\Omega,E,\leq\}}$, the category whose objects are $\{\Omega, E\}$-algebras, which have an order structure definable from the operations of Ω and the arrows are the respective $\{\Omega, E\}$-morphisms.

The notion of free algebra over a poset relative to $\mathbf{Alg}_{\{\Omega,E,\leq\}}$ can be defined as follows:

Definition 1. Let $(X, \leq) = X_{\leq}$ be a poset. We shall say that $\mathbf{Free}_{\mathbf{Alg}_{\{\Omega,E,\leq\}}}(X_{\leq})$ is the free algebra over $(X, \leq)$ if the following conditions are satisfied:

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(F1) there is a one-to-one order-preserving function $g : X_{\leq} \longrightarrow \mathbf{Free}_{\mathbf{Alg}_{\{\Omega,E,\leq\}}}(X_{\leq})$,

(F2) for each $A \in \mathbf{Alg}_{\{\Omega,E,\leq\}}$ and each one-to-one order-preserving function $f : X \to A$, there is a unique morphism $h : \mathbf{Free}_{\mathbf{Alg}_{\{\Omega,E,\leq\}}}(X_{\leq}) \longrightarrow A$ such that $h \circ g = f$.

If $X_{\leq}$ is an antichain, then the algebra $\mathbf{Free}_{\mathbf{Alg}_{\{\Omega,E,\leq\}}}(X_{\leq})$ is the usual free algebra (see [3]). The construction of this free algebra can be presented using techniques of the universal algebra (see [3]) or using the well–known Freyd's Adjoint Functor theorem of category theory (see [9]).

A general construction in varieties finitely generated in the following way is given [8]:

Let $\mathcal{V}$ be a variety generated by n algebras S_i, $n < \omega$ and suppose $C = \prod_{i=1}^{n} S_i$ is not an antichain. Let I be a non-empty poset and let E be the set of all increasing functions from I to the $\mathcal{V}$-algebra C. Besides, let $g : I \to C^E$ be defined by $g(i) = G_i$ where $G_i(f) = f(i)$, for all $f \in E$ and $i \in I$. Then, $L = [G]_{\mathcal{V}}$ is the free $\mathcal{V}$-algebra over I, where $G = \{G_i : i \in I\}$ and [G] is the $\mathcal{V}$-algebra generated by G. Indeed, it follows easily that $i \leq j$ implies $G_i \leq G_j$, for all $i, j \in I$. On the other hand, let us suppose that there are $i, j \in I$ such that $G_i \leq G_j$ and $i \nleq j$. Now, let us consider $a, b \in C$, $a < b$ and define $f^* : I \to C$ by

$$f^*(k) = \begin{cases} b & \text{if } k \geq i \\ a & \text{otherwise} \end{cases} .$$

Hence, we have that $f^* \in E$, $f^*(i) = b$ and $f^*(j) = a$. These statements imply that $G_i(f^*) \nleq G_j(f^*)$, which is a contradiction. Thus, g is an order-embedding. Besides, by the definition of g we get that $G = g(I)$ and so, $L = [g(I)]_{\mathcal{V}}$. Therefore, (F1) holds.

Now we assume that A is a $\mathcal{V}$-algebra and $f : I \to A$ is an increasing function. Since $\mathcal{V}$ is the variety generated by C, we have that A is isomorphic to a subalgebra A^* of C^X, where X is an arbitrary set. Then, there is an isomorphism $\varphi : A \to A^*$ defined by the prescription $\varphi(a) = H_a$, where $H_a \in C^X$ for all $a \in A$ and so, let us consider the function $\varphi^* = \varphi \circ f$ where $\varphi^*(i) = \varphi(f(i)) = H_{f(i)}$. We claim that there is a homomorphism $h : L \to A^*$ such that $h \circ g = \varphi^*$. Indeed, for each $x_0 \in X$ we define $\alpha_{x_0} : I \to C$ by $\alpha_{x_0}(i) = H_{f(i)}(x_0)$. Then, we infer that $\alpha_{x_0} \in E$. This assertion allows us to consider the function $k : X \to E$, defined by $k(x) = \alpha_x$ for all $x \in X$. Hence, it is routine to check that $h : L \to C^X$, where $h(F) = \overline{F}$ being $\overline{F}(x) = F(k(x))$ is a homomorphism. Moreover, we have that $(h \circ g)(i) = h(G_i) = \overline{G_i}$. Thus, for all $x \in X$ we infer that $\overline{G_i}(x) = G_i(k(x)) = G_i(\alpha_x) = \alpha_x(i) = H_{f(i)}(x) =$

