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C^r-RIGHT EQUIVALENCE OF ANALYTIC FUNCTIONS

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Abstract. Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be analytic functions. We will show that if $\nabla f(0) = 0$ and $g \quad f \in (f)^{r+2}$ then f and g are C^r-right equivalent, where (f) denote ideal generated by f and $r \in \mathbb{N}$.

1. Introduction and result

By $\mathbb{N}$ we denote the set of positive integers. A norm in $\mathbb{R}^n$ we denote by $|\cdot|$ and by $\operatorname{dist}(x, V)$ - the distance of a point $x \in \mathbb{R}^n$ to a set $V \subset \mathbb{R}^n$ (put $\operatorname{dist}(x, V) = 1$ if $V = \emptyset$).

Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be analytic functions. We say that f and g are C^r*-right equivalent* if there exists a C^r diffeomorphism $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ \varphi$ in a neighbourhood of 0.

Let $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ be an analytic function. By $\mathcal{J}_f$ we denote the ideal generated by $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ in the set of analytic functions $(\mathbb{R}^n, 0) \to \mathbb{R}$. The ideal $\mathcal{J}_f$ is called the *Jacobi ideal*. Moreover, by (f) we denote the ideal in set of analytic functions $(\mathbb{R}^n, 0) \to \mathbb{R}$ generated by f.

The aim of this paper is to prove the following theorem

Main Theorem. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be analytic functions and let $\nabla f(0) = 0$. If $g \quad f \in (f)^{r+2}$ then f and g are C^r-right equivalent, where $r \in \mathbb{N}$.*

The above theorem is a modification of author's result about C^r-right equivalence of C^{r+1} functions. In [8, Theorem 5] and [9, Theorem 1] it has been proved

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Theorem 1. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be C^k functions, $k, r \in \mathbb{N}$ be such that $k \geq r + 1$ and let $\nabla f(0) = 0$. If $g - f \in (\mathcal{J}_f C^{k-1}(n))^{r+2}$ then f and g are C^r-right equivalent. By $\mathcal{J}_f C^{k-1}(n)$ we mean the Jacobi ideal defined in the set of C^{k-1} functions $(\mathbb{R}^n, 0) \to \mathbb{R}$.*

Methods of proofs of above theorems are similar. First we construct suitable vector field of class C^r and next we integrate this vector field. The idea of construct vector field is descended from N. H. Kuiper, T. C. Kuo ([4], [5]). Whereas, integration of vector field is descended from Ch. Ehresmann ([2], see also [3]).

There exists one more result which deals with C^r-right equivalence of functions with similar condition for $g - f$. Namely, J. Bochnak has proved the following theorem ([1, Theorem 1])

Theorem 2. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be C^k functions, $k, r \in \mathbb{N}$ be such that $k \geq r + 2$ and let $\nabla f(0) = 0$. If $g - f \in \mathfrak{m}(\mathcal{J}_f C^{k-1}(n))^2$ then f and g are C^r-right equivalent. By $\mathcal{J}_f C^{k-1}(n)$ and $\mathfrak{m}$ we mean respectively the Jacobi ideal and maximal ideal defined in the set of C^{k-1} functions $(\mathbb{R}^n, 0) \to \mathbb{R}$.*

Proof of this theorem bases on Tougeron's Implicit Theorem ([10]).

Comparing the above results, we see that Theorem 1 deals with C^r-right equivalence of C^{r+1} functions, whereas Theorem 2 deals with C^r-right equivalence of C^{r+2} functions. Since in the last Theorem, the power of Jacobi ideal does not depend on r, it is difficult to say which Theorem is stronger. In addition, since in Main Theorem $g - f$ belongs to some power of ideal generated by f, whereas in Theorem 1 and Theorem 2 $g - f$ belongs to some power of ideal generated by partial derivatives of f, these results are of completely different type.

2. Auxiliary results

First, we define Łojasiewicz exponent in the gradient inequality.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function. It is known that there exist a neighbourhood U of $0 \in \mathbb{R}^n$ and constants $C > 0$, $\eta \in [0, 1)$ such that the following *Łojasiewicz gradient inequality* holds

$$|\nabla f(x)| \geq C|f(x)|^\eta, \qquad \text{for } x \in U.$$

The smallest exponent η in the above inequality is called the *Łojasiewicz exponent in the gradient inequality* and is denoted by $\varrho_0(f)$ (cf. [6], [7]).

