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FLAT APPROXIMATIONS OF SURFACES ALONG CURVES

Dedicated to Professor Stanisław Janeczko for his 60th birthday and to Professor Peter Giblin for his 70th birthday.

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Abstract. We consider a developable surface tangent to a surface along a curve on the surface. We call it an *osculating developable surface* along the curve on the surface. We investigate the uniqueness and the singularities of such developable surfaces. We discover two new invariants of curves on a surface which characterize these singularities. As a by-product, we show that a curve is a contour generator with respect to an orthogonal projection or a central projection if and only if one of these invariants constantly equal to zero.

1. Introduction

In this paper, we consider a curve on a surface in Euclidean 3-space and a developable surface tangent to the surface along the curve. Such a developable surface, if it exists, is called an *osculating developable surface along* the curve. If the curve is a boundary of a surface with boundaries, it is the flat extension of the surface with boundaries [4]. We consider the existence and the uniqueness of osculating developable surfaces along curves. The notion of Darboux frames along curves on surfaces has been known for some time. We have a special direction in the Darboux frame at each point of the curve which is directed by a vector in the tangent plane of the surface. We can show that this vector field has a constant direction if and only if the osculating developable surface is a generalized cylinder. We call this vector field an *osculating Darboux vector* field along the curve. On the other hand, there are three invariants associated with the Darboux frame of a curve on a surface. Under a certain condition of those invariants, we

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can show that there exists an osculating developable surface along the curve which is given as the envelope of tangent spaces of the surface along the curve. It follows that an osculating developable surface is a ruled surface whose rulings are directed by the osculating Darboux vector field along the curve. By using such invariants, we introduce two new invariants which related to the singularities of osculating developable surfaces. Actually, one of these invariants is constantly equal to zero if and only if the osculating Darboux vector field has a constant direction which means that the osculating developable surface is a cylindrical surface. In this case, the curve is a contour generator with respect to an orthogonal projection. Therefore, this invariant characterize a curve as the contour generator with respect to an orthogonal projection (cf., Theorem 3.1, (A)). Moreover, under the condition that this first invariant never vanished, another invariant is constantly equal to zero if and only if the osculating developable surface is a conical surface. In this case, the curve is a contour generator with respect to a central projection (cf., Theorem 3.1, (B)). The notion of contour generators plays an important role in the computer vision theory [2]. There have been no differential geometric characterization of contour generators so far as we know. We give a classification of the singularities of the osculating developable surface along a curve on a surface by using those two invariants (Theorem 3.3). In §6, we consider curves on special surfaces. Firstly, we consider the case when the surface itself a developable surface. We show that the osculating developable surface of a curve on a developable surface is equal to the original developable surface where the curve is located (Theorem 6.1). Therefore, the uniqueness of the osculating developable surface holds for the same conditions as the above (Corollary 6.2). Moreover, if the uniqueness does not hold, then the curve is a straight line (Corollary 6.4). We give some examples of curves on the unit sphere and the graph of a function in §6.2 and 6.3.

2. Basic concepts

We consider a surface $M = \boldsymbol{X}(U)$ given locally by an embedding $\boldsymbol{X} : U \to \mathbb{R}^3$, where $\mathbb{R}^3$ is Euclidean space and $U \subset \mathbb{R}^2$ is an open set. Let $\overline{\boldsymbol{\gamma}} : I \to U$ be an embedding, where $\overline{\boldsymbol{\gamma}}(t) = (u(t), v(t))$ and I is an open interval. Then we have a regular curve $\boldsymbol{\gamma} = \boldsymbol{X} \circ \overline{\boldsymbol{\gamma}} : I \to M \subset \mathbb{R}^3$ on the surface M. On the surface, we have the unit normal vector field $\boldsymbol{n}$ defined by

$$\boldsymbol{n}(p) = \frac{\boldsymbol{X}_u \times \boldsymbol{X}_v}{\|\boldsymbol{X}_u \times \boldsymbol{X}_v\|}(u, v),$$

where $p = \boldsymbol{X}(u, v)$. Here, $\boldsymbol{a} \times \boldsymbol{b}$ is the exterior product of $\boldsymbol{a}, \boldsymbol{b}$ in $\mathbb{R}^3$. Since $\boldsymbol{\gamma}$ is a space curve in $\mathbb{R}^3$, we adopt the arc-length parameter as usual and denote $\boldsymbol{\gamma}(s) = \boldsymbol{X}(u(s), v(s))$. Then we have the unit tangent vector field $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$

of $\boldsymbol{\gamma}(s)$, where $\boldsymbol{\gamma}'(s) = d\boldsymbol{\gamma}/ds(s)$. We have $\boldsymbol{n}_{\boldsymbol{\gamma}}(s) = \boldsymbol{n} \circ \boldsymbol{\gamma}(s)$, which is the unit normal vector field of M along $\boldsymbol{\gamma}$. Moreover, we define $\boldsymbol{b}(s) = \boldsymbol{n}_{\boldsymbol{\gamma}}(s) \times \boldsymbol{t}(s)$. Then we have a orthonormal frame $\{\boldsymbol{t}(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s), \boldsymbol{b}(s)\}$ along $\boldsymbol{\gamma}$, which is called the *Darboux frame* along $\boldsymbol{\gamma}$. Then we have the following Frenet–Serret type formulae:

$$\begin{cases} \boldsymbol{t}'(s) &= \kappa_g(s)\boldsymbol{b}(s) + \kappa_n(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s), \\ \boldsymbol{b}'(s) &= \quad \kappa_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s), \\ \boldsymbol{n}_{\boldsymbol{\gamma}}'(s) &= \quad \kappa_n(s)\boldsymbol{t}(s) \quad \tau_g(s)\boldsymbol{b}(s). \end{cases}$$