$\varphi^*(i)(x)$, which enables us to conclude that $(h \circ g)(i) = \varphi^*(i)$, for all $i \in I$. Finally, we have that $h(L) \subseteq A^*$. Indeed, since $L' = \{F \in C^E : h(F) \in A^*\}$ is a $\boldsymbol{\mathcal{V}}$-subalgebra of C^E and $G_i \in L'$ for all $i \in I$, then $L \subseteq L'$ and consequently $h(L) \subseteq A^*$. Therefore, (F2) holds.

On the other hand, in 1969, the free Łukasiewicz–Moisil algebras was studied by R. Cignoli in his Ph. D. thesis (see [1]), where he gave a construction for the free algebra, among other things.

In this work, we shall study the free algebra over a poset in the variety of Łukasiewicz–Moisil algebras. The paper is organized as follows. In Section 2, we recall definitions and properties of these algebras to facilitate the reading of this work; also, we cite many results needed in the rest of the paper. Section 3 contains a detailed description of the free algebra over a finite poset. Finally, we exhibit some examples and we calculate the cardinality of free algebra over specials finite posets.

2. Łukasiewicz–Moisil algebras

Łukasiewicz–Moisil algebras have been widely studied, the development of these algebras are available in [1, 2, 4, 6].

Definition 2. An n-valued Łukasiewicz–Moisil algebra with n integer and $n \geq 2$, (shortly LM_n-algebra) is an algebra $\mathcal{L} = (L, \wedge, \vee, \sim, 0, 1, \{\varphi_i\}_{1\leq i\leq n-1})$ of type $(2, 2, 1, 0, 0, \{1\}_{1\leq i\leq n-1})$ satisfying the following conditions:

LM 1. $\langle L, \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra,
$\varphi_1, \cdots, \varphi_{n-1} : L \longrightarrow L$ are bounded lattice morphisms such that for every $x, y \in L$:
LM2. $\varphi_i(x) \vee \sim \varphi_i(x) = 1$, for every $i = 1, \cdots, n-1$,
LM3. $\varphi_i(x) \wedge \sim \varphi_i(x) = 0$, for every $i = 1, \cdots, n-1$,
LM4. $\varphi_i\varphi_j(x) = \varphi_j(x)$, for every $i, j = 1, \cdots, n-1$,
LM5. $\varphi_i(\sim x) = \sim \varphi_j(x)$, for every $i, j = 1, \cdots, n-1$, $i + j = n$,
LM6. $\varphi_1(x) \leq \varphi_2(x) \leq \cdots \leq \varphi_{n-1}(x)$,
LM7. if $\varphi_i(x) = \varphi_i(y)$, for every $i = 1, \cdots, n-1$, then $x = y$. (the Moisil's determination principle)

As consequences of the principle, we have:

LM8. If $x, y \in L$, then $x \leq y$ iff $\varphi_i(x) \leq \varphi_i(y)$, for all $i = 1, \cdots, n-1$,
LM9. $\varphi_1(x) \leq x \leq \varphi_{n-1}(x)$, for all $x \in L$.

In what follows, we will expose known results on LM_n-algebras in order to facilitate the reading of this paper, but they can be found in the references cited.

