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DEMONSTRATIO MATHEMATICA
Vol. XLVIII No 2 2015

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ON THE LINKS OF SIMPLE SINGULARITIES, SIMPLE ELLIPTIC SINGULARITIES AND CUSP SINGULARITIES

Communicated by S. Izumiya

Abstract. This is a survey article about the study of the links of some complex hypersurface singularities in $\mathbb{C}^3$. We study the links of simple singularities, simple elliptic singularities and cusp singularities, and the canonical contact structures on them. It is known that each singularity link is diffeomorphic to a compact quotient of a 3-dimensional Lie group $SU(2)$, Nil^3 or Sol^3, respectively. Moreover, the canonical contact structure is equivalent to the contact structure invariant under the action of each Lie group. We show a new proof of this fact using the moment polytope of S^5. Our proof gives a new aspect to the relation between simple elliptic singularities and cusp singularities, and visualizes how the singularity links are embedded in S^5 as codimension two contact submanifolds.

1. Introduction

This is a survey article about the study of the links of some complex hypersurface singularities in $\mathbb{C}^3$ (for this topic, see also [17], [30]). V. I. Arnol'd started the classification of hypersurface singularities up to stable equivalence (see [1]). He introduced the concept of *modality* and classified all the singularities of modality $m \leq 2$. The functions of modality $m = 0, 1, 2$ are said to be *simple*, *unimodal* and *bimodal*, respectively. We are interested in simple singularities and unimodal singularities, which are listed below.

(1) Simple singularities.

$$A_n : z_1^2 + z_2^2 + z_3^{n+1} = 0,\ n \geq 1,$$
$$D_n : z_1^2 + z_2^2 z_3 + z_3^{n\ \ 1} = 0,\ n \geq 4,$$
$$E_6 : z_1^2 + z_2^3 + z_3^4 = 0,$$
$$E_7 : z_1^2 + z_2^3 + z_2 z_3^3 = 0,$$
$$E_8 : z_1^2 + z_2^3 + z_3^5 = 0.$$

2010 *Mathematics Subject Classification*: 57R17, 32S25.
Key words and phrases: contact structures, complex singularities.

DOI: 10.1515/dema-2015-0021

These singularities are also called *ADE singularities, Kleinian singularities, du Val singularities*, or *rational double points.*

(2) Unimodal singularities.

(i) Simple elliptic singularities (parabolic singularities)

$$\begin{aligned}
\tilde{E}_6 = P_8 = T_{333} &: z_1^3 + z_2^3 + z_3^3 + \lambda_1 z_1 z_2 z_3 = 0, \quad {\lambda_1}^3 + 27 \neq 0,\\
\tilde{E}_7 = X_9 = T_{244} &: z_1^2 + z_2^4 + z_3^4 + \lambda_2 z_1 z_2 z_3 = 0, \quad {\lambda_2}^4 - 64 \neq 0,\\
\tilde{E}_8 = J_{10} = T_{236} &: z_1^2 + z_2^3 + z_3^6 + \lambda_3 z_1 z_2 z_3 = 0, \quad {\lambda_3}^6 - 432 \neq 0,
\end{aligned}$$

(ii) cusp singularities (hyperbolic singularities)

$$T_{pqr} : z_1^p + z_2^q + z_3^r + \lambda z_1 z_2 z_3 = 0, \ \ \lambda \neq 0, \ p^{-1} + q^{-1} + r^{-1} < 1,$$

(iii) 14 exceptional singularities.

We study the links of simple singularities, simple elliptic singularities and cusp singularities, and the canonical contact structures on them. First, we remind the definitions of the link of a complex hypersurface singularity and the canonical contact structure on it.

Let $(V, \mathbf{0}) \hookrightarrow (\mathbb{C}^{n+1}, \mathbf{0})$ be a germ of complex analytic manifold with an isolated singularity at the origin. The intersection K of V and a sufficiently small sphere S_ε^{2n+1} centered at the origin is called the *link* of the singularity $(V, \mathbf{0})$. The *standard contact structure* ξ_0 *on* S^{2n+1} is defined by the complex tangency

$$\xi_0 = TS^{2n+1} \cap JTS^{2n+1},$$

where J is the standard complex structure on $\mathbb{C}^{n+1}$. The *canonical contact structure on the link* K is also given by the complex tangency, and it is the restriction of ξ_0 to K. Hence, the link K is a codimension two contact submanifold of the standard contact sphere (S^{2n+1}, ξ_0). Caubel, Nemethi, and Popescu-Pampu call it a *Milnor fillable contact structure.* They proved in [4] that an oriented 3-manifold admits at most one Milnor fillable contact structure. It is also known that a Milnor fillable contact structure is Stein fillable and universally tight (for the proof of universal tightness, see [16]).

If we obtain the *minimal good resolution* of an isolated surface singularity, we can detect the singularity link as a 3-manifold by plumbing circle bundles according to the *dual resolution graph* (§3). For simple singularities, the dual resolution graphs correspond to the Dynkin diagrams of A_n, D_n, E_6, E_7 and E_8. By Kirby calculus, it turns out that the corresponding 3-manifold is a Seifert manifold which fibers over S^2 with two or three exceptional fibers. For a simple elliptic singularity, the dual resolution graph consists of one elliptic curve. Thus, the minimal resolution is a complex line bundle over the elliptic curve and the link is an associated circle bundle over T^2. It is diffeomorphic to some parabolic T^2 bundle over S^1, hence, to a *Nil-manifold.*

For a cusp singularity, the dual resolution graph is a cycle of rational curves. Hence, the link is a hyperbolic T^2 bundle over S^1 and it is a *Sol-manifold*. It is the standard way to understand the topology of singularity links.

However, in order to know about deeper structures, we need to see the relations between these singularities and 3-dimensional Lie groups (§4). The starting point is Klein's theorem about simple singularities ([12]). He showed that a simple singularity is isomorphic to the quotient of $(\mathbb{C}^2, \mathbf{0})$ by a finite subgroup of $SU(2)$. He explained it by the polyhedral groups and the invariant polynomials. Milnor in [19] emphasized the geometrical meaning of Klein's theorem and gave a careful proof. He also extended this viewpoint to all the *Brieskorn singularities* $z_1^p + z_2^q + z_3^r = 0$ and showed that the Brieskorn 3-manifold carries the structure of $SU(2)$, Nil^3, or $\widetilde{SL}(2;\mathbb{R})$ according as the rational number $p^{\ 1} + q^{\ 1} + r^{\ 1} \quad 1$ is positive, zero, or negative. Namely, the Brieskorn 3-manifold is diffeomorphic to a quotient $\Pi\backslash G$, where G is $SU(2)$, Nil^3, or $\widetilde{SL}(2;\mathbb{R})$, and Π is a discrete subgroup of G.

Milnor's work was later extended to the so-called *quasi-homogeneous singularities*, which are singularities with good $\mathbb{C}^*$-actions. Simple singularities, simple elliptic singularities and Brieskorn singularities are all included in them. Their geometry and structure has been widely studied by many researchers, for example, Saito [28], Pinkham [27], Orlik and Wagreich [26], [32], Dolgachev [5] and Neumann [25]. On the other hand, cusp singularities are not quasi-homogeneous. By Laufer's work on cusp singularities ([15]) and Hirzebruch's work on *Hilbert modular cusps* ([9]), it follows that the link of a cusp singularity carries the structure of Sol^3. Neumann summarized these relations between 3-dimensional Lie groups and complex surface singularities. He also showed that the CR structures on these singularity links given by the complex tangency are induced by left-invariant CR structures on the Lie groups ([25], [7]). Hence, the Milnor fillable contact structures on these links are induced by left-invariant contact structures on the Lie groups. In such a sense, they are already well understood. However, there are still some problems.

Problem 1.1. Is there any new aspect which treats simple elliptic singularities and cusp singularities uniformly?

As we stated above, these two classes have different geometries, Nil^3 and Sol^3. In Arnold's list, however, the simple elliptic singularities

$$\tilde{E}_6 = T_{3,3,3} = P_8,\ \tilde{E}_7 = T_{2,4,4} = X_9,\ \tilde{E}_8 = T_{2,3,6} = J_{10}$$

are related with the minimal hyperbolic singularities

$$T_{3,3,4} = P_9,\ T_{2,4,5} = X_{10},\ T_{2,3,7} = J_{11}$$

by the so-called adjacency. Hence, we would like to know about the geometrical meaning of the relation. This is the motivation of Problem 1.1.