By using the matrix representation, we have

$$\begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{b}' \\ \boldsymbol{n}_{\boldsymbol{\gamma}}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ \kappa_g & 0 & \tau_g \\ \kappa_n & \tau_g & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{b} \\ \boldsymbol{n}_{\boldsymbol{\gamma}} \end{pmatrix}.$$

Here,

$$\begin{aligned} \kappa_g(s) &= \langle \boldsymbol{t}'(s), \boldsymbol{b}(s) \rangle = \det\left(\boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s)\right), \\ \kappa_n(s) &= \langle \boldsymbol{t}'(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s) \rangle = \langle \boldsymbol{\gamma}''(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s) \rangle, \\ \tau_g(s) &= \langle \boldsymbol{b}'(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s) \rangle = \det(\boldsymbol{\gamma}'(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s), \boldsymbol{n}_{\boldsymbol{\gamma}}'(s)) \end{aligned}$$

and $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ is the canonical inner product of $\mathbb{R}^3$. We call $\kappa_g(s)$ a *geodesic curvature*, $\kappa_n(s)$ a *normal curvature* and $\tau_g(s)$ a *geodesic torsion* of $\boldsymbol{\gamma}$, respectively. It is known that

1) $\boldsymbol{\gamma}$ is an asymptotic curve of M if and only if $\kappa_n = 0$,
2) $\boldsymbol{\gamma}$ is a geodesic of M if and only if $\kappa_g = 0$,
3) $\boldsymbol{\gamma}$ is a principal curve of M if and only if $\tau_g = 0$.

We define a vector field $D_o(s)$ along $\boldsymbol{\gamma}$ by

$$D_o(s) = \tau_g(s)\boldsymbol{t}(s) \quad \kappa_n(s)\boldsymbol{b}(s),$$

which is called an *osculating Darboux vector* along $\boldsymbol{\gamma}$. If $\kappa_n^2 + \tau_g^2 \neq 0$, we can define the *normalized osculating Darboux vector field* as

$$\overline{D_o}(s) = \frac{\tau_g(s)\boldsymbol{t}(s) \quad \kappa_n(s)\boldsymbol{b}(s)}{\sqrt{\kappa_n(s)^2 + \tau_g(s)^2}}.$$

On the other hand, we briefly review the notions and basic properties of ruled surfaces and developable surfaces. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ and $\boldsymbol{\xi} : I \to \mathbb{R}^3 \backslash \{\boldsymbol{0}\}$ be C^∞-mappings. Then we define a mapping $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})} : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}(u, v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\xi}(u).$$

We call the image of $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ a *ruled surface*, the mapping $\boldsymbol{\gamma}$ a *base curve* and the mapping $\boldsymbol{\xi}$ a *director curve*. The line defined by $\boldsymbol{\gamma}(u) + v\boldsymbol{\xi}(u)$ for a fixed $u \in I$ is called a *ruling*. We call the ruled surface with vanishing Gaussian curvature on the regular part a *developable surface*. It is known that a ruled surface $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a developable surface if and only if

$$\det\big(\dot{\boldsymbol{\gamma}}(u), \boldsymbol{\xi}(u), \dot{\boldsymbol{\xi}}(u)\big) = 0,$$

where $\dot{\boldsymbol{\gamma}}(u) = (d\boldsymbol{\gamma}/du)(u)$(cf., [5]). If the direction of the director curve $\boldsymbol{\xi}$ is constant, we call $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ a (*generalized*) *cylinder*. If we denote that $\widetilde{\boldsymbol{\xi}}(u) = \boldsymbol{\xi}(u)/\|\boldsymbol{\xi}(u)\|$, then we have $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}(I \times \mathbb{R}) = F_{(\boldsymbol{\gamma},\widetilde{\boldsymbol{\xi}})}(I \times \mathbb{R})$. In this case $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a cylinder if and only if $\dot{\widetilde{\boldsymbol{\xi}}}(u) \equiv 0$. We say that $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is *non-cylindrical* if $\dot{\widetilde{\boldsymbol{\xi}}}(u) \neq 0$. Suppose that $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is non-cylindrical. Then a *striction curve* is defined to be

$$\boldsymbol{\sigma}(u) = \boldsymbol{\gamma}(u) \quad \frac{\langle \dot{\boldsymbol{\gamma}}(u), \dot{\widetilde{\boldsymbol{\xi}}}(u)\rangle}{\langle \dot{\widetilde{\boldsymbol{\xi}}}(u), \dot{\widetilde{\boldsymbol{\xi}}}(u)\rangle}\widetilde{\boldsymbol{\xi}}(u).$$

It is known that a singular point of the non-cylindrical ruled surface is located on the striction curve [5]. A non-cylindrical ruled surface $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a *cone* if the striction curve $\boldsymbol{\sigma}$ is constant. In general, a *wave front* in $\mathbb{R}^3$ is a (singular) surface which is a projection image of a Legendrian submanifold in the projective cotangent bundle $\pi : PT^*(\mathbb{R}^3) \quad \to \mathbb{R}^3$. It is known (cf., [5]) that a non-cylindrical developable surface $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a wave front if and only if

$$\det\Big(\boldsymbol{\xi}(u), \dot{\boldsymbol{\xi}}(u), \ddot{\boldsymbol{\xi}}(u)\Big) \neq 0.$$

In this case we call $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ a (*non-cylindrical*) *developable front*.

We now briefly review the notion of contour generators. Let $M \subset \mathbb{R}^3$ be a surface and $\boldsymbol{n}$ be a unit normal vector field on M. For a unit vector $\boldsymbol{k} \in S^2$, the *contour generator* of the orthogonal projection with the direction $\boldsymbol{k}$ is defined to be

$$\{p \in M \mid \langle \boldsymbol{n}(p), \boldsymbol{k}\rangle = 0\}.$$

It is actually the singular set of the orthogonal projection with the direction $\boldsymbol{k}$. Moreover, for a point $\boldsymbol{c} \in \mathbb{R}^3$, the *contour generator of the central projection* with the center $\boldsymbol{c}$ is defined to be

$$\{p \in M \mid \langle p \quad \boldsymbol{c}, \boldsymbol{n}(p)\rangle = 0 \}.$$

It is also the singular set of the central projection with the center $\boldsymbol{c}$. The notion of contour generators play an important role in the vision theory [2].