From the above inequality, we obtain immediately that there exist a neighbourhood U of $0 \in \mathbb{R}^n$ and a constant $C > 0$ such that

$$|\nabla f(x)| \geq C|f(x)|, \qquad \text{for } x \in U. \tag{1}$$

Let $M, m, r \in \mathbb{N}$, $M > r$. Moreover, let $p, q_1, \dots, q_m : (\mathbb{R}^n, 0) \to \mathbb{R}$ be analytic functions and let $\mathcal{Q}$ denote the ideal generated by $q_1, \dots, q_m$.

Lemma 1. (cf. [9]) *If $p \in \mathcal{Q}^M$ then*

(i) $\frac{\partial^r p}{\partial x_{i_1} \dots \partial x_{i_r}} \in \mathcal{Q}^{M-r}$, *for* $i_1, \dots, i_r \in \{1, \dots, n\}$,

(ii) $|p(x)| \le C|(q_1(x), \dots, q_n(x))|^M$ *in a neighbourhood of* 0 *and for some positive constant* C.

Lemma 2. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function. Then there exist a neighbourhood U at $0 \in \mathbb{R}^n$ and a constant $C > 0$ such that for any $x \in U$, $|f(x)| \le C \operatorname{dist}(x, V_f)$ (V_f denote zero set of f).*

Proof. To the contrary, let us assume that for any neighbourhood U and for any $C > 0$ there exists $x \in U$, $|f(x)| > C \operatorname{dist}(x, V_f)$. In particular, for any $\nu \in \mathbb{N}$ there exists x_ν, such that $|x_\nu| < \frac{1}{\nu}$, $|f(x_\nu)| > \nu \operatorname{dist}(x_\nu, V_f)$. Moreover, there exists $u_\nu \in V_f$, that $\operatorname{dist}(x_\nu, V_f) = |x_\nu - u_\nu|$. Then we have $|f(x_\nu) - f(u_\nu)| > \nu |x_\nu - u_\nu|$. This contradicts the Lipschitz condition for function f. ∎

Lemma 3. *Let $\xi, \eta : U \to \mathbb{R}$ be analytic functions such that*

$$C_1|\eta(x)|^2 \le |\xi(x)| \le C_2|\eta(x)|^2, \quad |\partial\xi(x)| \le C_3|\eta(x)|, \quad x \in U, \tag{2}$$

where C_1, C_2, C_3 are some positive constants and $U \in \mathbb{R}^n$ is some neighbourhood of the origin. Then

$$\left|\partial^k\left(\frac{1}{\xi(x)}\right)\right| \le B|\eta(x)|^{-|k|-2}, \quad x \in U, \tag{3}$$

for some constant $B > 0$, $k \in \mathbb{N}_0^n$.

Proof. Let $m = |k|$. By induction it is easy to show that

$$\partial^k\left(\frac{1}{\xi}\right) = \frac{1}{\xi^{m+1}}\Big(\sum_{j=1}^{m} \xi^{m-j} \sum_{|i_1|+\dots+|i_j|=m} C_{i_1,\dots,i_j} \partial^{i_1}\xi \cdots \partial^{i_j}\xi\Big), \tag{4}$$

where $i_1, \dots, i_j \in \mathbb{N}_0^n$, $i_1 + \dots + i_j = k$, $|i_j| \ge 1$ and for some constants $C_{i_1,\dots,i_j} \ge 0$ ($C_{i_1,\dots,i_j} = 0$, when $i_1 + \dots + i_j \ne k$).

