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Zbigniew Jelonek

POLYNOMIAL MAPPINGS WITH SMALL DEGREE

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Abstract. Let X^n be an affine variety of dimension n and Y^n be a quasi-projective variety of the same dimension. We prove that for a quasi-finite polynomial mapping $f : X^n \to Y^n$, every non-empty component of the set $Y^n \setminus f(X^n)$ is closed and it has dimension greater or equal to $n \quad \mu(f)$, where $\mu(f)$ is a geometric degree of f. Moreover, we prove that generally, if $f : X^n \to Y^n$ is any polynomial mapping, then either every non-empty component of the set $\overline{Y^n \setminus f(X^n)}$ is of dimension $\geq n \quad \mu(f)$ or f contracts a subvariety of dimension $\geq n \quad \mu(f) + 1$.

1. Introduction

Let k be an algebraically closed field. Let X^n be an affine variety and Y^n be a quasi-projective variety (in this paper Z^n denotes an algebraic set of pure dimension n). Let $f : X^n \to Y^n$ be a generically-finite mapping. In general, $f(X)$ is only a constructible subset of Y and it is difficult to describe the subset $Y \setminus f(X)$. The aim of this note is to estimate the dimension of irreducible components of $\overline{Y \setminus f(X)}$ in some special cases. We prove that for a quasi-finite polynomial mapping $f : X^n \to Y^n$ of n-dimensional varieties, every non-empty component of the set $Y^n \setminus f(X^n)$ is closed and it has dimension greater or equal to $n \quad \mu(f)$, where $\mu(f)$ is a geometric degree of f, i.e., the number of points in the sufficiently general fiber of f. Moreover, we prove that if $f : X^n \to Y^n$ is any polynomial mapping, then either every non-empty component of the set $\overline{Y^n \setminus f(X^n)}$ is of dimension $\geq n \quad \mu(f)$ or f contracts a subvariety of dimension $\geq n \quad \mu(f) + 1$, i.e., there exists a subvariety $S \subset X$ of dimension $\geq n \quad \mu(f) + 1$, such that $\dim \overline{f(S)} < \dim S$. Of course, our results are interesting, only when the mapping f has relatively small geometric degree, i.e., when $\mu(f) < n$. Here, by an affine set X we mean any

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closed and reduced sub-scheme of some affine space $\mathbb{A}^m(k)$. An affine variety is an affine set which is irreducible. Similarly, an algebraic set is a closed and reduced sub-scheme of some quasi-projective variety and algebraic variety means a quasi-projective variety.

2. Main result

Let us recall the definition:

Definition 2.1. Let $f : X \to Y$ be a polynomial mapping of affine algebraic varieties. We say that f *is finite at a point* $y \in Y$ if there exists a Zariski open neighborhood U of y such that the mapping $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite.

Remark 2.2. Note that f is finite if and only if it is finite at every $y \in Y$.

Let X, Y be affine varieties. We have the following description of the set of points at which a dominant, generically-finite polynomial map $f : X \to Y$ is not finite (see [4], [6]):

Theorem 2.3. *Let X, Y be an affine varieties. Let $f : X \to Y$ be a dominant generically-finite polynomial mapping. Then the set S_f of points at which f is not finite is either the empty set or a hypersurface in Y.*

In this paper, by a hypersurface in Y we mean an algebraic subset of pure codimension one in Y.

Remark 2.4. Now assume that X^n is an arbitrary affine set of pure dimension n. Let Y^n be an affine variety and $f : X \to Y$ be a polynomial mapping. Assume that f restricted to any irreducible component of X is a generically finite mapping. Then Theorem 2.3 still holds. Indeed, if $X = \bigcup X_i$ is a decomposition of X into irreducible components and $f_i = f|_{X_i}$, then it is easy to see that $S_f = \bigcup S_{f_i}$.

The next result is well known. We give the proof since we were not able to find an appropriate reference.