3. Osculating developable surfaces

In this section, we introduce a flat approximation surface of a given surface along a curve. For a regular curve $\boldsymbol{\gamma} = \boldsymbol{X} \circ \overline{\gamma} : I \quad \to M \subset \mathbb{R}^3$ on a surface

M with $\kappa_n^2(s) + \tau_g^2(s) \neq 0$, we define a map $OD_{\boldsymbol{\gamma}} : I \times \mathbb{R} \quad \to \mathbb{R}^3$ by

$$OD_{\boldsymbol{\gamma}}(s,u) = \boldsymbol{\gamma}(s) + u\overline{D_o}(s) = \boldsymbol{\gamma}(s) + u\frac{\tau_g(s)\boldsymbol{t}(s) \quad \kappa_n(s)\boldsymbol{b}(s)}{\sqrt{\kappa_n(s)^2 + \tau_g(s)^2}}.$$

This is a ruled surface and we have

$$\overline{D_o}' = \left(\kappa_g + \frac{\kappa_n\tau_g' \quad \kappa_n'\tau_g}{\kappa_n^2 + \tau_g^2}\right)\frac{\kappa_n\boldsymbol{t} + \tau_g\boldsymbol{b}}{\sqrt{\kappa_n^2 + \tau_g^2}}$$

so that we have

$$\begin{aligned}\det(\boldsymbol{\gamma}', \overline{D_o}, \overline{D_o}') &= \det\left(\boldsymbol{t}, \frac{\tau_g\boldsymbol{t} \quad \kappa_n\boldsymbol{b}}{\sqrt{\kappa_n^2 + \tau_g^2}}, \left(\kappa_g + \frac{\kappa_n\tau_g' \quad \kappa_n'\tau_g}{\kappa_n^2 + \tau_g^2}\right)\frac{\kappa_n\boldsymbol{t} + \tau_g\boldsymbol{b}}{\sqrt{\kappa_n^2 + \tau_g^2}}\right) \\ &= 0.\end{aligned}$$

This means that $OD_{\boldsymbol{\gamma}}(I \times \mathbb{R})$ is a developable surface. We call $OD_{\boldsymbol{\gamma}}$ an *osculating developable surface* of M along $\boldsymbol{\gamma}$. Moreover, we introduce two invariants $\delta(s), \sigma(s)$ of $(M, \boldsymbol{\gamma})$ as follows:

$$\begin{aligned}\delta(s) &= \kappa_g(s) + \frac{\kappa_n(s)\tau_g'(s) \quad \kappa_n'(s)\tau_g(s)}{\kappa_n^2(s) + \tau_g^2(s)}, \\ \sigma(s) &= \frac{\tau_g(s)}{\sqrt{\kappa_n^2(s) + \tau_g^2(s)}} \quad \left(\frac{\kappa_n(s)}{\delta(s)\sqrt{\kappa_n^2(s) + \tau_g^2(s)}}\right)', \text{ (when } \delta(s) \neq 0).\end{aligned}$$

By the above calculation, $\delta(s) = 0$ if and only if $\overline{D_o}'(s) = \mathbf{0}$. We can also calculate that

$$\frac{\partial OD_{\boldsymbol{\gamma}}}{\partial s} \times \frac{\partial OD_{\boldsymbol{\gamma}}}{\partial u} = \quad \left(\frac{\kappa_n}{\sqrt{\kappa_n^2 + \tau_g^2}} + u\delta\right)\boldsymbol{n}_{\boldsymbol{\gamma}}.$$

Therefore, $(s_0, u_0) \in I \times \mathbb{R}$ is a singular point of $OD_{\boldsymbol{\gamma}}$ if and only if $\delta(s_0) \neq 0$ and

$$u_0 = \quad \frac{\kappa_n(s_0)}{\delta(s_0)\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)}}.$$

If $(s_0, 0)$ is a regular point (i.e., $\kappa_n(s_0) \neq 0$), the normal vector of $OD_{\boldsymbol{\gamma}}$ at $OD_{\boldsymbol{\gamma}}(s_0, 0) = \boldsymbol{\gamma}(s_0)$ has the same direction of the normal vector of M at $\boldsymbol{\gamma}(s_0)$. This is the reason why we call $OD_{\boldsymbol{\gamma}}$ the osculating developable surface of M along $\boldsymbol{\gamma}$. On the other hand, these two invariants characterize contour generators of M as follows:

Theorem 3.1. *Let $\gamma : I \quad \to M \subset \mathbb{R}^3$ be a unit speed curves on M with $\kappa_n^2(s) + \tau_g^2(s) \neq 0$. Then we have the following*:

(A) *The following are equivalent*:

(1) OD_γ *is a cylinder,*

(2) $\delta(s) \equiv 0$,

(3) $\boldsymbol{\gamma}$ *is a contour generator with respect to an orthogonal projection.*

(B) *If* $\delta(s) \not\equiv 0$, *then the following are equivalent*:

(1) OD_γ *a conical surface,*

(2) $\sigma(s) \equiv 0$,

(3) $\boldsymbol{\gamma}$ *is a contour generator with respect to a central projection.*

Proof. (A) By definition, $OD_{\boldsymbol{\gamma}}$ is a cylinder if and only if $\overline{D_o}(s)$ is constant. Since

$$\overline{D_o}'(s) = \delta(s)\frac{\kappa_n(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{b}(s)}{\sqrt{\kappa_n^2(s) + \tau_g^2(s)}},$$