Now we will prove (3). Let us take $k \in \mathbb{N}_0^n$ and let $|k| = m$. First, consider the case when m is even.

$$\begin{aligned}
&\Big|\frac{1}{\xi^{m+1}}\Big(\sum_{j=1}^{m}\xi^{m-j}\sum_{|i_1|+\cdots+|i_j|=m}C_{i_1,\dots,i_j}\partial^{i_1}\xi\cdots\partial^{i_j}\xi\Big)\Big|\\
&\qquad\le\Big|\frac{1}{\xi^{m+1}}\Big(\sum_{j=1}^{\frac{1}{2}m}\xi^{m-j}\sum_{|i_1|+\cdots+|i_j|=m}C_{i_1,\dots,i_j}\partial^{i_1}\xi\cdots\partial^{i_j}\xi\Big)\Big|\\
&\qquad\quad+\Big|\frac{1}{\xi^{m+1}}\Big(\sum_{j=\frac{1}{2}m+1}^{m}\xi^{m-j}\sum_{|i_1|+\cdots+|i_j|=m}C_{i_1,\dots,i_j}\partial^{i_1}\xi\cdots\partial^{i_j}\xi\Big)\Big|.
\end{aligned}$$

Note that for $m\ge j\ge\frac{1}{2}m+1$ and for any sequence $i_1,\dots,i_j\in\mathbb{N}_0^n$, $|i_j|\ge 1$, such that $|i_1|+\cdots+|i_j|=m$, there exist at least $2j-m$ elements of this sequence which modules are equal 1. Therefore, we can assume that $|i_{m-j+1}|=\dots|i_j|=1$ for $m\ge j\ge\frac{1}{2}m+1$. From this and (2), we obtain

$$\begin{aligned}
&\Big|\frac{1}{\xi^{m+1}}\Big(\sum_{j=1}^{m}\xi^{m-j}\sum_{|i_1|+\cdots+|i_j|=m}C_{i_1,\dots,i_j}\partial^{i_1}\xi\cdots\partial^{i_j}\xi\Big)\Big|\\
&\le A_1|\eta|^{-2m-2}\sum_{j=1}^{\frac{1}{2}m}|\xi^{m-\frac{1}{2}m}|\Big|\sum_{|i_1|+\cdots+|i_j|=m}C_{i_1,\dots,i_j}\partial^{i_1}\xi\cdots\partial^{i_j}\xi\Big|\\
&\ +A_1|\eta|^{-2m-2}\Big|\sum_{j=\frac{1}{2}m+1}^{m}\xi^{m-j}\sum_{|i_1|+\cdots+|i_j|=m}C_{i_1,\dots,i_j}\partial^{i_1}\xi\cdots\partial^{i_{m-j}}\xi\partial^{i_{m-j+1}}\xi\cdots\partial^{i_j}\xi\Big|\\
&\le A_1|\eta|^{-2m-2}A_2B_1|\eta|^{2(m-\frac{1}{2}m)}+A_1|\eta|^{-2m-2}\sum_{j=\frac{1}{2}m+1}^{m}|\xi^{m-j}|B_2|\eta|^{2j-m}\\
&\le A_1A_2B_1|\eta|^{-m-2}+A_1|\eta|^{-2m-2}\sum_{j=\frac{1}{2}m+1}^{m}A_3B_2|\eta|^{2(m-j)+2j-m}\\
&=B_3|\eta|^{-m-2},
\end{aligned}$$

where A_i, B_i are some positive constants.

Let us consider the case when m is odd. Note that for $m\ge j\ge\frac{1}{2}(m+1)$ and for any sequaence $i_1,\dots,i_j\in\mathbb{N}_0^n$, $|i_j|\ge 1$, such that $|i_1|+\cdots+|i_j|=m$, there exist at least $2j-m$ elements of this sequence which modules are equal 1. Knowing this fact, similar as previously, we show

$$\Big|\frac{1}{\xi^{m+1}}\Big(\sum_{j=1}^{m}\xi^{m-j}\sum_{|i_1|+\cdots+|i_j|=m}C_{i_1,\dots,i_j}\partial^{i_1}\xi\cdots\partial^{i_j}\xi\Big)\Big|$$

$$\leq \left|\frac{1}{\xi^{m+1}}\Big(\sum_{j=1}^{\frac{1}{2}(m\quad 1)} \xi^{m\quad j} \sum_{|i_1|+\quad +|i_j|=m} C_{i_1,\ldots,i_j}\partial^{i_1}\xi\cdots\partial^{i_j}\xi\Big)\right|$$
$$+\left|\frac{1}{\xi^{m+1}}\Big(\sum_{j=\frac{1}{2}(m+1)}^{m} \xi^{m\quad j} \sum_{|i_1|+\quad +|i_j|=m} C_{i_1,\ldots,i_j}\partial^{i_1}\xi\cdots\partial^{i_j}\xi\Big)\right|$$
$$\leq B_4|\eta|^{\quad 2m\quad 2}|\eta|^{2(m\quad \frac{1}{2}m+\frac{1}{2})} + B_5|\eta|^{\quad m\quad 2},$$