Proposition 2.5. *Let X^n be an algebraic set of pure dimension n and let Y^n be an algebraic variety of the same dimension. Let $f : X \to Y$ be a polynomial mapping. Then there exists a non-empty open subset $U \subset Y^n$ such that for every $y \in U$, the number $\#f^{-1}(y)$ is constant (we denote this number by $\mu(f)$). Moreover, if the variety Y is normal then for every point $y \in Y$ with a finite fiber, $\#f^{-1}(y) \leq \mu(f)$.*

Proof. We can assume that X, Y are affine varieties. Since the set of normal points is non-empty and open in Y^n, we can assume that Y^n is normal. First assume that X is irreducible. We can assume that f is a generically-finite mapping. Consider the graph $\Gamma_f \subset \overline{X} \times Y$, where $\overline{X}$ is a projective closure of X. Let $Z = \overline{\Gamma_f}$. Hence $X \subset Z$ and we have a natural projection $\pi : Z \to Y$, such that $\pi|_X = f$. By the Stein Factorization Theorem (see e.g., [2, III, 11, Corollary 11.5]) one can factor π into $h \circ g$, where $g : Z \to Z'$ is a projective morphism with connected fibers, and $h : Z' \to Y$ is a finite morphism. This shows that our problem reduces to the case where f is a finite mapping. Now we follow [7, II.6, Theorem 3 and 4] with some necessary modification. Let $k[X] = A, k[Y] = B, k(X) = K, k(Y) = L$ and let W be the maximal separable extension of the field L that is contained in the field K.

First we show that for every $y \in Y$ we have $\#f^{-1}(y) \leq [W : L] = [K : L]_s$. Indeed, let $f^{-1}(y) = \{x_1, ..., x_n\}$. Since the field k is algebraically closed, hence infinite, there exists a function $\alpha \in A$ such that α separates all x_i. Let $p = char\ k$. There exists a number $w = p^r$ such that $\alpha^w \in W$ (if $char\ k = 0$ we put $w = 1$). Let $F \in B[T]$ be a minimal polynomial of α^w. The polynomial F has degree $\deg F \leq [K : L]_s$ which shows that $\#f^{-1}(y) \leq [K : L]_s$.

Now we show that there exists an open subset $U \subset Y^n$ such that for every $y \in U$, the number $\#f^{-1}(y)$ is constant and equal to $[K : L]_s$. Let $W = k(\alpha)$ for $\alpha \in K$. We have $\alpha = f/g$, where $f, g \in A$. Take $H = V(g)$ and consider varieties $Y' = Y \backslash f(H)$ and $X' = X \backslash f^{-1}(f(H))$. Without losing generality we can assume that $X' = X$ and $Y' = Y$. Now $\alpha \in A$. The mapping $\Phi : X \ni x \to (f(x), \alpha(x)) \in Y \times k$ is finite. Hence $Im\Phi$ is a hypersurface in $Y \times k$. Let $F \in B[T] \cong k[Y \times \mathbb{A}^1]$ be the minimal equation of α over L. We can treat F as an equation of the hypersurface $Im\Phi$. Since the discriminant of F is non zero in B, we have that for a generic $y \in Y$ the polynomial $F(y, t)$ has all distinct zeroes with respect to variable t. Note that all these zeroes are of the form $\alpha(x)$, $x \in f^{-1}(y)$ because on the hypersurface $Im\Phi$ over y lies only values of this type. Hence $\#f^{-1}(y) \geq [K : L]_s$.

The case when X is not irreducible we leave to the reader. ∎

Definition 2.6. Let X^n be an algebraic set of pure dimension n and let Y^n be an algebraic variety of the same dimension. Let $f : X \to Y$ be a polynomial mapping. Let $\mu(f)$ be a number as in Proposition 2.5. Then we call $\mu(f)$ a geometric degree of the mapping f. If Y^n is an arbitrary algebraic set of pure dimension with irreducible components $Y_1, ..., Y_m$, then $\mu(f) = \max_{i=1}^m \{\mu(f_i)\}$, where $f_i = f|_{f^{-1}(Y_i)} : f^{-1}(Y_i) \to Y_i$.

Remark 2.7. If X, Y are algebraic varieties and f is a generically-finite mapping, then the number $\mu(f)$ is equal to $[k(X) : k(Y)]_s$. In particular, in characteristic zero it coincides with usual degree $\deg f$ as defined e.g. in [7]. In a positive characteristic we have $\mu(f) \leq \deg f$.

Corollary 2.8. *Let X^n be an affine set of pure dimension n and let Y^n be an affine normal variety of the same dimension. Let $f : X \to Y$ be a polynomial mapping, such that the restriction of f to every component of*

X is generically finite. Let S_f be the set of non-properness of the mapping f. Let S be the union of all $n-1$ dimensional components of the set $f^{-1}(S_f)$ and put $g = f|_S : S \to S_f$. Then $\mu(g) < \mu(f)$.