Problem 1.2. Visualize how a singularity link is embedded in S^5 as a codimension two contact submanifold.

The signature of the page of the Milnor fibration associated with $f(z_1, z_2) + z_3^N$ is computed by Nemethi ([23]). The regular homotopy class of the embedding of the link in S^5 can be computed by the signature of the page. In particular, we know the regular homotopy class of the embedding of a Brieskorn singularity link. However, this is not a direct grasp of the embedding in S^5. We would like to know how the link is embedded.

The moment polytope of S^5 provides answers to these problems (§5.1 and 5.2). The key point is the recent work of Ryo Furukawa. He constructed contact embeddings of parabolic and hyperbolic T^2 bundles over S^1 in the standard contact 5-sphere (S^5, ξ_0) using the moment polytope. We show that the link of a simple elliptic singularity or a cusp singularity can be perturbed to his model as a contact submanifold (Theorem 5.2). This result is contained in [11]. The method is also useful for visualizing the structure of a Brieskorn singularity (§5.3). In the moment polytope, we can draw the fundamental domain of the triangle group which appears in Milnor's construction. This is an original contribution of this article.

2. Nil-manifolds and Sol-manifolds

In this section, we remind the definitions and properties of Nil-manifolds and Sol-manifolds. Let $\binom{x}{y}$ be the coordinates on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $(\binom{x}{y}, z)$ be the coordinates on $T^2 \times [0,1]$.

Definition 2.1. (Mapping tori) Let A be an element in $SL(2;\mathbb{Z})$. We define an equivalence relation $\sim$ on $T^2 \times [0,1]$ by $(A\binom{x}{y}, 0) \sim (\binom{x}{y}, 1)$. The quotient $T_A = T^2 \times [0,1]/\sim$ is called a *mapping torus of* $A \in SL(2;\mathbb{Z})$.

2.1. Sol-manifolds. Let A be a hyperbolic element of $SL(2,\mathbb{Z})$, that is, such that $\mathrm{tr}(A) > 2$. Then the matrix A has two positive eigenvalues a and $a^{\ 1}$ and the corresponding eigenvectors v_+ and $v_{\ }$, where $a > 1$ and $dx \wedge dy(v_+, v_{\ }) = 1$.

Definition 2.2. (Sol-manifolds) The Lie group Sol^3 is the split extension $1 \to \mathbb{R}^2 \to Sol^3 \to \mathbb{R} \to 1$ whose group structure is given by

$$(u, v; w) \cdot (u', v'; w') = (u + e^w \cdot u', v + e^{\ w} \cdot v'; w + w') \quad \text{on} \quad \mathbb{R}^2 \times \mathbb{R}.$$

There is a left invariant metric $e^{\ 2w} du \otimes du + e^{2w} dv \otimes dv + dw \otimes dw$ on Sol^3. Let Γ be a cocompact discrete subgroup of Sol^3. The compact quotient $M^3 = \Gamma\backslash Sol^3$ is called a *Sol-manifold*.

The mapping torus T_A of a hyperbolic matrix $A \in SL(2,\mathbb{Z})$ is a Sol-manifold. The left invariant 1-forms $e^{-w}du$ and $-e^{w}dv$ on Sol^3 induce the 1-forms $\beta_+ = a^{-z}dx \wedge dy(v_+,\,\cdot\,)$ and $\beta_- = -a^{z}dx \wedge dy(v_-,\,\cdot\,)$ on T_A. The flow $\phi_t(x,y,z) = (x,y,z+t)$ is an Anosov flow with respect to a Riemannian metric $g = \beta_+ \otimes \beta_+ + \beta_- \otimes \beta_- + dz \otimes dz$. It is easy to see that the 1-forms β_+ and β_- define foliations, so-called *Anosov foliations*. Moreover, $\beta_+ + \beta_-$ is a positive contact form and $\beta_+ - \beta_-$ is a negative contact form. They form a pair of contact structures on T_A which is called a *bi-contact structure* ([20]).

Remark 2.3. The 1-forms $\beta_+ + \beta_-$ and $\beta_+ - \beta_-$ are induced by left invariant contact forms $e^{-w}du - e^{w}dv$ and $e^{-w}du + e^{w}dv$ on Sol^3, respectively. The universal covering of $(T_A, \ker(\beta_+ + \beta_-))$ is $(Sol^3, \ker(e^{-w}du - e^{w}dv))$, which is the standard positive contact structure on $\mathbb{R}^3$. Thus $(T_A, \ker(\beta_+ + \beta_-))$ is universally tight. Similarly, $(T_A, \ker(\beta_+ - \beta_-))$ is a negative universally tight contact structure.

2.2. Nil-manifolds.

Definition 2.4. (Nil-manifolds) The Lie group Nil^3 is the central extension $1 \to \mathbb{R} \to Nil^3 \to \mathbb{R}^2 \to 1$ whose group structure is given by

$$(u,v;w)\cdot(u',v';w') = (u+u', v+v'; w+w'+uv') \quad \text{on} \quad \mathbb{R}^2 \times \mathbb{R}.$$

Let Γ be a cocompact discrete subgroup of Nil^3. The compact quotient $M^3 = \Gamma\backslash Nil^3$ is called a *Nil-manifold*.

We note that Nil^3 is isomorphic to the Heisenberg group of real matrices and a Nil-manifold is a parabolic mapping torus T_A, where $A = \left(\begin{smallmatrix}1 & 0\\ l & 1\end{smallmatrix}\right)$ for some $l \in \mathbb{Z}$. On a Nil-manifold T_A, there is a left invariant positive contact form $\alpha_l = dy + lzdx$. The contact structure $(T_A, \ker \alpha_l)$ is universally tight. We also note that there is no Anosov flow on Nil-manifolds ([20]).

3. Minimal resolutions and graph manifolds

In this section, we remind minimal resolutions and the graph manifolds associated with weighted dual graphs. We see that the link of a simple singularity, a simple elliptic singularity and a cusp singularity is diffeomorphic to a Seifert manifold, a Nil-manifold and a Sol-manifold, respectively.

Definition 3.1. (Resolutions, exceptional divisors) Let $(X,0)$ be a normal surface singularity. Then there exists a non-singular complex surface $\tilde{X}$ and a proper analytic map $\pi : \tilde{X} \to X$ satisfying the following conditions (1) and (2).

(1) $E = \pi^{-1}(0)$ is a union of 1-dimensional compact curves in $\tilde{X}$, and
(2) the restriction of π to $\pi^{-1}(X\backslash\{0\})$ is a biholomorphic map between $\tilde{X} - E$ and $X\backslash\{0\}$.

The surface $\tilde{X}$ is called a *resolution* of the singularity of X, $\pi : \tilde{X} \to X$ is called a *resolution map*, and E is called the *exceptional divisor*.

A resolution always exists and it can be obtained by a finite sequence of blow ups. By performing more blow ups, if necessary, we obtain a resolution such that E consists of non-singular irreducible components and has only normal crossings. Such a resolution is said to be *good*.

Definition 3.2. (Minimal resolutions) A resolution $\pi : \tilde{X} \to X$ is *minimal* if for any resolution $\pi' : \tilde{X}' \to X$, there is a proper analytic map $p : \tilde{X}' \to X$ such that $\pi' = \pi \circ p$.

By Castelnuovo's criterion, minimality of a resolution is equivalent to the condition that the exceptional set contains no non-singular rational curves with self-intersection -1. From the minimal resolution, we obtain a good one by performing blow ups, if necessary, and there is a unique *minimal good resolution*.

Let $E = E_1 \cup \cdots \cup E_n$ be the exceptional divisor of the minimal good resolution of $(X, 0)$, where each E_i denotes an irreducible component of it. We associate a graph to the resolution in the following way. In the graph, each divisor E_i is represented by a vertex with the weight E_i^2, and two divisors E_i and E_j are connected by an edge if they intersect transversely at one point. In the case where $E_i^2 = -2$, the weight is often omitted. This graph is called the *weighted dual graph* of the resolution.