$\overline{D_o}(s)$ is constant if and only if $\delta(s) \equiv 0$. Therefore, the condition (1) is equivalent to the condition (2). Suppose that the condition (3) holds. Then there exists a vector $\boldsymbol{k} \in S^2$ such that $\langle \boldsymbol{n}_{\boldsymbol{\gamma}}(s), \boldsymbol{k}\rangle \equiv 0$. Then there exist $\lambda, \mu \in \mathbb{R}$ such that $\boldsymbol{k} = \lambda \boldsymbol{t}(s) + \mu \boldsymbol{b}(s)$. Since $\langle \boldsymbol{n}'_{\boldsymbol{\gamma}}(s), \boldsymbol{k}\rangle \equiv 0$, we have $\quad \kappa_n(s)\lambda \quad \tau_g(s)\mu = 0$, so that we have $\boldsymbol{k} = \pm\overline{D_o}(s)$. The condition (1) holds. Suppose that $\overline{D_o}(s)$ is constant. Then we choose $\boldsymbol{k} = \overline{D_o}(s) \in S^2$. By the definition of $\overline{D_o}(s)$, we have $\langle \boldsymbol{n}_{\boldsymbol{\gamma}}(s), \boldsymbol{k}\rangle = \langle \boldsymbol{n}_{\boldsymbol{\gamma}}(s), \overline{D_o}(s)\rangle \equiv 0$. Thus, the condition (1) implies the condition (3).

(B) The condition (1) means that the singular value set of $OD_{\boldsymbol{\gamma}}$ is a constant vector. We consider a vector valued function $\boldsymbol{f}(s)$ defined by

$$\boldsymbol{f}(s) = \boldsymbol{\gamma}(s) \quad \frac{\kappa_n(s)}{\delta(s)\sqrt{\kappa_n^2(s) + \tau_g^2(s)}}\overline{D_o}(s).$$

Then the condition (1) is equivalent to the condition that $\boldsymbol{f}'(s) \equiv 0$. We can calculate that

$$\begin{aligned}
\boldsymbol{f}' &= \boldsymbol{t} \quad \left(\frac{\kappa_n}{\delta\sqrt{\kappa_n^2 + \tau_g^2}}\right)'\overline{D_o} \quad \frac{\kappa_n}{\delta\sqrt{\kappa_n^2 + \tau_g^2}}\overline{D_o}' \\
&= \boldsymbol{t} \quad \left(\frac{\kappa_n}{\delta\sqrt{\kappa_n^2 + \tau_g^2}}\right)'\overline{D_o} \quad \frac{\kappa_n}{\sqrt{\kappa_n^2 + \tau_g^2}}\frac{\kappa_n\boldsymbol{t} + \tau_g\boldsymbol{b}}{\sqrt{\kappa_n^2 + \tau_g^2}} \\
&= \left(\frac{\tau_g}{\sqrt{\kappa_n^2 + \tau_g^2}} \quad \left(\frac{\kappa_n}{\delta\sqrt{\kappa_n^2 + \tau_g^2}}\right)'\right)\overline{D_o} \\
&= \sigma\overline{D_o}.
\end{aligned}$$

It follows that the conditions (1) and (2) are equivalent. By the definition of the contour generator with respect to a central projection, the condition (3) means that there exists $\boldsymbol{c} \in \mathbb{R}^3$ such that $\langle \boldsymbol{\gamma}(s) - \boldsymbol{c}, \boldsymbol{n}_{\boldsymbol{\gamma}}(s)\rangle \equiv 0$. If the condition (1) holds, then $\boldsymbol{f}(s)$ is constant. For the constant point $\boldsymbol{c} = \boldsymbol{f}(s) \in \mathbb{R}^3$, we have

$$\begin{aligned}\langle \boldsymbol{\gamma}(s) - \boldsymbol{c}, \boldsymbol{n}_{\boldsymbol{\gamma}}(s)\rangle &= \langle \boldsymbol{\gamma}(s) - \boldsymbol{f}(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s)\rangle \\ &= \left\langle \frac{\kappa_n(s)}{\delta(s)\sqrt{\kappa_n^2(s) + \tau_g^2(s)}}\overline{D_o}(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s)\right\rangle = 0.\end{aligned}$$

This means that the condition (3) holds. For the converse, by the condition (3), there exists a point $\boldsymbol{c} \in \mathbb{R}^3$ such that $\langle \boldsymbol{\gamma}(s) - \boldsymbol{c}, \boldsymbol{n}_{\boldsymbol{\gamma}}(s)\rangle = 0$. Taking the derivative of the both side, we have $0 = \langle \boldsymbol{\gamma}(s) - \boldsymbol{c}, \boldsymbol{n}_{\boldsymbol{\gamma}}(s)\rangle' = \langle \boldsymbol{\gamma}(s) - \boldsymbol{c}, -\kappa_n \boldsymbol{t}(s) - \tau_g \boldsymbol{b}(s)\rangle$. Then there exists $\lambda \in \mathbb{R}$ such that $\boldsymbol{\gamma}(s) - \boldsymbol{c} = \lambda \overline{D_o}(s)$. Taking the derivative again, we have

$$\begin{aligned}0 &= \langle \boldsymbol{\gamma} - \boldsymbol{c}, \boldsymbol{n}_{\boldsymbol{\gamma}}\rangle'' = \langle \boldsymbol{t}, -\kappa_n \boldsymbol{t} - \tau_g \boldsymbol{b}\rangle + \langle \boldsymbol{\gamma} - \boldsymbol{c}, (-\kappa_n \boldsymbol{t} - \tau_g \boldsymbol{b})'\rangle \\ &= -\kappa_n + \lambda\delta\sqrt{\kappa_n^2 + \tau_g^2}.\end{aligned}$$

It follows that

$$\boldsymbol{c} = \boldsymbol{\gamma}(s) - \lambda\overline{D_o}(s) = \boldsymbol{\gamma}(s) - \frac{\kappa_n(s)}{\delta(s)\sqrt{\kappa_n^2(s) + \tau_g^2(s)}}\overline{D_o}(s) = \boldsymbol{f}(s).$$

Therefore, $\boldsymbol{f}(s)$ is constant, so that the condition (1) holds. This completes the proof. ■

Corollary 3.2. *The osculating developable surface $OD_{\boldsymbol{\gamma}}$ is non-cylindrical if and only if $\delta(s) \neq 0$.*

We remark that developable surfaces are classified into cylinders, cones or tangent surfaces of space curves (cf., [8]). Hartman and Nirenberg [3] showed that a cylinder is only one non-singular (complete) developable surface. Hence, (complete) tangent surfaces have always singularities. By the results of Theorem 3.1, two invariants $\delta(s)$ and $\sigma(s)$ might be related to the singularities of osculating developable surfaces. Actually, we can classify the singularities of osculating developable surfaces of M along curves by using theses two invariants $\delta(s)$ and $\sigma(s)$.