We denoted by $C(L)$, the set of all complemented elements of the bounded lattice $\langle L, \wedge, \vee, 0, 1\rangle$ and we call it the center of L; it is easy to see that $\langle C(L), \wedge, \vee, \sim, 0, 1\rangle$ is a Boolean algebra.

Lemma 3. *Let L be an LM_n-algebra. The following are equivalent:*

(i) $e \in C(L)$,
(ii) *there are $i \in I_{n-1}$ and $x \in L$ such that $e = \varphi_i(x)$,*
(iii) *there is $i \in I_{n-1}$ such that $e = \varphi_i(e)$,*
(iv) $e = \varphi_i(e)$ *for every $i \in I_{n-1}$,*
(v) $\varphi_i(e) = \varphi_j(e)$ *for every $i, j \in I_{n-1}$.*

Let us note that $\varphi_i(x) \in C(L)$ for all $x \in L$ and $i \in I_{n-1}$.

Theorem 4. *Let L be an LM_n-algebra. The following are equivalent:*

(i) $C(L) = \{0, 1\}$,
(ii) *L is a chain,*
(iii) *L is subdirectly irreducible.*

Important examples of LM_n-algebras are the algebras $\langle L_n, \wedge, \vee, \sim, \{\varphi_i\}_{i\in I_n}, 1\rangle$ where $L_n = \{0, \frac{1}{n-1}, \cdots, \frac{n-2}{n-1}, 1\}$ is a chain with lattices operations, $\sim x = 1 - x$ and $\varphi_i(\frac{j}{n-1}) = 0$ if $i + j < n$ and 1 otherwise, for every $i, j \in I_{n-1}$.

The subalgebras of L_n are characterized in [1] as follows:

Theorem 5. (Cignoli [1]) *Let L_n be as above, and let $S \subseteq L_n$. If S is a subalgebra of L_n, then one and only one of the following conditions holds:*

(i) *If n is even, then S is an LM_k-algebra with k elements, where k is even and $k \leq n$.*
(ii) *If n is odd, then S is an LM_k-algebra with k elements, where k can be even or odd and $k \leq n$.*

Corollary 6. (Cignoli [1])

(i) *If n is even, then L_n has $\Pi(t) = \binom{\frac{n-2}{2}}{t-1}$ subalgebras with $2t$-elements, with $t = 1, \cdots, \frac{n-1}{2}$.*
(ii) *If n is odd, then L_n has $\Pi_1(t) = \binom{\frac{n-3}{2}}{t-1}$ subalgebras with $(2t+1)$-elements and $\Pi_2(t) = \binom{\frac{n-2}{2}}{t-1}$ subalgebras with $2t$ elements, with $t = 1, \cdots, \frac{n-1}{2}$.*

3. Free LM_n-algebra over a poset

In this section, we investigate the free algebra over a poset in variety of LM_n-algebra. Let $I_\leq = (I, \leq)$ be a finite poset and let E be the following set:

$$E = \{f : I_\leq \longrightarrow L_n;\ f \text{ is an order-preserving function }\},$$

Let us suppose that S_k is an arbitrary, but fix, subalgebra of L_n with k elements. Suppose that $\{E_2, \cdots, E_n\}$ is a partition of E, where $E_k = \{f \in E : [f(I)] \simeq S_k\}$ for all $2 \leq k \leq n$ and $[X]$ is the notation for the subalgebra generated by X .

Next, we will consider the following set, which will be very useful for what follows:

$$S = \{F \in L_n^E : \forall k(f \in E_k \Rightarrow F(f) \in S_k)\},$$

where L_n^E is the set of all functions from E to L_n. Now, we are going to prove the following lemma:

Lemma 7.

(i) *S is a subalgebra of L_n^E,*

(ii) *$S \simeq \bigotimes_{k=2}^{n} S_k^{E_k}$, where $\otimes$ is the direct product of algebras.*

Proof. It is easy to check that (i) holds, we are going to prove (ii).