Example 3.3. We calculate the minimal resolution of A_n singularity. We consider the algebraic surface $S : z_1^2 + z_2^2 + z_3^{n+1} = 0$. Let $\tilde{U} \subset \mathbb{C}^3 \times \mathbb{C}P^2$ be the one point blow up of $\mathbb{C}^3$ at the origin and $\pi : \tilde{U} \to \mathbb{C}^3$ the resolution map. That is, $\tilde{U} = \{((z_1, z_2, z_3), [x_1 : x_2 : x_3]) \mid x_1 z_2 - x_2 z_1 = 0,\ x_2 z_3 - x_3 z_2 = 0,\ x_3 z_2 - x_1 z_3 = 0\}$. We put $\tilde{U}_j = \tilde{U} \cap \{x_j \neq 0\}$, then $\tilde{U} = \tilde{U}_1 \cup \tilde{U}_2 \cup \tilde{U}_3$. On $\tilde{U}_1$, we can take coordinates

$$(z_1, u_2 = x_2/x_1, u_3 = x_3/x_1).$$

Similarly, $(v_1 = x_1/x_2, z_2, v_3 = x_3/x_2)$ and $(w_1 = x_1/x_3, w_2 = x_2/x_3, z_3)$ are coordinates on $\tilde{U}_2$ and $\tilde{U}_3$. With respect to these coordinates, the pull-back $\tilde{S} = \pi^{-1}(S)$ is given by the following equations:

$$\begin{aligned} z_1^2(1 + u_2^2 + z_1^{n-1} u_3^{n+1}) &= 0 \quad \text{on } \tilde{U}_1, \\ z_2^2(v_1^2 + 1 + z_2^{n-1} v_3^{n+1}) &= 0 \quad \text{on } \tilde{U}_2, \\ z_3^2(w_1^2 + w_2^2 + z_3^{n-1}) &= 0 \quad \text{on } \tilde{U}_3. \end{aligned}$$

If $n = 1, 2$, this surface does not have a singularity. In the case where $n = 1$, the exceptional set E is given by $x_1^2 + x_2^2 + x_3^2 = 0$ in $\mathbb{C}P^2$. It is a non-singular rational curve. Now we show that $E^2 = -2$. The self-intersection number

E^2 is equal to the normal Chern number of $E \subset \tilde{S} = \pi^{-1}(S)$. In order to know the normal Chern number, we take a normal vector field $(\frac{\partial}{\partial z_1})$ along the curve $z_1 = 0, 1 + u_2^2 + u_3^2 = 0$ in $\tilde{U_1}$. By coordinate transformations,

$$\left(\frac{\partial}{\partial z_1}\right) = \frac{1}{v_1}\left(\frac{\partial}{\partial z_2}\right) \quad \text{on } \tilde{U_1} \cap \tilde{U_2},$$
$$\left(\frac{\partial}{\partial z_1}\right) = \frac{1}{w_1}\left(\frac{\partial}{\partial z_3}\right) \quad \text{on } \tilde{U_3} \cap \tilde{U_1}.$$

Hence, the normal vector field has two poles of order 1. Therefore, the normal Chern number of $E \subset \tilde{S}$ is equal to -2. In the case where $n = 2$, the exceptional set is given by $x_1^2 + x_2^2 = 0$. It consists of two complex lines E_1 and E_2 which transversely intersect each other at one point. One may show similarly as above that the self-intersection numbers E_1^2 and E_2^2 are both equal to -2. In the case where $n > 2$, the exceptional set of $\pi^{-1}(S)$ is the same as in the case where $n = 2$. However, the surface $w_1^2 + w_2^2 + z_3^{n-1} = 0$ has A_{n-2} singularity and we need to continue performing point blow ups. By induction, the exceptional set of the minimal resolution of A_n singularity consists of n non-singular rational curves $E_1, \cdots, E_n$ such that $E_j^2 = -2$ and $E_k \cdot E_{k+1} = 1$ for all $1 \leq j \leq n$, $1 \leq k \leq n-1$. Thus, we obtain the following dual resolution graph.

Fig. 1. A_n

Since E_i^2 corresponds to the Chern number of the normal bundle of $E_i \subset \tilde{X}$, the tubular neighborhood of E_i is diffeomorphic to the D^2 bundle $D(E_i)$ over E_i with Chern number E_i^2. Hence, the tubular neighborhood of the exceptional set E is a union of $D(E_i)$ for all i. When E_i and E_j intersect transversely, the intersection $D(E_i) \cap D(E_j)$ is diffeomorphic to $D^2 \times D^2$. Let $D^2_{\varepsilon,i} \subset E_i$ be a small 2-disk which contains the intersection $E_i \cap E_j$. The bundle $D(E_i)$ restricted to $D^2_{\varepsilon,i} \subset E_i$ is diffeomorphic to $D^2_{\varepsilon,i} \times D^2$. Similarly, the bundle $D(E_j)$ restricted to a small 2-disk $D^2_{\varepsilon,j} \subset E_j$ containing the intersection point is diffeomorphic to $D^2_{\varepsilon,j} \times D^2$. Pasting the D^2 bundles by the identification

$$D^2_{\varepsilon,i} \times D^2 \to D^2_{\varepsilon,j} \times D^2; (x, y) \mapsto (y, x),$$

for each i, j, we obtain a 4-manifold U diffeomorphic to the tubular neighborhood of $E \subset \tilde{X}$. Such an operation is called a *plumbing* according to the weighted dual graph and the resulting manifold is called a *plumbed manifold*. Note that ∂U is diffeomorphic to the singularity link. Let us denote $\partial D(E_i)$

by $S(E_i)$. The singularity link can be obtained by a cut-and-paste operation on the S^1-bundles $S(E_i)$ which is also called *plumbing*. First, we cut out the solid torus $D^2_{\varepsilon,i} \times \partial D^2$ from $S(E_i)$ for each i. We paste the boundary tori by the identification

$$\partial D^2_{\varepsilon,i} \times \partial D^2 \to \partial D^2_{\varepsilon,j} \times \partial D^2; (x, y) \mapsto (y, x),$$

if E_i and E_j intersect transversely at one point. Then we obtain a description of the singularity link as a graph 3-manifold. It is well known that the diffeomorphism type of a graph 3-manifold is invariant under the so-called *Kirby moves*. The Kirby moves are originally operations on a framed link in S^3 defined as follows.

Definition 3.4. (Kirby moves) Let L be a framed link in S^3. The following two operations on L and their inverses are called *Kirby moves*.

(1) Add an unknotted circle with framing ± 1 which is contained in a 3-ball that does not intersect any component of L.
(2) Let L_1 and L_2 be two link components framed by n_1 and n_2, respectively, and L'_2 a longitude defining the framing n_2 of the knot L_2. Replace the pair $L_1 \cup L_2$ by $L_\sharp \cup L_2$, where $L_\sharp$ is the connected sum of L_1 and L'_2. The new framing of $L_\sharp$ is given by

$$n_1 + n_2 + 2\ \mathrm{lk}(L_1, L_2),$$

where $\mathrm{lk}(L_1, L_2)$ is the linking number of L_1 and L_2. Note that L_1 and L_2 have not been oriented so far. In order to compute $\mathrm{lk}(L_1, L_2)$, we orient L_1 and L_2 so that their orientations define an orientation on $L_\sharp$. Thus, we can compute $\mathrm{lk}(L_1, L_2)$.

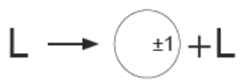

Fig. 2. Kirby move1

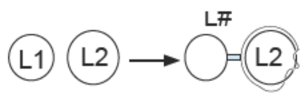

Fig. 3. Kirby move2

We note that the Kirby moves can be seen as operations on a 4-dimensional handle body. The move (1) with framing $+1$ (resp. 1) corresponds to taking the connected sum with $\mathbb{C}P^2$ (resp. $\overline{\mathbb{C}P^2}$). The move (2) corresponds to handle sliding. Hence, these moves preserve the diffeomorphism type of the boundary of a 4-dimensional manifold.

Neumann in [24] interpreted the Kirby moves in terms of weighted dual graphs. He listed the allowable moves of weighted dual graphs which preserve the graph 3-manifold. We also call such operations *Kirby moves*. In particular, the operations represented in Figures 4 and 5 preserve the graph 3-manifold. They are called *blow up* and *blow down*.