Theorem 3.3. *Let $\boldsymbol{\gamma} : I \to M \subset \mathbb{R}^3$ be a unit speed curve with $\kappa_n^2(s) + \tau_g^2(s) \neq 0$. Then we have the following*:

(1) *The image of osculating developable surface $OD_{\boldsymbol{\gamma}}$ of M along $\boldsymbol{\gamma}$ is non-singular at (s_0, u_0) if and only if*

$$\frac{\kappa_n(s_0)}{\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)}} + u_0\delta(s_0) \neq 0.$$

(2) *The image of osculating developable surface $OD_{\boldsymbol{\gamma}}$ of M along $\boldsymbol{\gamma}$ is locally diffeomorphic to the cuspidaledge $C \times \mathbb{R}$ at (s_0, u_0) if*

(i) $\delta(s_0) \neq 0, \sigma(s_0) \neq 0$ *and*

$$u_0 = \quad \frac{\kappa_n(s_0)}{\delta(s_0)\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)}},$$

or

(ii) $\delta(s_0) = \kappa_n(s_0) = 0,\ \delta'(s_0) \ddagger 0$ *and*

$$u_0 \ddagger \quad \frac{\kappa_n'(s_0)}{2\kappa_g(s_0)\tau_g'(s_0) + \kappa_g'(s_0)\tau_g(s_0) \quad \kappa_n''(s_0)},$$

or

(iii) $\delta(s_0) = \kappa_n(s_0) = 0$ *and* $\kappa_n'(s_0) \ddagger 0$.
We remark that if $\delta'(s_0) \ddagger 0$*, then*

$$2\kappa_g(s_0)\tau_g'(s_0) + \kappa_g'(s_0)\tau_g(s_0) \quad \kappa_n''(s_0) \neq 0.$$

(3) *The image of osculating developable surface $OD_{\boldsymbol{\gamma}}$ of M along $\boldsymbol{\gamma}$ is locally diffeomorphic to the swallowtail SW at (s_0, u_0) if $\delta(s_0) \neq 0, \sigma(s_0) = 0, \sigma'(s_0) \neq 0$ and*

$$u_0 = \quad \frac{\kappa_n(s_0)}{\delta(s_0)\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)}}.$$

Here, $C \times \mathbb{R} = \{(x_1, x_2, x_3) | {x_1}^2 = {x_2}^3\}$ is the *cuspidaledge* (c.f., Fig.1) and $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the *swallowtail* (c.f., Fig.2).

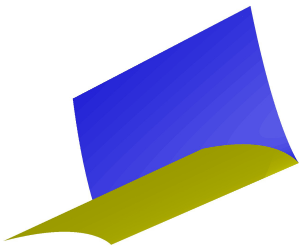

Fig. 1. The cuspidaledge

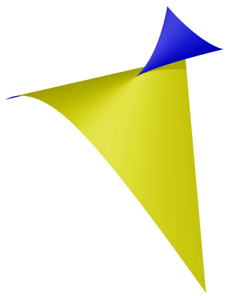

Fig. 2. The swallowtail

4. Support functions

In this section, we introduce a family of functions on a curve which is useful for the study of invariants of curves on surfaces. For a unit speed curve $\boldsymbol{\gamma} : I \quad \to M \subset \mathbb{R}^3$, we define a function $G : I \times \mathbb{R}^3 \quad \to \mathbb{R}$ by $G(s, \boldsymbol{x}) = \langle \boldsymbol{x} \quad \boldsymbol{\gamma}(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s)\rangle$. We call G a *support function* on $\boldsymbol{\gamma}$ with respect to $\boldsymbol{n}_{\boldsymbol{\gamma}}$. We denote that $g_{\boldsymbol{x}_0}(s) = G(s, \boldsymbol{x}_0)$ for any $\boldsymbol{x}_0 \in \mathbb{R}^3$. Then we have the following proposition.

Proposition 4.1. *Let $\boldsymbol{\gamma} : I \quad \to M \subset \mathbb{R}^3$ be a unit speed curve with $\kappa_n^2 + \tau_g^2 \neq 0$. Then we have the followings:*

(1) *$g_{\boldsymbol{x}_0}(s_0) = 0$ if and only if there exist $u, v \in \mathbb{R}$ such that $\boldsymbol{x}_0 \quad \boldsymbol{\gamma}(s_0) = u\boldsymbol{t}(s_0) + v\boldsymbol{b}(s_0)$.*

(2) *$g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = 0$ if and only if there exists $u \in \mathbb{R}$ such that*

$$\boldsymbol{x}_0 \quad \boldsymbol{\gamma}(s_0) = u\frac{\tau_g(s_0)\boldsymbol{t}(s_0) \quad \kappa_n(s_0)\boldsymbol{b}(s_0)}{\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)}}.$$