Note first that if n is even, L_n has no subalgebras S_k with k odd elements, so some of algebras of the direct product above may not appear; but in what follows, we will consider $(n-1)$-tuples to facilitate reading.

Let $\alpha : \bigotimes_{k=2}^{n} S_k^{E_k} \to S$ be a function given by $\alpha(\mathcal{F}) = M_\mathcal{F}$ where $\mathcal{F} = (F_2, \cdots, F_n) \in \bigotimes_{k=2}^{n} {S_k}^{E_k}$ and $M : E \longrightarrow {}_n$ is defined by:

$$M_\mathcal{F}(f) = \begin{cases} F_2(f) & \text{if } f \in E_2 \\ \vdots & \\ F_n(f) & \text{if } f \in E_n \end{cases}.$$

We are going to prove now that α is an isomorphism. Indeed, let $\mathcal{F}, \mathcal{G} \in \bigotimes_{k=2}^{n} {S_k}^{E_k}$ be such that $\mathcal{F} = (F_2, \cdots, F_n)$ and $\mathcal{G} = (G_2, \cdots, G_n)$. Suppose now that $\alpha(\mathcal{F}) = \alpha(\mathcal{G})$, then $F_k(f) = G_k(f)$ for all $f \in E_k$ and $k = 2, \cdots, n$; which implies that $\mathcal{F} = \mathcal{G}$. Then α is one-to-one. Supppose now that $R \in S$, then it is not hard to see $R/E_k : E_k \to S_k$, with $k = 2, \cdots, n$ and $\alpha(\mathcal{R}) = R$ where $\mathcal{R} = (R/E_2, \cdots, R/E_n) \in \bigotimes_{k=2}^{n} {S_k}^{E_k}$. Then α is a bijective function.

It is not difficult to see that $M_{\mathcal{P}\vee\mathcal{G}}(f) = M_{\mathcal{P}}(f) \vee M_{\mathcal{G}}(f) \in S_k$, for all $\mathcal{P}, \mathcal{G} \in \bigotimes_{k=2}^{n} {S_k}^{E_k},\ f \in E_k,\ k = 2, \cdots, n$. Then $\alpha(\mathcal{P} \vee \mathcal{G}) = \alpha(\mathcal{P}) \vee \alpha(\mathcal{G})$. In a similar way to the previous step, we obtain that α is a homomorphism. ∎

We are going to described the relationship between the free algebra over a finite poset $I_{\leq}$ and algebra S. We shall denote this free algebra by $\mathbf{Free}_n(I_{\leq})$.

Let now $G = \{G_j\}_{j\in I}$ be the set of generators of the free algebra over the poset $I_{\leq}$, where $G_j(f) = f(j)$, for all $f \in E$, $j \in I$. Then it is easy to see that $\mathbf{Free}_n(I_{\leq}) \subseteq S$.

On the other hand, let now $P \in S$ and suppose that $P \neq 0$. Then there exists $f \in E$ such that $P(f) \neq 0$. Let us consider then the set $X = \{g \in E : P(g) \neq 0\}$ therefore, we have $X = \bigcup B_{\frac{i}{n-1}}$ where

$$B_{\frac{i}{n-1}} = \{f \in E : P(f) = \frac{i}{n-1}\}, i = 1, \cdots, n-1.$$

We prove now that (1) $P = \bigvee_{i=1}^{n-1} \bigvee_{f\in B_{\frac{i}{n-1}}} F_{f,\frac{i}{n-1}}$, where $F_{f,\frac{i}{n-1}} : E \longrightarrow {}_n$, with $f \in B_{\frac{i}{n-1}}$, is defined by:

$$F_{f,\frac{i}{n-1}}(g) = \begin{cases} \frac{i}{n-1} & \text{si } f = g \\ 0 & \text{si } f \neq g \end{cases}.$$