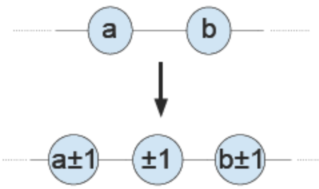

Fig. 4. ±1 blow up

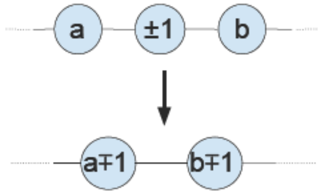

Fig. 5. ±1 blow down

Example 3.5. We show using blow ups and blow downs that the link of A_n singularity is diffeomorphic to the lens space $L(n+1, n)$. We already obtained the dual resolution graph in Example 3.3. Each exceptional divisor is a non-singular rational curve with self-intersection 2. After performing the $+1$ blow up once (Figure 6), we repeat the 1 blow downs n times. Then, we get one rational curve with self-intersection $n+1$ (Figure 8). It represents the lens space $L(n+1, n)$.

Fig. 6.

Fig. 7.

Fig. 8.

3.1. Simple singularities. For any simple singularity, the minimal resolution can be obtained by a sequence of blow ups of points. The dual resolution graphs for simple singularities are represented in the Figures 9, 10, 11, 12, and 13. Note that each vertex represents a non-singular rational curve with self-intersection number 2. We already showed that the link of A_n singularity is diffeomorphic to the lens space $L(n+1, n)$ (Example 3.5). For the other cases, it is easily proved by blow ups and blow downs that the link is a Seifert fibered manifold over S^2 with three exceptional fibers. In particular, the link of E_8 singularity is the Poincaré homology 3-sphere.

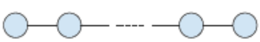

Fig. 9. A_n

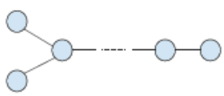

Fig. 10. D_n

Remark 3.6. (1) The names A_n, D_n, E_6, E_7 and E_8 derive from simple Lie algebras. Note that the above graphs coincide with the Dynkin diagrams of simply laced Lie groups. The graph corresponds also to the Milnor lattice, namely, the intersection form on the second homology group of the Milnor fiber. This illustrates the fact that the minimal resolution and the Milnor fiber are diffeomorphic. This is a specific property of simple singularities.

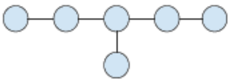

Fig. 11. E_6

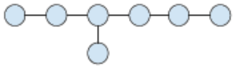

Fig. 12. E_7

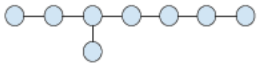

Fig. 13. E_8

(2) Brieskorn [3] and Slodowy [31] explained the reason of the correspondence between simple singularities and simple Lie algebras by constructing the semi-universal deformation of a simple singularity and its simultaneous resolution in terms of the corresponding simple Lie algebra. Another explanation was given by McKay in [18] and it is called the McKay correspondence.

(3) Durfee described in [6] many other characterizations of simple singularities.

3.2. Simple elliptic singularities. For simple elliptic singularities, the minimal resolutions are given by the following graphs. In these cases, a vertex represents a non-singular elliptic curve.

Fig. 14. $\tilde{E}_6$

Fig. 15. $\tilde{E}_7$

Fig. 16. $\tilde{E}_8$

It is clear that the link is a S^1 bundle over T^2. Therefore, it is a Nil-manifold.

Remark 3.7. (1) Originally, Saito defined a simple elliptic singularity to be a singularity such that the exceptional divisor of its minimal resolution is a non-singular elliptic curve. Moreover, he classified simple elliptic singularities which can be surface singularities in $\mathbb{C}^3$. He proved that $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ are the only simple elliptic hypersurface singularities ([29]).

(2) By Laufer (§4 in [13]), each minimal resolution was described in terms of elliptic functions on the exceptional curve. The defining equations of $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ essentially derive from the equation

$$(\wp'(z))^2 - 4(\wp(z))^3 + g_2\wp(z) + g_3 = 0,$$

where $\wp(z)$ is the Weierstrass $\wp$-function.

(3) The names $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ derive from the Dynkin diagrams of the Milnor lattices ([8]). They are completely different from the dual graphs of the minimal resolutions.

3.3. Cusp singularities. The minimal resolutions of the T_{pqr} singularities are given by Figures 17, 18, 19, 20, 21, 22, 23, 24, and 25 (see [15]). Each vertex represents a non-singular rational curve E_i $(1 \leq i \leq k)$. We put $b_i = -E_i^2$ if $k \geq 2$ and $b_1 = -E_1^2 + 2$ if $k = 1$. Plumbing S^1 bundles according to this graph, we obtain the T^2 bundle over S^1 with monodromy

$$(JH_k)\cdots(JH_1) = \begin{pmatrix} b_k & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_1 & 1 \\ 1 & 0 \end{pmatrix},$$

where $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $H_i = \left(\begin{smallmatrix} 1 & 0 \\ b_i & 1 \end{smallmatrix}\right)$. The matrix $(JH_k)\cdots(JH_1)$ is hyperbolic by the condition $b_i \geq 2$ for all i and $b_i > 2$ for some i. Therefore, the link of T_{pqr} is a hyperbolic mapping torus, namely, a Sol-manifold.

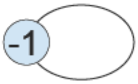

Fig. 17. T_{237}

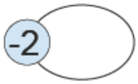

Fig. 18. T_{245}

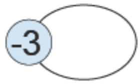

Fig. 19. T_{334}

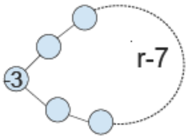

Fig. 20. T_{23r}

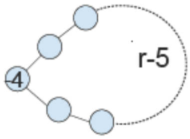

Fig. 21. T_{24r}

Fig. 22. T_{33r}

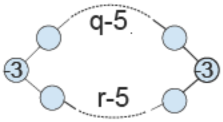

Fig. 23. T_{2qr}

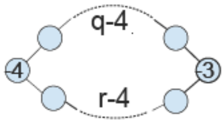

Fig. 24. T_{3qr}

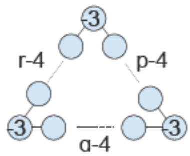

Fig. 25. T_{pqr}

Remark 3.8. (1) For T_{237}, T_{245}, T_{334}, T_{238}, T_{246} and T_{335}, the minimal resolution is not good. In the former three cases, the exceptional divisor consists of one rational curve with a transversal self-intersection. That is why we define b_1 by $E_1^2 + 2$ (not by E_1^2). Namely, the Euler number of the corresponding S^1 bundle over S^2 is $b_1 = E_1^2$ 2. In the latter three cases, the exceptional divisor consists of two non-singular rational curves which intersect transversally at two points. Hence, we can do plumbing S^1 bundles as it is.

(2) A normal singularity is a cusp singularity if the exceptional divisor of the minimal resolution is a cycle of rational curves. This is the original definition of a cusp singularity. Hirzebruch in [9] explicitly constructed the minimal resolution of a complex 2-dimensional Hilbert modular cusp. Consequently, he gave a one-to-one correspondence between the weighted

dual graphs satisfying the above condition for b_i and the minimal resolutions of Hilbert modular cusps. On the other hand, Laufer proved in [14] that a cusp singularity is taut, that is, any two cusp singularities whose minimal resolutions have the same weighted dual graph, are analytically equivalent. Therefore, we have a one-to-one correspondence between Hilbert modular cusps and cusp singularities.

(3) Laufer in [15] proved that T_{pqr} is a cusp singularity and the link is the mapping torus of

$$A = \begin{pmatrix} p & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r & -1 \\ 1 & 0 \end{pmatrix}.$$

Karras in [10] proved that T_{pqr} are the only cusp singularities which are surface singularities in $\mathbb{C}^3$.

(4) As is the case of simple elliptic singularities, the name T_{pqr} derives from the Dynkin diagram of the Milnor lattice ([8]).

4. 3-dimensional Lie groups and complex surface singularities

In §4.1, we explain Milnor's theorem about Brieskorn singularities. His theorem is an essential part of the relation between quasi-homogeneous singularities and the Lie groups $SU(2)$, Nil^3, and $\widetilde{SL}(2;\mathbb{R})$. In §4.2, we explain Hilbert modular cusps, which are analytically equivalent to cusp singularities. Note that cusp singularities are not quasi-homogeneous singularities. It can be easily seen that their links carry the structure of the Lie group Sol^3.

4.1. Milnor's construction. Let $M(p,q,r)$ be the 3-manifold obtained by intersecting the complex surface $V(p,q,r) = \{(z_1,z_2,z_3) \mid z_1^p + z_2^q + z_3^r = 0\}$ with the 5-sphere $\{(z_1,z_2,z_3) \mid |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$. It is called a Brieskorn manifold. Let G be $SU(2)$, Nil^3, or $\widetilde{SL}(2;\mathbb{R})$ according as the number $p^{-1}+q^{-1}+r^{-1}-1$ is positive, zero, or negative. Milnor showed the following theorem.