(A) *Suppsoe that $\delta(s_0) \neq 0$. Then we have the following:*

(3) *$g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = g''_{\boldsymbol{x}_0}(s_0) = 0$ if and only if*

$$(*) \quad \boldsymbol{x}_0 \quad \boldsymbol{\gamma}(s_0) = \quad \frac{\kappa_n(s_0)}{\delta(s_0)\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)}} \frac{\tau_g(s_0)\boldsymbol{t}(s_0) \quad \kappa_n(s_0)\boldsymbol{b}(s_0)}{\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)}}.$$

(4) *$g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = g''_{\boldsymbol{x}_0}(s_0) = g^{(3)}_{\boldsymbol{x}}(s_0) = 0$ if and only if $\sigma(s_0) = 0$ and (*).*

(5) *$g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = g''_{\boldsymbol{x}_0}(s_0) = g^{(3)}_{\boldsymbol{x}}(s_0) = d^{(4)}_{\boldsymbol{x}_0}(s_0) = 0$ if and only if*

$$\sigma(s_0) = \sigma'(s_0) = 0 \text{ and } (*).$$

(B) *Suppose that $\delta(s_0) = 0$. Then we have the following:*

(6) *$g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = g''_{\boldsymbol{x}_0}(s_0) = 0$ if and only if $\kappa_n(s_0) = 0$ (i.e., $\kappa_n(s_0) = 0, \kappa'_n(s_0) = \kappa_g(s_0)\tau_g(s_0)$) and there exists $u \in \mathbb{R}$ such that*

$$\boldsymbol{x}_0 \quad \boldsymbol{\gamma}(s_0) = u\boldsymbol{t}(s_0).$$

(7) *$g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = g''_{\boldsymbol{x}_0}(s_0) = g^{(3)}_{\boldsymbol{x}}(s_0) = 0$ if and only if one of the following conditions holds:*

(a) $\delta'(s_0) \neq 0, \kappa_n(s_0) = 0$
(i.e., $\kappa_n(s_0) = 0, \kappa'_n(s_0) = \kappa_g(s_0)\tau_g(s_0), 2\kappa_g(s_0)\tau'_g(s_0) + \kappa'_g(s_0)\tau_g(s_0) \quad \kappa''_n(s_0) \neq 0$) and

$$\boldsymbol{x}_0 \quad \boldsymbol{\gamma}(s_0) = \quad \frac{\kappa'_n(s_0)}{2\kappa_g(s_0)\tau'_g(s_0) + \kappa'_g(s_0)\tau_g(s_0) \quad \kappa''_n(s_0)}\boldsymbol{t}(s_0).$$

(b) $\delta'(s_0) = 0, \kappa_n(s_0) = \kappa'_n(s_0) = 0$
(i.e., $\kappa_g(s_0) = \kappa_n(s_0) = \kappa'_n(s_0) = 0, \kappa''_n(s_0) = \kappa'_g(s_0)\tau_g(s_0)$) and there exists $u \in \mathbb{R}$ such that

$$\boldsymbol{x}_0 \quad \boldsymbol{\gamma}(s_0) = u\boldsymbol{t}(s_0).$$

Proof. Since $g_{\boldsymbol{x}_0}(s) = \langle \boldsymbol{x}_0 - \boldsymbol{\gamma}(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s)\rangle$, we have the following calculations:

(α) $g_{\boldsymbol{x}_0} = \langle \boldsymbol{x}_0 - \boldsymbol{\gamma}, \boldsymbol{n}_{\boldsymbol{\gamma}}\rangle$,

(β) $g'_{\boldsymbol{x}_0} = \langle \boldsymbol{x}_0 - \boldsymbol{\gamma}, -\kappa_n \boldsymbol{t} - \tau_g \boldsymbol{b}\rangle$,

(γ) $g''_{\boldsymbol{x}_0} = \kappa_n + \langle \boldsymbol{x}_0 - \boldsymbol{\gamma}, (-\kappa'_n + \kappa_g\tau_g)\boldsymbol{t} - (\tau'_g + \kappa_g\kappa_n)\boldsymbol{b} - (\kappa_n^2 + \tau_g^2)\boldsymbol{n}_{\boldsymbol{\gamma}}\rangle$,

$$\begin{aligned}(\delta)\ g^{(3)}_{\boldsymbol{x}_0} &= 2\kappa'_n - \kappa_g\tau_g \\ &\quad + \langle \boldsymbol{x}_0 - \boldsymbol{\gamma}, \big(\kappa_n(\kappa_g^2 + \kappa_n^2 + \tau_g^2) + (\kappa'_g\tau_g + 2\kappa_g\tau'_g) - \kappa''_n\big)\,\boldsymbol{t} \\ &\quad + \big(\tau_g(\kappa_g^2 + \kappa_n^2 + \tau_g^2) - (\kappa'_g\kappa_n + 2\kappa_g\kappa'_n) - \tau''_g\big)\,\boldsymbol{b} - 3(\kappa_n\kappa'_n + \tau_g\tau'_g)\boldsymbol{n}_{\boldsymbol{\gamma}}\rangle,\end{aligned}$$

$$\begin{aligned}(\epsilon)\ g^{(4)}_{\boldsymbol{x}_0} &= 3\kappa''_n - 2\kappa'_g\tau_g - 3\kappa_g\tau'_g + \kappa_n(\kappa_g^2 + \kappa_n^2 + \tau_g^2) \\ &\quad + \langle \boldsymbol{x} - \boldsymbol{\gamma}, \big(\kappa'_n(3\kappa_g^2 + \kappa_n^2 + \tau_g^2) + \kappa_n(3\kappa_g\kappa'_g + 5\kappa_n\kappa'_n + 5\tau_g\tau'_g) \\ &\qquad - \kappa_g\tau_g(\kappa_g^2 + \kappa_n^2 + \tau_g^2) + (\kappa''_g\tau_g + 3\kappa'_g\tau'_g + 3\kappa_g\tau''_g) - \kappa'''_n\big)\,\boldsymbol{t} \\ &\quad + \big(\tau'_g(3\kappa_g^2 + \kappa_n^2 + \tau_g^2) + \tau_g(3\kappa_g\kappa'_g + 5\kappa_n\kappa'_n + 5\tau_g\tau'_g) \\ &\qquad + \kappa_g\kappa_n(\kappa_g^2 + \kappa_n^2 + \tau_g^2) - (\kappa''_g\kappa_n + 3\kappa'_g\kappa'_n + 3\kappa_g\kappa''_n) - \tau'''_g\big)\,\boldsymbol{b} \\ &\quad + \big((\kappa_g^2 + \kappa_n^2)(\kappa_g^2 + \kappa_n^2 + \tau_g^2) + 2\kappa_g(\kappa_n\tau'_g - \kappa'_n\tau_g) - 3(\kappa'^2_n + \tau'^2_g) \\ &\qquad - 4(\kappa_n\kappa''_n + \tau_g\tau''_g)\big)\boldsymbol{n}_{\boldsymbol{\gamma}}\rangle.\end{aligned}$$