for some positive constants B_4, B_5. Finally, we proved (3). ∎

3. Proof of Main Theorem

Let Z be the zero set of ∇f and let $U \in \mathbb{R}^n$ be a neighbourhood of 0 such that f and g are well defined. By Lemma 2 there exists a positive constant A such that

(5) $$|\nabla f(x)| \leq A\,\mathrm{dist}(x, Z), \qquad \text{for } x \in U.$$

Define the function $F : \mathbb{R}^n \times U \to \mathbb{R}$ by the formula

$$F(\xi, x) = f(x) + \xi(g \quad f)(x),$$

obviously

$$\nabla F(\xi, x) = ((g \quad f)(x), \nabla f(x) + \xi\nabla(g \quad f)(x)).$$

Let $G = \{(\xi, x) \in \mathbb{R} \times U : |\xi| < \delta\}$ where $\delta \in \mathbb{N}$, $\delta > 2$. From the above, diminishing U if necessary, we have that there exists a constant $C_1 > 0$ such that

(6) $$|\nabla f(x)| \leq C_1|\nabla F(\xi, x)|, \qquad \text{for } (\xi, x) \in G.$$

Indeed,

$$|\nabla F(\xi, x)| \geq |\nabla f(x) \quad \xi\nabla(g \quad f)(x)| \geq |\nabla f(x)| \quad |\xi||\nabla(g \quad f)(x)|.$$

Since $(g \quad f) \in (f)^{r+2}$ and $r \geq 1$, so from Lemma 1 and (1), we get

$$|\nabla(g \quad f)(x)| \leq C_2'|f(x)|^{r+1} \leq C_2|\nabla f(x)|^{r+1} \leq C_2|\nabla f(x)|^2,$$

for some positive constants C_2, C_2'. Hence, diminishing U if necessary,

$$|\nabla F(\xi, x)| \geq |\nabla f(x)| \quad |\xi|C_2|\nabla f(x)|^2 \geq \frac{1}{C_1}|\nabla f(x)|, \qquad \text{for } (\xi, x) \in G.$$

Moreover, from definition of ∇F we get at once, that there exists a positive constant C_3 such that

(7) $$|\nabla f(x)| \geq C_3|\nabla F(\xi, x)|, \qquad \text{for } (\xi, x) \in G.$$

Now we will show that the mapping $X : G \to \mathbb{R}^n \times \mathbb{R}$ defined by

$$X(\xi, x) = (X_1, \ldots, X_{n+1}) = \begin{cases} \frac{(g\quad f)(x)}{|\nabla F(\xi,x)|^2}\nabla F(\xi, x), & \text{for } x \notin Z, \\ 0, & \text{for } x \in Z \end{cases}$$

is a C^r mapping. The proof of this fact will be divided into several steps.

Step 1. *The mapping X is continuous in G.*

Indeed, let us fix ξ and let $h_i(\xi,x) = (g \circ f)(x)\frac{\partial F}{\partial x_i}(\xi,x)$. Then for $x \in U$ and $x \notin Z$, from (1) and Lemma 1, we have $|X_i(\xi,x)| \le A_1|\nabla f(x)|^{r+1} \le A' \operatorname{dist}(x,Z)^{r+1}$ for some positive constants A_1, A'. The above inequality also holds for $x \in Z$. Since A' does not depend on the choice of ξ so for $(\xi,x) \in G$, we obtain

$$|X(\xi,x)| \le A' \operatorname{dist}(x,Z)^{r+1}. \tag{8}$$

Therefore X is continuous in G.