Theorem 4.1. (Milnor [19]) *The manifold $M(p,q,r)$ is diffeomorphic to a coset space of the form $\Pi\backslash G$, where Π is a certain discrete subgroup of G.*

We devote this subsection to a review of Theorem 4.1 following [19].

Let P denote the 2-sphere S^2 if $p^{-1}+q^{-1}+r^{-1} > 1$, the Euclidean plane $\mathbb{R}^2$ if $p^{-1}+q^{-1}+r^{-1} = 1$, and the hyperbolic plane $\mathbb{H}$ if $p^{-1}+q^{-1}+r^{-1} < 1$. The isometry group of P is the orthogonal group $O(3)$, the affine group $E(2)$, and the Möbius group Möb$(2;\mathbb{R})$, respectively. That is,

(1) $P = S^2$ (spherical) if $(p,q,r) = (2,2,r)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$,
(2) $P = \mathbb{R}^2$ (Euclidean) if $(p,q,r) = (2,3,6)$, $(2,4,4)$, $(3,3,3)$,
(3) $P = \mathbb{H}$ (hyperbolic) otherwise.

We consider a triangle with interior angles π/p, π/q and π/r. This triangle $T(p,q,r)$ lies in the plane P. Let σ_1, σ_2, σ_3 be the reflections in the three edges of $T(p,q,r)$.

Definition 4.2. (The Schwarz triangle groups) The *full Schwarz triangle group* $\Sigma^*(p,q,r)$ is the group of isometries of P generated by σ_1, σ_2, σ_3. The Schwarz triangle group $\Sigma(p,q,r)$ is the index 2 subgroup of $\Sigma^*(p,q,r)$ consisting of all orientation preserving elements.

Theorem 4.3. (Poincaré) *The group $\Sigma^*(p,q,r)$ has the following presentation;*

$$\Sigma^*(p,q,r) = < \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^2, \sigma_2^2, \sigma_3^2, (\sigma_1\sigma_2)^p, (\sigma_2\sigma_3)^q, (\sigma_3\sigma_1)^r > .$$

The triangle $T(p,q,r)$ is a fundamental domain for the action of $\Sigma^*(p,q,r)$ on P.

We put $\tau_1 = \sigma_1\sigma_2$, $\tau_2 = \sigma_2\sigma_3$, $\tau_3 = \sigma_3\sigma_1$. Then, we obtain the following.

Corollary 4.4. *The group $\Sigma(p,q,r)$ has the following presentation;*

$$\Sigma(p,q,r) = < \tau_1, \tau_2, \tau_3 \mid \tau_1^p, \tau_2^q, \tau_3^r, \tau_1\tau_2\tau_3 > .$$

Note that $\Sigma(p,q,r)$ is a discrete subgroup of $\bar{G} = SO(3)$, $E^+(2)$, or $PSL(2;\mathbb{R})$.

Definition 4.5. (The centrally extended triangle group) The full inverse image of $\Sigma(p,q,r) \subset \bar{G}$ in the universal covering group of $\bar{G}$ is called the *centrally extended triangle group* $\Gamma(p,q,r)$.

Lemma 4.6. *The group $\Gamma(p,q,r)$ has the following presentation;*

$$\Gamma(p,q,r) = < \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1^p = \gamma_2^q = \gamma_3^r = \gamma_1\gamma_2\gamma_3 > .$$

In spherical and hyperbolic cases, the discrete group Π in Theorem 4.1 can be characterized as the commutator subgroup of $\Gamma(p,q,r)$. In the following, we mainly explain the spherical cases. The classification of finite subgroups of $SO(3)$ is well-known, and the list is as follows:

(1) the cyclic group of order r,
(2) the dihedral group $\Sigma(2,2,r)$ of order $2r$,
(3) the tetrahedral group $\Sigma(2,3,3)$ of order 12,
(4) the octahedral group $\Sigma(2,3,4)$ of order 24,
(5) the icosahedral group $\Sigma(2,3,5)$ of order 60.

Except for the case (1), the triangle $T(p,q,r)$ can be drawn on the corresponding regular polyhedron as follows.

Since we have a 2-fold covering $SU(2) \to SO(3)$, each finite subgroup of $SO(3)$ lifts to a finite subgroup of $SU(2)$. The cyclic group of order r lifts to the cyclic group of order $2r$ and the triangle group $\Sigma(p,q,r)$ lifts

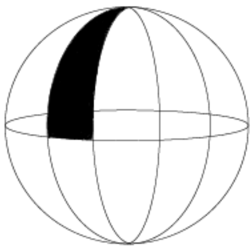

Fig. 26. $(2, 2, r)$

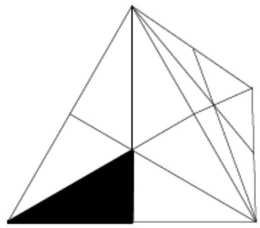

Fig. 27. $(2, 3, 3)$

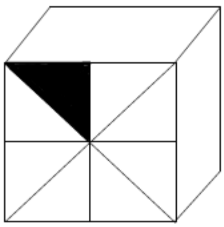

Fig. 28. $(2, 3, 4)$

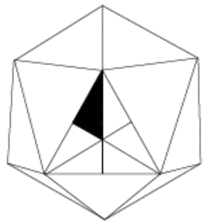

Fig. 29. $(2, 3, 5)$

to $\Gamma(p, q, r)$. The group $SU(2)$ naturally acts on $\mathbb{C}^2$, hence on $\mathbb{C}P^1$. Now, we consider $\mathbb{C}[z_1, z_2]^{\Pi}$, the space of polynomials invariant under the action of $\Pi = [\Gamma(p, q, r), \Gamma(p, q, r)]$. Milnor showed that it is generated by three homogeneous polynomials f_1, f_2, f_3 such that

$$f_i(\gamma(z_1, z_2)) = \chi_i(\gamma) f_i(z_1, z_2),$$

for any $\gamma \in \Gamma(p, q, r)$ and characters $\chi_i : \Gamma(p, q, r) \to U(1)$ satisfying $\chi_1^p = \chi_2^q = \chi_3^r$. Since $\mathbb{C}P^1$ is diffeomorphic to S^2, we can mark the zeros of the homogeneous polynomial f_i on S^2. They correspond to the $\Sigma(p, q, r)$-orbits of the vertices of the triangle $T(p, q, r)$. After having been multiplied by suitable constants, the three polynomials satisfy

$$f_1^p + f_2^q + f_3^r = 0.$$

They define an injective map $(f_1, f_2, f_3) : \Pi \backslash \mathbb{C}^2 \to \mathbb{C}^3$ whose image is the Brieskorn variety $V(p, q, r)$ and the image of its restriction to $\Pi \backslash S^3$ is diffeomorphic to the Brieskorn manifold $M(p, q, r)$. We note that

(1) $[\Gamma(2, 2, r), \Gamma(2, 2, r)] = \mathbb{Z}_r$,
(2) $[\Gamma(2, 3, 3), \Gamma(2, 3, 3)] = \Gamma(2, 2, 2)$,
(3) $[\Gamma(2, 3, 4), \Gamma(2, 3, 4)] = \Gamma(2, 3, 3)$,
(4) $[\Gamma(2, 3, 5), \Gamma(2, 3, 5)] = \Gamma(2, 3, 5)$.

In particular, $M(2,3,5)$ is diffeomorphic to $\Gamma(2,3,5)\backslash S^3$. On the other hand, the singularities $\Gamma(2,3,4)\backslash\mathbb{C}^2$ and $\Gamma(2,2,r)\backslash\mathbb{C}^2$ are not Brieskorn singularities but quasi-homogeneous singularities $z_1^2+z_2^3+z_2z_3^3=0$ and $z_1^2+z_2^2z_3+z_3^{r+1}=0$ (see [12]). In this way, we described all the simple singularities.

In the hyperbolic cases, Milnor used a similar method.

Definition 4.7. (A differential form of fractional degree) A *differential form of fractional degree* $\alpha \in \mathbb{Q}$ on $\mathbb{H}$ is a complex valued function of the form

$$\phi(z,w) = f(z)w^{\alpha},$$

where f is a holomorphic function on $\mathbb{H}$ and w varies over the universal covering group $\widetilde{\mathbb{C}^*}$ of $\mathbb{C}^*$.