Proof. First assume that X is irreducible. Consider the graph $\Gamma_f \subset \overline{X} \times Y$, where $\overline{X}$ is a projective closure of X. Let $Z = \overline{\Gamma_f}$. Hence $X \subset Z$ and we have a natural projection $\pi : Z \to Y$, such that $\pi|_X = f$. By the Stein Factorization Theorem (see e.g., [2, III, 11, Corollary 11.5]) one can factor π into $h \circ g$, where $g : Z \to Z'$ is a projective morphism with connected fibers, and $h : Z' \to Y$ is a finite morphism. We have $S_f = \pi(Z \backslash X)$.

Note that every fiber of π has not more than $\mu(f)$ connected components. Assume that $y \in S_f$ and the fiber $f^{-1}(y)$ is finite. We have $\pi^{-1}(y) = f^{-1}(y) \cup (\pi^{-1}(y) \cap (Z \backslash X))$. By the assumption, the set $\pi^{-1}(y) \cap (Z \backslash X)$ is non-empty, which implies that $\# f^{-1}(y) < \mu(f)$. Since $g^{-1}(y) \subset f^{-1}(y)$ this finishes the proof.

The case when X is not irreducible we leave to the reader. ∎

In the sequel, the following result will be also useful:

Lemma 2.9. *Let X, Y be affine normal varieties of dimension n. Let $f : X \to Y$ be a quasi-finite dominant mapping. Let $S \subset Y$ be a hypersurface in Y. Then the set $f^{-1}(S)$ if non-empty is also a hypersurface.*

Proof. By the Zariski Main Theorem (version of Grothendieck-see [1]) there is a normal variety Z, such that $X \subset Z$ is a dense open subveriety, and there is a finite mapping $g : Z \to Y$ such that $g|_X = f$. Now, since the set S is a hypersurface and the mapping g is finite, we have that the set $S' := g^{-1}(S)$ is also a hypersurface (the going down theorem). But $f^{-1}(S) = S' \cap X$. Since X is open dense subset in Z, we conclude that the set $f^{-1}(S)$ is a hypersurface. ∎

It is worth to note that for not quasi-finite mappings this theorem is no longer true:

Example 2.10. Let k be an algebraically closed field. We show that there exist affine normal varieties X, Y and a dominant generically-finite polynomial mapping $G : X \to Y$, such that the set $G^{-1}(S_G)$ is not a hypersurface.

Indeed, let us take a line $l \subset k^n$, where $n > 2$. By Theorem 5.4 from [4], there is a generically finite polynomial mapping $F : k^n \to k^n$, such that $F^{-1}(0) = \{l\}$ and all other fibers of F are finite. By the Stein factorization theorem (see [3, Thm. 2.26, p.141]), there is a normal variety W, which is affine such that the mapping F factors through a mapping $G : k^n \to W$ and for $w \in W$ the fiber $G^{-1}(w)$ is either empty, or one point or the line l. In particular, by the Zariski Main Theorem, the mapping G restricted to $k^n \backslash l$ is an open embedding. Consequently, G is proper over every point from the set

$G(k^n \setminus l)$. It means that $S_G = (W \setminus G(k^n)) \cup G(l)$ and consequently we have that the set $G^{-1}(S_G) = l$ is not a hypersurface.

Now we are in a position to prove our first result:

Theorem 2.11. *Let X^n be an affine set of pure dimension n and let Y^n be an algebraic variety of the same dimension. Let $f : X \to Y$ be a quasi-finite polynomial mapping. Then the set $Y^n \setminus f(X^n)$ is closed and every non-empty component of this set has dimension greater or equal to $n - \mu(f)$.*

Proof. Note that we can assume that Y is also affine. Indeed, since Y is a quasi-projective variety, for a point $a \in Y$ there is an ample effective Cartier divisor D such that $a \in Y \setminus Supp(D)$. Note that the divisor $f^*(D)$ is also effective Cartier divisor. Take $Y_0 = Y \setminus Supp(D)$ and $X_0 := X \setminus Supp(f^*(D))$. We have that X_0 is affine (see e.g. [5]), Y_0 is quasi-affine and $f^{-1}(Y_0) = X_0$. Let $Y_0 = A \setminus E$, where A is an affine variety and E is a closed subset of A. For a given point $a \in Y_0$, there is a regular function $h \in k[A]$ such that $h_{|E} = 0$ and $h(a) = 1$. The set A_h is an open affine subvariety of Y_0 which contains the point a. Moreover, $f^{-1}(A_h) = (X_0)_{f^*(h)}$ is also an open affine subvariety of X_0. Hence we can take $X = (X_0)_{f^*(h)}$ and $Y = A_h$.