By definition and the formula (α), the assertion (1) follows.

By the formula (β), $g_{\boldsymbol{x}_0}(s_0) = g'_{v_0}(s_0) = 0$ if and only if $\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = u\boldsymbol{t}(s_0) + v\boldsymbol{b}(s_0)$ and $-\kappa_n(s_0)u - \tau_g(s_0)v = 0$. If $\kappa_n(s_0) \neq 0$, $\tau_g(s_0) \neq 0$, then we have

$$u = -v\frac{\tau_g(s_0)}{\kappa_n(s_0)},\ v = -u\frac{\kappa_n(s_0)}{\tau_g(s_0)},$$

so that there exists $\lambda \in \mathbb{R}$ such that

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = \lambda\frac{\tau_g(s_0)\boldsymbol{t}(s_0) - \kappa_n(s_0)\boldsymbol{b}(s_0)}{\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)}}.$$

Suppose that $\kappa_n(s_0) = 0$. Then we have $\tau_g(s_0) \neq 0$, so that $\tau_g(s_0)v = 0$. Therefore, we have

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = u\boldsymbol{t}(s_0) = \pm u\frac{\tau_g(s_0)\boldsymbol{t}(s_0) - \kappa_n(s_0)\boldsymbol{b}(s_0)}{\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)}}.$$

If $\tau_g(s_0) = 0$, then we have $\boldsymbol{x}_0 - \gamma(s_0) = v\boldsymbol{b}(s_0)$. Therefore the assertion (2) holds.

By the formula (γ), $g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = g''_{\boldsymbol{x}_0}(s_0) = 0$ if and only if

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = \lambda\frac{\tau_g(s_0)\boldsymbol{t}(s_0) - \kappa_n(s_0)\boldsymbol{b}(s_0)}{\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)}},$$

and

$$\kappa_n(s_0) + \lambda \frac{\tau_g(\kappa_g\tau_g \quad \kappa_n') + \kappa_n(\kappa_g\kappa_n + \tau_g')}{\sqrt{\kappa_n^2 + \tau_g^2}}(s_0) = 0.$$

It follows that

$$\frac{\kappa_n}{\sqrt{\kappa_n^2 + \tau_g^2}}(s_0) + \lambda \left(\kappa_g + \frac{\kappa_n\tau_g' \quad \kappa_n'\tau_g}{\kappa_n^2 + \tau_g^2} \right)(s_0) = 0.$$

Thus,

$$\delta(s_0) = \kappa_g(s_0) + \frac{\kappa_n\tau_g' \quad \kappa_n'\tau_g}{\kappa_n^2 + \tau_g^2}(s_0) \neq 0 \text{ and } \lambda = \quad \frac{\kappa_n}{\delta\sqrt{\kappa_n^2 + \tau_g^2}}(s_0)$$

or $\delta(s_0) = 0, \kappa_n(s_0) = 0$. This completes the proof of the assertion (A), (3) and (B), (6).

Suppose that $\delta(s_0) \ddagger 0$. By the formula (δ), $g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = g''_{\boldsymbol{x}_0}(s_0) = g^{(3)}_{\boldsymbol{x}_0}(s_0) = 0$ if and only if

$$2\kappa_n' \quad \kappa_g\tau_g \quad \frac{\kappa_n}{\delta\sqrt{\kappa_n^2+\tau_g^2}} \Bigg(\frac{\tau_g}{\sqrt{\kappa_n^2+\tau_g^2}} \big(\kappa_n(\kappa_g^2+\kappa_n^2+\tau_g^2)+(\kappa_g'\tau_g+2\kappa_g\tau_g') \quad \kappa_n''\big)$$
$$\frac{\kappa_n}{\sqrt{\kappa_n^2+\tau_g^2}} \big(\tau_g(\kappa_g^2+\kappa_n^2+\tau_g^2) \quad (\kappa_g'\kappa_n+2\kappa_g\kappa_n') \quad \tau_g''\big) \Bigg) = 0$$

at $s = s_0$. It follows that

$$2\kappa_n'(s_0) \quad \kappa_g(s_0)\tau_g(s_0) \quad \frac{\kappa_n(s_0)}{\delta(s_0)} \left(\kappa_g' + 2\kappa_g \frac{\kappa_n\kappa_n' + \tau_g\tau_g'}{\kappa_n^2+\tau_g^2} + \frac{\kappa_n\tau_g'' \quad \kappa_n''\tau_g}{\kappa_n^2+\tau_g^2} \right)(s_0) = 0.$$

Since

$$\delta' = \kappa_g' \quad 2\frac{(\kappa_n\kappa_n'+\tau_g\tau_g')(\kappa_n\tau_g' \quad \kappa_n'\tau_g)}{(\kappa_n^2+\tau_g^2)^2} + \frac{\kappa_n\tau_g'' \quad \kappa_n''\tau_g}{\kappa_n^2+\tau_g^2},$$

$$2\kappa_n'(s_0) \quad \kappa_g(s_0)\tau_g(s_0) \quad \kappa_n(s_0)\frac{\delta'(s_0)}{\delta(s_0)} \quad 2\kappa_n(s_0)\frac{\kappa_n(s_0)\kappa_n'(s_0)+\tau_g(s_0)\tau_g'(s_0)}{\kappa_n^2(s_0)+\tau_g^2(s_0)} = 0.$$