The action of $PSL(2;\mathbb{R})$ on the tangent bundle $T\mathbb{H} \cong \mathbb{H}\times\mathbb{C}$ lifts to the action of the universal covering group $\widetilde{SL}(2;\mathbb{R})$ on $\mathbb{H}\times\widetilde{\mathbb{C}^*}$. Using this action, the pull-back $\gamma^*\phi$ of a differential form ϕ by an element $\gamma \in \widetilde{SL}(2;\mathbb{R})$ is defined by

$$\gamma^*\phi(z,w) = \phi\left(\gamma(z), \widetilde{\frac{d\gamma}{dz}}(z)\cdot w\right).$$

Definition 4.8. (Automorphic forms) Given a discrete subgroup Γ of $\widetilde{SL}(2;\mathbb{R})$ and a character $\chi : \Gamma \to U(1)$, a differential form $\phi(z,w) = f(z)w^{\alpha}$ on $\mathbb{H}$ is χ-*automorphic* if $\gamma^*\phi = \chi(\gamma)\phi$ for every $\gamma \in \Gamma$. If χ is the trivial character, ϕ is said to be Γ-*automorphic*.

Automorphic forms play the role of Γ-invariant polynomials in the spherical cases. Milnor showed that there are generators ϕ_1, ϕ_2, ϕ_3 of the space of Π-automorphic forms such that $\phi_1^p + \phi_2^q + \phi_3^r = 0$. The map $(\phi_1,\phi_2,\phi_3) : \Pi\backslash\mathbb{H}\times\widetilde{\mathbb{C}^*} \to \mathbb{C}^3$ gives an injective holomorphic map into the Brieskorn variety $V(p,q,r)$ and the diffeomorphism between $\Pi\backslash\widetilde{SL}(2;\mathbb{R})$ and the Brieskorn manifold $M(p,q,r)$.

In the Euclidean cases, the situation is a little different.

Theorem 4.9. (Milnor [19]) *If* $l.c.m.(p,q) = l.c.m.(q,r) = l.c.m.(r,p) = m$*, then the manifold* $M(p,q,r)$ *is a* S^1 *bundle with Euler class* $\;pqr/m^2$ *over an orientable closed surface* B *with* $\chi(B) = (pq+qr+pr \quad pqr)/m$*.*

By this theorem, the manifolds $M(2,3,6)$, $M(2,4,4)$, $M(3,3,3)$ are diffeomorphic to the S^1 bundles over T^2 with Euler class $\;1$, $\;2$, $\;3$. Hence, they are not the quotients of $\widetilde{E^+(2)}$, but the quotients of the Heisenberg group Nil^3.

4.2. Hilbert modular cusps. Let K be a totally real algebraic field of degree 2 over $\mathbb{Q}$. Then we have two distinct embeddings $x \mapsto x^{(i)}$ $(i = 1,2)$

of K in $\mathbb{R}$. Let H be an additive subgroup of K of rank 2, and let V be a multiplicative subgroup of U_H^+ of rank 1, where U_H^+ is the group of totally positive units e with $eH = H$. Let

$$G(H, V) = \left\{ \begin{pmatrix} e & b \\ 0 & 1 \end{pmatrix} \mid b \in H, e \in V \right\}.$$

Then $G(H, V)$ acts properly discontinuously and without fixed points on the product $\mathbb{H}^2$ of 2 copies of the upper half plane $\mathbb{H}$ by

$$(z_1, z_2) \mapsto (e^{(1)} z_1 + b^{(1)}, e^{(2)} z_2 + b^{(2)}).$$

Since $e \in V$ is a totally positive unit, we have $e^{(1)} e^{(2)} = 1$. Hence, the action of $G(H, V)$ is identified with the action of Sol^3. We consider $\overline{\mathbb{H}^2/G(H, V)}$ which is the completion of $\mathbb{H}^2/G(H, V)$ by adding the point ∞. The basis of open neighborhoods of ∞ is given by the sets

$$\Big(\text{int } W(d)/G(H, V)\Big) \cup \{\infty\},$$

where, for any positive d,

$$W(d) = \left\{(z_1, z_2) \in \mathbb{H}^2 \mid \text{Im} z_1 \cdot \text{Im} z_2 \geq d\right\}.$$

Then $\overline{\mathbb{H}^2/G(H, V)}$ is a normal complex space. The singularity ∞ is called a *Hilbert modular cusp*. The function

$$\varphi(z_1, z_2) = \frac{1}{\text{Im} z_1 \cdot \text{Im} z_2}$$

induces a strictly pluri-subharmonic function on $\overline{\mathbb{H}^2/G(H, V)}$. By general theory of Stein manifolds, we obtain the following. Let J be the standard complex structure on $\mathbb{H}^2$ and we put

$$\lambda = \quad J^* d\varphi, \;\; \omega = \quad dJ^* d\varphi, \;\; g(u, v) = \omega(u, Jv).$$

Then ω is a symplectic form on $\mathbb{H}^2$ compatible with J, and g is a J-invariant Riemannian metric. Moreover, $\alpha = \lambda \mid \partial W(d)$ is a contact form on $\partial W(d)$. Since φ is $G(H, V)$-invariant, there is an induced contact structure $(\partial W(d)/G(H, V), \ker \tilde{\alpha})$. It is the canonical contact structure on the link of the singularity ∞. The contact form $\tilde{\alpha}$ on the link $\partial W(1)/G(H, V)$ is given by

$$\alpha = \frac{dx_1}{y_1} + \frac{dx_2}{y_2},$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. By an explicit computation, we can confirm that $(\partial W(d)/G(H, V), \ker \tilde{\alpha})$ is contactomorphic to $(T_A, \beta_+ + \beta \;\;)$ for some hyperbolic matrix A (Theorem 4.5 in [11]).

5. A new approach

5.1. The moment polytope of S^5. Let $(r_1, \theta_1, r_2, \theta_2, r_3, \theta_3)$ be the polar coordinates on $S^5 \subset \mathbb{C}^3$, where

$$(z_1, z_2, z_3) = (r_1 e^{2\pi i \theta_1}, r_2 e^{2\pi i \theta_2}, r_3 e^{2\pi i \theta_3}) \in \mathbb{C}^3,\ S^5 = \left\{r_1^2 + r_2^2 + r_3^2 = 1\right\}.$$

The standard contact form on S^5 is $\alpha_0 = r_1^2 d\theta_1 + r_2^2 d\theta_2 + r_3^2 d\theta_3$. Let $\phi\colon S^5 \to \mathbb{R}^3$ be the projection, where

$$\phi(r_1, \theta_1, r_2, \theta_2, r_3, \theta_3) = (r_1^2, r_2^2, r_3^2).$$

Then the image $\phi(S^5) = \{x_1 + x_2 + x_3 = 1, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$ is a regular triangle in $\mathbb{R}^3$. It is called the *moment polytope* Δ. We note that the projection ϕ is a T^3 fibration over the interior of Δ, a T^2 fibration over the three edges and a S^1 fibration over the three vertexes.

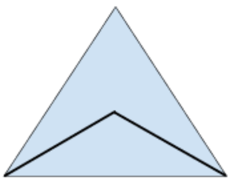

Fig. 30. Reeb foliation

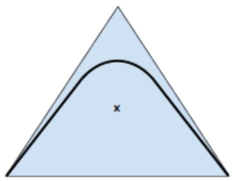

Fig. 31. OT 3-sphere

A. Mori in [22] constructed a contact embedding of an overtwisted (OT) contact 3-sphere into the standard contact 5-sphere using the moment polytope. The key point of his construction was that he connected the rotation around the barycenter of the triangle Δ with the non-integrability of the induced contact structure. Ryo Furukawa extended this principle to the rotation around a weighted excenter of the triangle. As a result, he obtained the following example.