Now we proceed by induction with respect to $\mu(f)$. Assume first that $\mu(f) = 1$. Assume that $Y \neq f(X)$ and let S be a component of the set $\overline{Y^n \setminus f(X^n)}$. Since $\mu(f) = 1$, we have that X is irreducible.

Let $(\tilde{X}, v_X), (\tilde{Y}, v_Y)$ be normalizations of X and Y. Let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be the normalization of f. Since the mappings v_X, v_Y are finite (hence closed) and the following diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \downarrow v_X & & \downarrow v_Y \\ X & \xrightarrow{f} & Y \end{array}$$

we have $(v_Y)^{-1}(S) := \tilde{S} \subset \overline{\tilde{Y} \setminus \tilde{f}(\tilde{X})}$. Hence we can assume that X and Y are normal. Of course $\mu(\tilde{f}) = 1$. Let S_f be the set of points over which the mapping f is not finite. By the Zariski Main Theorem in version Grothendieck, there exists an affine variety Z with an open embedding $\iota : X \to Z$ and a finite mapping $\overline{f} : Z \to Y$, such that $f = \overline{f}|_X$. Moreover, $S_f = \overline{f}(Z \setminus X)$. Since $\mu(\overline{f}) = 1$ this implies that for every $y \in S_f$ we have $f^{-1}(y) = \emptyset$. Hence $\overline{Y^n \setminus f(X^n)} = Y^n \setminus f(X^n) = S_f$ and $\dim \overline{Y^n \setminus f(X^n)} = n - 1$ by Theorem 2.3.

Now assume that $\mu(f) > 1$. Let $(\tilde{X}, v_X), (\tilde{Y}, v_Y)$ be the normalization of X and Y (if $X = \bigcup X_i$ then $\tilde{X} = \bigsqcup \tilde{X}_i$ and $v_X = \bigsqcup v_{X_i}$). Let $\tilde{f} = \bigsqcup f_i : \tilde{X} \to \tilde{Y}$ be a normalization of f. As before we can assume that $X = \tilde{X}, Y = \tilde{Y}$ (note that the normalization is a finite mapping). Let $R \subset Y \setminus f(X)$ be an irreducible component of the (potentially constructible) set $Y \setminus f(X)$. Then of course $R \subset S_f$. Since the set S_f is a hypersurface, there exists an irreducible hypersurface S of Y contained in S_f such that $R \subset S$. Let $X' = f^{-1}(S)$. If $X' = \emptyset$, then $R = S$ and the conclusion follows. In the other case X' is an affine set of pure dimension $n - 1$ (see Lemma 2.9) and we have a mapping $f' = f|_{X'} : X' \to S$. Moreover, by Corollary 2.8 we have $\mu(f') \le \mu(f) - 1$. Now by the induction principle, the set R is closed and $\dim R \ge n - 1 - (\mu(f) - 1) = n - \mu(f)$. ∎

To prove our second result we need:

Lemma 2.12. *Let X^n, Y^n be affine normal varieties of dimension n. Let $f : X \to Y$ be a dominant generically-finite mapping. Let $S \subset Y$ be a hypersurface in Y. Assume that a variety R is a non-empty irreducible component of the set $f^{-1}(S)$. If R is not a hypersurface, then f contracts R.*

Proof. Let Q be the union of positive dimensional components of all fibers of the mapping f. The set Q is closed. Indeed, every component P of the set $\overline{Q}$ is contracted by f, because the union of positive dimensional fibers of the mapping $f|_P$ is dense in P and consequently every fiber of the mapping $f|_P$ is infinite. This implies that $P \subset Q$.