Step 2. *Let $\alpha = (\alpha_0,\dots,\alpha_n) \in \mathbb{N}_0^{n+1}$ be a multi-index such that $|\alpha| \le r$, then, diminishing U if necessary,*

$$|\partial^\alpha X_i(\xi,x)| \le A'' \operatorname{dist}(x,Z)^{r+1-|\alpha|}, \qquad \textit{for } x \notin Z,$$

where $\partial^\alpha X_i = \partial^{\alpha_0}\cdots\partial^{\alpha_{n+1}} X_i = \frac{\partial^{|\alpha|} X_i}{\partial\xi^{\alpha_0}\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$.

Indeed, from Leibniz rule we have

$$\partial^\alpha X_i(\xi,x) = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial^{\alpha-\beta}(h_i(\xi,x))\partial^\beta\Big(\frac{1}{|\nabla F(\xi,x)|^2}\Big). \tag{9}$$

Diminishing G if necessary, from Lemma 3, we obtain

$$\Big|\partial^\beta\Big(\frac{1}{|\nabla F(\xi,x)|^2}\Big)\Big| \le \frac{A''_\beta}{|\nabla F(\xi,x)|^{|\beta|+2}},$$

for some constants $A''_\beta > 0$. Therefore, from (9) we have

$$|\partial^\alpha X_i(\xi,x)| \le \sum_{\beta\le\alpha}\binom{\alpha}{\beta}|\partial^{\alpha-\beta}(h_i(\xi,x))|\frac{A''_\beta}{|\nabla F(\xi,x)|^{|\beta|+2}}. \tag{10}$$

Let us fix ξ. From Lemma 1, (7) and (1), we have

$$|\partial^{\alpha-\beta}(h_i(\xi,x))| \le B_{\alpha-\beta}|\nabla f(x)|^{r+3-|\alpha|+|\beta|}, \tag{11}$$

for some positive constant $B_{\alpha-\beta}$. Since $B_{\alpha-\beta}$ doesn't depend on the choice of ξ so this equality holds for $(\xi,x) \in G$. Finally, from (10), (11), (6), (7) and (5), we obtain

$$\begin{aligned}
|\partial^\alpha X_i(\xi,x)| &\le \sum_{\beta\le\alpha}\binom{\alpha}{\beta}B_{\alpha-\beta}|\nabla f(x)|^{r+3-|\alpha|+|\beta|}\frac{A''_\beta}{|\nabla F(\xi,x)|^{|\beta|+2}} \\
&\le \sum_{\beta\le\alpha}\binom{\alpha}{\beta}A''_\beta B_{\alpha-\beta}|\nabla f(x)|^{r+3-|\alpha|+|\beta|-|\beta|-2} \\
&\le \frac{A''}{A}|\nabla f(x)|^{r+1-|\alpha|} \le A'' \operatorname{dist}(x,Z)^{r+1-|\alpha|},
\end{aligned}$$

for some constant $A'' > 0$.

Step 3. *Partial derivatives $\partial^\alpha X_i$ vanish for $x \in Z$ and $|\alpha| \leq r$.*

Indeed, we will carry out induction with respect to $|\alpha|$. Let $t \in \mathbb{R}$, $x \in Z$ and let $x_m^t = (x_1, \ldots, x_m + t, \ldots, x_n)$. For $|\alpha| = 0$ hypothesis is obvious. Assume that hypothesis is true for $|\alpha| \leq r - 1$. Then from Step 2, we have

$$\frac{|\partial^\alpha X_i(\xi, x_m^t) - \partial^\alpha X_i(\xi, x)|}{|t|} = \frac{|\partial^\alpha X_i(\xi, x_m^t)|}{|t|} \leq \frac{A'' \operatorname{dist}(x_m^t, Z)^{r+1-|\alpha|}}{|t|}$$
$$\leq \frac{A''|t|^{r+1-|\alpha|}}{|t|} = A''|t|^{r-|\alpha|}.$$

Since $r - |\alpha| \geq r - r + 1 = 1$, we obtain $\partial^\gamma X_i(\xi, X) = 0$ for $x \in Z$ and $|\gamma| = |\alpha| + 1$. This completes Step 3.

In summary, from Step 1, 2 and 3, we obtain that X_i are C^r functions in G. Therefore, X is a C^r mapping in G.