Example 5.1 (Furukawa). One may embed in the same way the positive contact structure associated to the Anosov flow of T_A, where

$$A = \begin{pmatrix} p & 1 & 1 \\ & 1 & 0 \end{pmatrix} \begin{pmatrix} q & 1 & 1 \\ & 1 & 0 \end{pmatrix} \begin{pmatrix} r & 1 & 1 \\ & 1 & 0 \end{pmatrix}.$$

Let $c : [0,1] \to \Delta$ be a C^∞-curve such that

$$\begin{cases} c([0, \frac{1}{6}]) = \{x_2 = 0, \varepsilon \leq x_3 \leq 1 \quad \varepsilon\} = AB, \\ c([\frac{1}{3}, \frac{1}{2}]) = \{x_1 = 0, \varepsilon \leq x_2 \leq 1 \quad \varepsilon\} = CD, \\ c([\frac{2}{3}, \frac{5}{6}]) = \{x_3 = 0, \varepsilon \leq x_1 \leq 1 \quad \varepsilon\} = EF \end{cases}$$

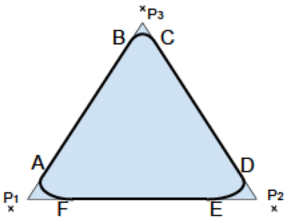

Fig. 32. Furukawa's example

and c rotates counter clockwise around the point P, where

$$P=\begin{cases} P_3=\left(\dfrac{-1}{r-3},\dfrac{-1}{r-3},\dfrac{r-1}{r-3}\right) & t\in[\frac{1}{12},\frac{5}{12}),\\ P_2=\left(\dfrac{-1}{q-3},\dfrac{q-1}{q-3},\dfrac{-1}{q-3}\right) & t\in[\frac{5}{12},\frac{3}{4}),\\ P_1=\left(\dfrac{p-1}{p-3},\dfrac{-1}{p-3},\dfrac{-1}{p-3}\right) & t\in[0,\frac{1}{12})\cup[\frac{3}{4},1]. \end{cases}$$

We define the submanifold X of S^5 as the slice section over c given by

$$\begin{cases} \{(r-1)\theta_3-\theta_1-\theta_2=0\}\,(t\in(\frac{1}{6},\frac{1}{3})),\\ \{(q-1)\theta_2-\theta_3-\theta_1=0\}\,(t\in(\frac{1}{2},\frac{2}{3})),\\ \{(p-1)\theta_1-\theta_2-\theta_3=0\}\,(t\in(\frac{5}{6},1)). \end{cases}$$

The pull backs $\phi^{-1}(c([0,\frac{1}{6}]))$, $\phi^{-1}(c([\frac{1}{3},\frac{1}{2}]))$ and $\phi^{-1}(c([\frac{2}{3},\frac{5}{6}]))$ are all $T^2\times I$ with the standard contact structure. The equation $(p-1)\theta_1-\theta_2-\theta_3=0$ is equivalent to the coordinate transformation $\binom{\theta_3}{\theta_1}=\begin{pmatrix} p-1 & -1\\ 1 & 0\end{pmatrix}\binom{\theta_1}{\theta_2}$. Similarly, the two other equations represent the coordinate transformations $\binom{\theta_1}{\theta_2}=\begin{pmatrix} q-1 & -1\\ 1 & 0\end{pmatrix}\binom{\theta_2}{\theta_3}$ and $\binom{\theta_2}{\theta_3}=\begin{pmatrix} r-1 & -1\\ 1 & 0\end{pmatrix}\binom{\theta_3}{\theta_1}$. Thus X is the resultant contact manifold of pasting the three pieces of $T^2\times I$ by the linear maps $\begin{pmatrix} p-1 & -1\\ 1 & 0\end{pmatrix}$, $\begin{pmatrix} q-1 & -1\\ 1 & 0\end{pmatrix}$, and $\begin{pmatrix} r-1 & -1\\ 1 & 0\end{pmatrix}$. In other words, it is obtained by pasting the boundary tori of $(T^2\times I,\ker(f(z)dx+g(z)dy))$ by the linear map A, where $\binom{f}{g}$ is a curve rotating clockwise around $\binom{0}{0}$ from $\binom{1}{0}$ to $A\binom{1}{0}$. Therefore, X is contactomorphic to $(T_A,\ker(\beta_++\beta_-))$. In the case of $(p,q,r)=(2,3,6),(2,4,4),(3,3,3)$, we can see that X is contactomorphic to $(T_A,\ker(dy+lzdx))(l=1,2,3)$ by the same argument.

5.2. Simple elliptic and cusp singularities. We prove the following theorem, which treats simple elliptic singularities and cusp singularities uniformly.

Theorem 5.2. *The link of the surface singularity* $z_1^p+z_2^q+z_3^r+z_1z_2z_3=0$ *such that* $p^{-1}+q^{-1}+r^{-1}\leq 1$ *is the mapping torus* T_A *of* A*, where*

$$A = \begin{pmatrix} p \quad 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \quad 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r \quad 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Moreover, the canonical contact structure on the link is contactomorphic to the contact manifold X of Example 5.1.

Proof. We consider the intersection $L_\lambda = S^5 \cap \{z_1^p + z_2^q + z_3^r \quad \lambda z_1 z_2 z_3 = 0\}$ for a sufficiently large positive real number λ. It is contactomorphic to the link of $\{z_1^p + z_2^q + z_3^r + z_1 z_2 z_3 = 0\}$. Since λ is sufficiently large, $|z_1 z_2 z_3|$ is small on L_λ. Indeed,

$$|z_1 z_2 z_3| < \frac{1}{\lambda}(|z_1|^p + |z_2|^q + |z_3|^r) < \frac{3}{\lambda}.$$

Thus L_λ is very close to $L = S^5 \cap \{z_1 z_2 z_3 = 0\}$ except on a neighborhood of the union of three circles $\{z_2 = z_3 = 0\} \cup \{z_3 = z_1 = 0\} \cup \{z_1 = z_2 = 0\}$. On the other hand, for sufficiently small ε, X of Example 5.1 is also very close to L except near the three circles. Moreover, L_λ is very close to X even on a neighborhood of the three circles. We can isotope L_λ to X as a contact submanifold, and by Gray stability, they are contactomorphic. This is the out line of the proof.

Let us prove the existence of a contact isotopy from L_λ to X. Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a bump function supported on $\{s \in \mathbb{R} \mid 1 \quad 2\delta \leq s\}$ and $\phi \equiv 1$ on $\left\{s \in \mathbb{R} \mid 1 \quad \delta \leq s\right\}$ with $0 < \delta < \frac{1}{5}$. We define $F_\lambda = z_1 z_2 z_3 \quad \frac{1}{\lambda}(z_1^p + z_2^q + z_3^r)$ and $G_\lambda = z_1 z_2 z_3 \quad \frac{1}{\lambda}(\phi(r_1^2) z_1^p + \phi(r_2^2) z_2^q + \phi(r_3^2) z_3^r)$. Note that $G_\lambda^{\ 1}(0)$ satisfies the condition of X of Example 5.1. Hence it is enough to find a contact isotopy between $F_\lambda^{\ 1}(0)$ and $G_\lambda^{\ 1}(0)$. We define $H_t = (1 \quad t)F_\lambda + tG_\lambda$. For sufficiently large λ, $H_t^{\ 1}(0)$ defines a contact isotopy. On the open set $\left\{|z_i| > \sqrt{1 \quad \delta}\right\} \subset S^5$, $H_t^{\ 1}(0)$ is a complex hypersurface singularity link for each $t \in [0, 1]$. Thus, it is a contact submanifold on the open set. On the other hand, $H_t^{\ 1}(0)$ is close to L on $U = \left\{|z_1|, |z_2|, |z_3| < \sqrt{1 \quad \frac{1}{2}\delta}\right\}$. Since $L \cap U$ is a contact submanifold of U and the contactness is an open condition, there exists λ such that $H_t^{\ 1}(0) \cap U$ is a contact submanifold of U for each $t \in [0, 1]$. For such a positive number λ, $H_t^{\ 1}(0)$ is a contact submanifold of the standard contact 5-sphere for each $t \in [0, 1]$. Hence it is a contact isotopy between $F_\lambda^{\ 1}(0)$ and $G_\lambda^{\ 1}(0)$. Therefore, $L_\lambda = F_\lambda^{\ 1}(0)$ is diffeomorphic to the mapping torus T_A of A, where

$$A = \begin{pmatrix} p \quad 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \quad 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r \quad 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and the canonical contact structure is the positive contact structure associated to the suspension Anosov flow on it. The above argument also works for the case of $(p,q,r) = (2,3,6),(2,4,4),(3,3,3)$, namely, the simple elliptic singularities. ∎

5.3. Brieskorn singularities. First, we reprove the following well-known theorem. It is a special case of Theorem 4.9.

Theorem 5.3. *The link of the singularity $z_1^p + z_2^p + z_3^p = 0$ is diffeomorphic to the S^1 bundle over the Riemann surface of genus $\frac{1}{2}(p-1)(p-2)$ with Euler class $-p$.*