Let $f^{-1}(S) = R \cup \bigcup_{i=1}^s R_i$. Take $R' = R \setminus \bigcup_{i=1}^s R_i$. Now assume the mapping $f|_R$ has a fiber with an isolated point $a \in R'$. This implies that also the mapping f has a fiber with an isolated point a. In particular $a \in X \setminus Q$. We can find an affine neighborhood $U \subset X \setminus Q$ of a such that the mapping $f|_U$ is quasi finite. Then by Lemma 2.9 the set $R \cap U$ is a hypersurface. Consequently R itself is a hypersurface. If R is not a hypersurface, we have that all fibers of the mapping $f|_R$ are infinite, i.e., R is contracted by f. ∎

Our second result is:

Theorem 2.13. *Let X^n be an affine set and let Y^n be an algebraic variety. If $f : X^n \to Y^n$ is a polynomial mapping, then either every non-empty component of the set $\overline{Y^n \setminus f(X^n)}$ is of dimension $\ge n - \mu(f)$ or f contracts a subvariety of dimension $\ge n - \mu(f) + 1$.*

Proof. As before, we can assume that X and Y are affine and normal. If $\mu(f) = 0$, then the result is obvious: we have $\overline{Y^n \backslash f(X^n)} = Y$. Now we apply induction.

Assume that $\mu(f) > 0$ and f does not contract any subvariety of dimension $\geq n - \mu(f)+1$. Let $X' \subset X$ be the union of all components of X, on which the mapping f is generically-finite. We can assume that $X = X'$ since otherwise f contract a subvariety of dimension n.

Let X'' be the union of all $n - 1$ dimensional components of the set $f^{-1}(S_f)$. If X'' is the empty set then the conclusion holds. Indeed, in this case we have $\overline{Y^n \backslash f(X^n)} = S_f$.

Hence we can assume that $X'' \neq \emptyset$. The mapping $f'' : X'' \to S_f$ induced by f satisfies $\mu(f'') \leq \mu(f) - 1$ (see Corollary 2.8). Let $a \in \overline{Y^n \backslash f(X^n)}$ and S be a fixed component of the set $\overline{S_f \backslash f(X'')}$, which contains the point a.

Since f'' does not contract any subvariety of dimension $\geq (n - 1) - \mu(f'')+1 \geq n - \mu(f)+1$, we have by induction that dim $S \geq n - 1 - \mu(f'') \geq n - 1 - (\mu(f) - 1) = n - \mu(f)$. Moreover, S is contained in $\overline{Y^n \backslash f(X^n)}$. Indeed, otherwise the set $f^{-1}(S)$ has to dominate S. Note that all fibers of f over S_f which are not included in X'' are positive dimensional - see Lemma 2.12. Consequently, we find a subvariety $R \subset f^{-1}(S)$, which is contracted by f and which dominates S. The subvariety R would have a dimension $\geq n - \mu(f)+1$.

Finally, let S' be any component of the set $\overline{Y^n \backslash f(X^n)}$. Choose a point $a \in S' \cap (Y^n \backslash f(X^n))$ which does not lie on any other component of this set. Of course $a \in S_f \backslash f(X'')$. By our previous considerations, we can find a component S of the set $\overline{S_f \backslash f(X'')}$ of dimension $\geq n - \mu(f)$, which contains the point a and which is contained in $\overline{Y^n \backslash f(X^n)}$. Of course $S \subset S'$. ∎

The proof of the latter result suggests the following :

Corollary 2.14. *Let X^n be an affine variety and let Y^n be an algebraic variety. Let $f : X^n \to Y^n$ be a generically finite dominant mapping. If all positive dimensional fibers of f have dimension $\geq \mu(f)$, then every non-empty component of the set $\overline{Y^n \backslash f(X^n)}$ is of dimension $\geq n - \mu(f)$.*

Proof. We can assume that X, Y are affine and normal. Let S be the set as in the end of the proof of Theorem 2.13. If the conclusion of the Corollary does not hold then there is a component R of the set $f^{-1}(S)$ which dominates S and which is contracted by f. Since $\dim S \geq n - \mu(f)$, we have dim $R \geq n - \mu(f) + \mu(f) = n$ - it is a contradiction. ∎

Corollary 2.15. *Let X, Y be affine varieties and $f : X \to Y$ be a dominant mapping with $\mu(f) = 1$. If f has a positive dimensional fiber, then the set $\overline{Y \backslash f(X)}$ is a hypersurface in Y.*

Proof. Indeed, by the assumption we have $S_f \neq \emptyset$. Moreover, we can assume that X, Y are normal. By Corollary 2.8, every component of the set $f^{-1}(S_f)$ has to be contracted by f. This means that $\overline{Y \setminus f(X)} = S_f$. ∎

The last result can be applied e.g., for purely inseparable mapping of affine varieties.

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Z. Jelonek
INSTYTUT MATEMATYCZNY
POLSKA AKADEMIA NAUK
ŚNIADECKICH 8
00-656 WARSZAWA, POLAND
E-mail: najelone@cyf-kr.edu.pl

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