Define a vector field $W : G \to \mathbb{R}^n$ by the formula

$$W(\xi, x) = \frac{1}{X_1(\xi, x) - 1}(X_2(\xi, x), \ldots, X_{n+1}(\xi, x)).$$

Diminishing U if necessary, we may assume that $A' \operatorname{dist}(x, Z) < \frac{1}{2}$. From (8) we obtain

$$|X_1(\xi, x) - 1| \geq 1 - |X(\xi, x)| \geq 1 - A' \operatorname{dist}(x, Z) > \frac{1}{2}, \quad (\xi, x) \in G.$$

Hence the field W is well defined and it is a C^r mapping.

Consider the following system of ordinary differential equations

$$\frac{dy}{dt} = W(t, y). \tag{12}$$

Since $r \geq 1$, then W is at least of class C^1 on G, so it is a lipschitzian vector field. As a consequence, the above system has a uniqueness of solutions property in G. Since $y_0(t) = 0$, $t \in (-2, 2)$ is one of the solutions of (12), then the above implies the existence of a neigbourhood $U \subset \mathbb{R}^n$ of 0 such that every integral solution y_x of (12) with $y_x(0) = x$, where $x \in U$, is defined at least in $[0, 1]$.

Now, let us define a mapping $\varphi : U \to \mathbb{R}^n$ by the formula

$$\varphi(x) = y_x(1),$$

where y_x stands for an integral solution of (12) with $y_x(0) = x$. This mapping is a C^r bijection. It gives a C^r diffeomorphism of some neighbourhood of the origin. Indeed, considering solutions $\bar{y}_x : [0, 1] \to \mathbb{R}^n$ of (12) with $\bar{y}_x(1) = x$, where x is from some neigbourhood of the origin, we get that $\varphi(\bar{y}_x(0)) = x$. Similar reasoning shows that the mapping $x \to \bar{y}_x(0)$ is class C^r in the neigbourhood of the origin. Consequently $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a C^r

diffeomorphism and maps a neighbourhood of the origin onto a neighbourhood of the origin.

Finally, note that for any $x \in U$,

$$F(t, y_x(t)) = const \quad \text{in } [0, 1]. \tag{13}$$

Indeed, from definition of W, we derive the formula

$$[1, W(\xi, x)] = \frac{1}{X_1(\xi, x) \quad 1}(X(\xi, x) \quad e_1), \qquad \text{for } (\xi, x) \in G,$$

where $e_1 = [1, 0, \ldots, 0] \in \mathbb{R}^{n+1}$ and $[1, W] : G \to \mathbb{R} \times \mathbb{R}^n$. Thus, if we denote by $\langle a, b\rangle$ the scalar product of two vectors a, b, then for $t \in [0, 1]$, we have

$$\frac{dF(t, y_x(t))}{dt} = \langle(\nabla F)(t, y_x(t)), [1, W(t, y_x(t))]\rangle$$
$$= \frac{1}{X_1(t, y_x(t)) \quad 1}\left(\langle(\nabla_x F)(t, y_x(t)), X(t, y_x(t))\rangle \quad \frac{\partial F}{\partial \xi}(t, y_x(t))\right)$$
$$= \frac{1}{X_1(t, y_x(t)) \quad 1}(g(y_x(t)) \quad f(y_x(t)) \quad g(y_x(t)) + f(y_x(t))) = 0.$$

This gives (13). Finally, (13) yields

$$f(x) = F(0, x) = F(0, y_x(0)) = F(1, y_x(1)) = F(1, \varphi(x)) = g(\varphi(x)),$$

for $x \in U$. This ends the proof. ■

4. Remark

In Main Theorem we can not omit the assumption about analtyticity of function f and g. It follows from the fact that the Łojasiewicz gradient inequality holds only for analytic functions.

Note that the condition $g \quad f \in (f)^{r+2}$ in Main Theorem can be replaced by $g = f(hf^{r+1} + 1)$, where $h : (\mathbb{R}^n, 0) \to \mathbb{R}$ is an analytic function. It seems natural to try to replace this condition by $g = hf$, where $h : (\mathbb{R}^n, 0) \to \mathbb{R}$ is an analytic function such that $h(0) \neq 0$. But then the theorem would not hold. Indeed, let $f(x) = x^2$, $g(x) = \quad x^2$ and $h(x) = \quad 1$, then $g = hf$ but f and g are not right equivalent.

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