Proof. First, we show that the Fermat curve $F_p = \{z_1^p + z_2^p + z_3^p = 0\}$ in the projective space $\mathbb{C}P^2$ is the closed orientable surface of genus $\frac{1}{2}(p-1)(p-2)$. We define a projection $\pi : \mathbb{C}P^2 \to \mathbb{R}^3$ by

$$\begin{aligned}&\pi([z_1 : z_2 : z_3])\\ &\quad = \left(\frac{|z_1|^2}{|z_1|^2+|z_2|^2+|z_3|^2}, \frac{|z_2|^2}{|z_1|^2+|z_2|^2+|z_3|^2}, \frac{|z_3|^2}{|z_1|^2+|z_2|^2+|z_3|^2}\right).\end{aligned}$$

The projections $\phi : S^5 \to \mathbb{R}^3$ and $\pi : \mathbb{C}P^2 \to \mathbb{R}^3$ are compatible with respect to the Hopf fibration $h : S^5 \to \mathbb{C}P^2$. Hence, Δ also denotes the image of π. The image of the Fermat curve $z_1^p + z_2^p + z_3^p = 0$ by π is represented in Figure 33. The boundary curves are defined by the equations

$$r_1^p = r_2^p + r_3^p, \quad r_2^p = r_3^p + r_1^p, \quad r_3^p = r_1^p + r_2^p,$$

where $r_j = |z_j|$. This is proved by the following argument. Using the polar coordinates (r_j, θ_j), the equation $z_1^p + z_2^p + z_3^p = 0$ is described as

$$r_1^p e^{2\pi i p\theta_1} + r_2^p e^{2\pi i p\theta_2} + r_3^p e^{2\pi i p\theta_3} = 0.$$

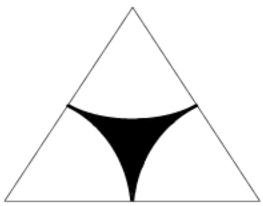

Fig. 33. Fermat curve

Each term represents a complex number of absolute value r_j^p. Therefore, we see a triangle whose edges are of lengths r_1^p, r_2^p, r_3^p in the complex plane. Such arguments θ_j exist if and only if

$$r_1^p \le r_2^p + r_3^p, \quad r_2^p \le r_3^p + r_1^p, \quad r_3^p \le r_1^p + r_2^p.$$

The region defined by these inequalities looks like the shaded region in the above figure. Let us denote it by τ. Then, τ is corresponding to the triangle

$T(p,p,p)$ and the pull-back $\pi^{-1}(\partial\tau)$ gives the 1-skeleton of a triangulation of F_p. It is easily seen that this triangulation consists of $2p^2$ faces, $3p^2$ edges and $3p$ points. Hence, we obtain the following equation about the genus g of F_p:

$$2-2g = 3p-3p^2+2p^2 \iff 2g = p^2-3p+2 \iff g = \tfrac{1}{2}(p-1)(p-2).$$

The Brieskorn manifold $M(p,p,p)$ carries the natural S^1-action

$$\theta\cdot(z_1,z_2,z_3) = (e^{2\pi i\theta}z_1, e^{2\pi i\theta}z_2, e^{2\pi i\theta}z_3),$$

and the quotient by this action is the closed orientable surface F_p. Hence, it is obviously a S^1 bundle over F_p. Now, we show that the Euler class of this S^1 bundle is equal to $-p$. The standard contact form $\alpha_0 = r_1^2 d\theta_1 + r_2^2 d\theta_2 + r_3^2 d\theta_3$ is the connection 1-form of the Hopf fibration $h : S^5 \to \mathbb{C}P^2$. The restriction of α_0 to $M(p,p,p)$ is also the connection 1-form of the S^1 bundle $M(p,p,p) \to F_p$. Hence, the Euler class is equal to the integral

$$\frac{1}{2\pi}\int_{F_p}\Omega,$$

where $d\alpha_0 = h^*\Omega$. We may assume that Ω is the Fubini-Study form on $\mathbb{C}P^2$. With respect to the decomposition $\mathbb{C}P^2 = \mathbb{C}^2 \cup \mathbb{C}P^1$, the symplectic form Ω is compatible with the standard symplectic form on $\mathbb{C}^2$ and the Brieskorn manifold $M(p,p,p)$ transversely intersects with $h^{-1}(\mathbb{C}P^1)$ at p distinct Hopf fibers $C_1,\cdots,C_p$. Let $s : \mathbb{C}^2 \to S^5$ be a section of $h : S^5 \to \mathbb{C}P^2$ over $\mathbb{C}^2 \subset \mathbb{C}P^2$ defined by

$$s(x,y) = \left(\frac{x}{\sqrt{|x|^2+|y|^2+1}}, \frac{y}{\sqrt{|x|^2+|y|^2+1}}, \frac{1}{\sqrt{|x|^2+|y|^2+1}}\right).$$

Then, we have $h \circ s(\{x^p+y^p+1=0\}) = F_p \cap \mathbb{C}^2$. By Stokes' theorem,

$$\int_{F_p}\Omega = \int_{s(\{x^p+y^p+1=0\})} d\alpha_0 = -\int_{C_1}\alpha_0 + \cdots + -\int_{C_p}\alpha_0 = -2p\pi.$$

Therefore, the Euler class of the S^1 bundle $M(p,p,p) \to F_p$ is equal to $-p$. ∎

In this way, we recovered Milnor's theorem for the case $p = q = r$ by using the moment polytope. Similarly, the image $\phi(\{z_1^p + z_2^q + z_3^r = 0\})$ is corresponding to the triangle $T(p,q,r)$ and the orbits of S^1 action

$$\theta\cdot(z_1,z_2,z_3) = (e^{2\pi ik\theta/p}z_1, e^{2\pi ik\theta/q}z_2, e^{2\pi ik\theta/r}z_3)$$

are the fibers of the Seifert fibration, where $k = \text{l.c.m.}(p,q,r)$. We take a defining 1-form of ξ_0,

$$\alpha(p,q,r) = \frac{\alpha_0}{2m\pi(|z_1|^2/p + |z_2|^2/q + |z_3|^2/r)}.$$

Then, on the manifold $M(p,q,r)$, the Reeb vector field $R_{\alpha(p,q,r)}$ corresponds to the velocity vector of the S^1 action. In the 3-dimensional case, two contact manifolds with the same Reeb vector fields are contactomorphic. Therefore, the Brieskorn manifold $M(p,q,r)$ is contactomorphic to the quotient of the left invariant positive contact structure on G by a cocompact lattice Π.

5.4. Problems. We propose some problems about the topology of singularity links. The first one is proposed by Yoshihiko Mitsumatsu.

Problem 5.4. Let p,q,r be positive integers such that $p^{-1}+q^{-1}+r^{-1}<1$. For the algebraic surface $V=\{z_1^p+z_2^q+z_3^r+z_1z_2z_3=0\}$, the singularity link $K=S^5_\varepsilon\cap V$ carries the structure of Sol^3, while the intersection $K_s=S^5_s\cap V$ is diffeomorphic to the Brieskorn manifold $M(p,q,r)$ when the radius s is large enough. Explain why the topology of K_s changes drastically.

If we change the radius s continuously from 0 to ∞, the change of topology of K_s happens when s is equal to a value $R>0$. Since the Milnor numbers of a cusp singularity and a Brieskorn singularity is $p+q+r-1$ and $(p-1)(q-1)(r-1)$, respectively, the intersection K_R carries $pqr(1-p^{-1}-q^{-1}-r^{-1})$ Morse singularities. Though we know that these Morse singularities are the cause of the change, we would like to give a more detailed account.

Problem 5.5. Mori connected the rotation around the barycenter of the moment polytope Δ with the non-integrability of the induced contact structure on a submanifold of S^5. This principle led to Furukawa's example (Example 5.1). Find such a principle for the Brieskorn manifold $M(p,q,r)$ with respect to the moment polytope.

Owing to the principle of Mori, Example 5.1 is also useful for the study of the links of mixed polynomial singularities $\overline{z_1}z_1^p+\overline{z_2}z_2^q+\overline{z_3}z_3^r+\overline{z_1z_2z_3}=0$. If one obtains a principle for the Brieskorn manifold $M(p,q,r)$, it might be useful for the research on some real singularities.

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Received June 11, 2014; revised version January 10, 2015.