Indeed, let $g \in E$ be such that $P(g) = 0$. Then $g \notin B_{\frac{i}{n-1}}$, and this implies that $F_{f,\frac{i}{n-1}}(g) = 0$ for every $i = 1, \cdots, n-1$. Also, if $g \in E$ such that $P(g) = \frac{i}{n-1}$ for some i, then $g \in B_{\frac{i}{n-1}}$ and $g \notin B_j$ with $\frac{i}{n-1} \neq j$. So we have $\bigvee_{f\in B_{\frac{i}{n-1}}} F_{f,\frac{i}{n-1}}(g) = \frac{i}{n-1}$ and $\bigvee_{f\in B_j} F_{f,\frac{i}{n-1}}(g) = 0$, for all $\frac{i}{n-1} \neq j$. Therefore (1) is verified.

Next, we show that $F_{f,\frac{i}{n-1}}$ belongs to free algebra over the poset $I_{\leq}$.

Theorem 8. *If $f \in E_k$, then $F_{f,\frac{l}{n-1}} \in \mathbf{Free}_n(I_{\leq})$, where $\frac{l}{n-1} \in S_k$ and $2 \leq k \leq n$.*

Proof. Suppose that $f \in E$, then we can see that:

$$F_{f,1} = \bigwedge_{j\in I}\Big(\bigwedge_{f(j)=1} \varphi_1 G_j \wedge \bigwedge_{f(j)=0} \varphi_1 \sim G_j\Big).$$

Let us consider E_k (see beginning of Section 3) and consider also S_k, the subalgebra associated with E_k, $2 \leq k \leq n$. Let now $f \in E_k$ and suppose that $\frac{l}{n-1} \in S_k$, for any $1 \leq l \leq n$, such that it is not a fixed point with respect to unary operation $\sim$. It is clear that $\sim \frac{l}{n-1} \in S_k$. On the other hand, since $[f(I)] = S_k$, then there exists $i_0 \in I$ such that $f(i_0) = \frac{l}{n-1}$ or $f(i_0) =\sim \frac{l}{n-1}$, so we have

$$F_{f,\frac{l}{n-1}} = \bigwedge_{j\in I} \bigwedge_{f(j)=\frac{l}{n-1}} (G_j \wedge \varphi_{n-l}(G_j \wedge \varphi_{l+1} \sim G_j)) \wedge \bigwedge_{f(j)=\sim\frac{l}{n-1}} (\sim G_j \wedge \varphi_{l+1}(G_j \wedge \varphi_{n-l} \sim G_j)),$$

and

$$F_{f,\sim\frac{l}{n-1}} = \bigwedge_{j\in I} \bigwedge_{f(j)=\frac{l}{n-1}} (\sim G_j \wedge \varphi_{n-l}(G_j \wedge \varphi_{l+1} \sim G_j)) \wedge \bigwedge_{f(j)=\sim\frac{l}{n-1}} (G_j \wedge \varphi_{l+1}(G_j \wedge \varphi_{n-l} \sim G_j)).$$

Besides, if k is odd, then S_k only has a fixed point; i.e., there exists $c \in S_k$ such that $\sim c = c$. Hence, there exists $i_0 \in I$ such that $f(i_0) = c$ and so,

$$F_{f,c} = \bigwedge_{j\in I} \bigwedge_{f(j)=c} (G_j \wedge \varphi_{\frac{n-1}{2}+1} G_j \wedge \varphi_{\frac{n-1}{2}+1} \sim G_j).$$

Therefore, we have that each function $F_{f,\frac{l}{n-1}}$ is a polynomial in terms of $\{G_i\}$, then the proof is complete. ■

Now, we are going to prove our main result.

Theorem 9. *Let n be an odd positive integer and $\mathbf{Free}_n(I_{\leq})$ the free algebra over finite poset I. Then*

$$\mathbf{Free}_n(I_{\leq}) \simeq \bigotimes_{t=2}^{\frac{n-1}{2}} {S_{2t}}^{E_{2t}} \otimes {S_{2t+1}}^{E_{2t+1}}$$

and let n be an even positive integer, then we have

$$\mathbf{Free}_n(I_{\leq}) \simeq \bigotimes_{t=1}^{\frac{n-1}{2}} {S_{2t}}^{E_{2t}}.$$

Proof. According to Lemma 7, we have that $S \simeq \bigotimes_{k=2}^{n} S_k^{E_k}$ and suppose that $P \in S$. Then, we infer that $P = \bigvee_{i=1}^{n-1} \bigvee_{f\in B_{\frac{i}{n-1}}} F_{f,\frac{i}{n-1}}$, where $B_{\frac{i}{n-1}} = \{f \in E : P(f) = \frac{i}{n-1}\}$ and $i = 1, \cdots, n-1$. From Theorem 9, we can write

$P \in \mathbf{Free}_n(I_{\leq})$ and therefore $\mathbf{Free}_n(I_{\leq}) \simeq \bigotimes_{k=2}^{n} S_k^{E_k}$. From the latter and by Theorem 5, we conclude the proof. ■

Examples

We are going to indicate the cardinality of $\mathbf{Free}_n(X_{\leq})$, where $X_{\leq}$ is the poset $X = \{b_1, \cdots, b_r, a_1, \cdots a_m\}$ such that $a_m < b_k$ for all $k = 1, \cdots, r$, $a_i < a_j$ if $i < j$ and the poset $\{b_1, \ldots, b_r\}$ is an antichain.

First case: *n* odd number. By the Theorem 9, we have

$$|\mathbf{Free}_n(X_{\leq})| = \prod_{k=1}^{\frac{n-1}{2}} |S_{2k}|^{\Pi_2(k)|E_{2k}|} \times |S_{2k+1}|^{\Pi_1(k)|E_{2k+1}|}$$
$$= \prod_{k=1}^{\frac{n-1}{2}} 2k^{\binom{\frac{n-3}{2}}{k-1}\alpha_{2k}} \times (2k+1)^{\binom{\frac{n-3}{2}}{k-1}\alpha_{2k+1}}$$

and let us consider that $\alpha_{2k} = |\{f : I \longrightarrow S_{2k};\ f$ is an order-preserving function$\}|$. Therefore, it is clear that $\alpha_{2k} = \binom{m+2k-1}{m} \sum_{i=1}^{k-1} (-1)^i \binom{k-1}{i} (2k-2i)^r$ and

$$\alpha_{2k+1} = \binom{m+2k}{m}\Big(\sum_{i=1}^{k-1}(-1)^i\binom{k-1}{i}(2k+1-2i)^r - \sum_{i=1}^{k-1}(-1)^i\binom{k-1}{i}(2k-2i)^r\Big).$$

Then,

$$|\mathbf{Free}_n(X_{\leq})| = \prod_{k=1}^{\frac{n-1}{2}} \Big(2k^{\binom{\frac{n-3}{2}}{k-1}\binom{m+2k-1}{m}\sum_{i=1}^{k-1}(-1)^i\binom{k-1}{i}(2k-2i)^r}$$
$$\times (2k+1)^{\binom{\frac{n-3}{2}}{k-1}\binom{m+2k}{m}(\sum_{i=1}^{k-1}(-1)^i\binom{k-1}{i}(2k+1-2i)^r - \sum_{i=1}^{k-1}(-1)^i\binom{k-1}{i}(2k-2i)^r)}\Big).$$

Second case: *n* even number. In a similar way, we have

$$|\mathbf{Free}_n(X_{\leq})| = \prod_{k=1}^{\frac{n}{2}} (2k)^{\binom{\frac{n-2}{2}}{k-1}\binom{m+2k-1}{m}2^r\sum_{i=1}^{k-1}(-1)^i\binom{k-1}{i}(k-i)^r}.$$

Note that if we take $m = 0$, we obtain the results found in [1] (see also [2, Theorems 3.9 and 3.10]) in a simple way.

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