Moreover, we apply the relation

$$\left(\frac{\kappa_n}{\sqrt{\kappa_n^2+\tau_g^2}} \right)' = \quad \frac{\tau_g}{\sqrt{\kappa_n^2+\tau_g^2}} \frac{\kappa_n\tau_g' \quad \kappa_n'\tau_g}{\kappa_n^2+\tau_g^2} = \quad \frac{\tau_g}{\sqrt{\kappa_n^2+\tau_g^2}}(\delta \quad \kappa_g)$$

to the above. Then we have

$$\delta(s_0)\sqrt{\kappa_n^2+\tau_g^2}(s_0)\left(\frac{\tau_g}{\sqrt{\kappa_n^2+\tau_g^2}} - \left(\frac{\kappa_n}{\delta\sqrt{\kappa_n^2+\tau_g^2}}\right)'\right)(s_0)$$

$$= \delta(s_0)\sigma(s_0)\sqrt{\kappa_n^2+\tau_g^2}(s_0) = 0,$$

so that $\sigma(s_0) = 0$. The converse assertion also holds.

Suppose that $\delta(s_0) = 0$. Then by the formulae (δ), $g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = g''_{\boldsymbol{x}_0}(s_0) = g^{(3)}_{\boldsymbol{x}_0}(s_0) = 0$ if and only if $\kappa_n(s_0) = 0$ (i.e., $\kappa_n(s_0) = 0, \kappa'_n(s_0) = \kappa_g(s_0)\tau_g(s_0)$), there exists $u \in \mathbb{R}$ such that

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = u\boldsymbol{t}(s_0)$$

and

$$2\kappa'_n(s_0) - \kappa_g(s_0)\tau_g(s_0) + u\left(2\kappa_g(s_0)\tau'_g(s_0) + \kappa'_g(s_0)\tau_g(s_0) - \kappa''_n(s_0)\right) = 0.$$

Since $\delta(s_0) = 0$ and $\kappa_n(s_0) = 0$, we have $\kappa_g(s_0)\tau_g(s_0) = \kappa'_n(s_0)$, so that

$$\kappa'_n(s_0) + u(2\kappa_g(s_0)\tau'_g(s_0) + \kappa'_g(s_0)\tau_g(s_0) - \kappa''_n(s_0)) = 0.$$

It follows that

$$2\kappa_g(s_0)\tau'_g(s_0) + \kappa'_g(s_0)\tau_g(s_0) - \kappa''_n(s_0) \neq 0$$

and

$$u = -\frac{\kappa'_n(s_0)}{2\kappa_g(s_0)\tau'_g(s_0) + \kappa'_g(s_0)\tau_g(s_0) - \kappa''_n(s_0)}$$

or

$$2\kappa_g(s_0)\tau'_g(s_0) + \kappa'_g(s_0)\tau_g(s_0) - \kappa''_n(s_0) = 0 \text{ and } \kappa'_n(s_0) = 0.$$

Therefore we have (B), (7), (a) or (b).

By the similar arguments to the above, we have the assertion (A), (5). This completes the proof. ∎

In order to prove Theorem 3.3, we use some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book [1]. Let $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f, where $f(s) = F_{x_0}(s, x_0)$. We say that f has an *A_k-singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that f has an *$A_{\geq k}$-singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let F be an unfolding of f and $f(s)$ has an A_k-singularity ($k \geq 1$) at s_0. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s, x_0))(s_0) = \sum_{j=0}^{k-1}\alpha_{ji}(s-s_0)^j$ for $i = 1, \ldots, r$. Then F is called an *$\mathcal{R}$-versal unfolding* if the $k\times r$ matrix of coefficients $(\alpha_{ji})_{j=0,\ldots,k-1;i=1,\ldots,r}$ has rank k $(k \leq r)$. We introduce an important set concerning the unfoldings relative to the above notions. The *discriminant set* of F is the set

$$\mathcal{D}_F = \{x \in \mathbb{R}^r | \text{there exists } s \text{ with } F = \frac{\partial F}{\partial s} = 0 \text{ at } (s, x)\}.$$

Then we have the following classification (cf., [1]).

Theorem 4.2. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has an A_k singularity at s_0. Suppose that F is an $\mathcal{R}$-versal unfolding.*

(1) *If $k = 2$, then $\mathcal{D}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r\ \ 1}$.*
(2) *If $k = 3$, then $\mathcal{D}_F$ is locally diffeomorphic to $SW \times \mathbb{R}^{r\ \ 2}$.*

For the proof of Theorem 3.3, we have the following propositions.

Proposition 4.3. *Let $\boldsymbol{\gamma} : I \quad \to M \subset \mathbb{R}^3$ be a unit speed curve with $\kappa_n^2 + \tau_g^2 \neq 0$ and let $G : I \times \mathbb{R}^3 \quad \to \mathbb{R}$ be the support function on γ with respect to $\boldsymbol{n_\gamma}$. If $g_{\boldsymbol{x}_0}$ has an A_k-singularity ($k = 2, 3$) at s_0, then G is an $\mathcal{R}$-versal unfolding of $g_{\boldsymbol{x}_0}$.*

Proof. We denote that $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{n_\gamma}(s) = (n_1(s), n_2(s), n_3(s))$. Then we have

$$G(s, \boldsymbol{x}) = n_1(s)x_1 + n_2(s)x_2 + n_3(s)x_3,$$

so that

$$\frac{\partial G}{\partial x_i}(s, \boldsymbol{x}) = n_i(s), \quad (i = 1, 2, 3).$$

Therefore the 2-jet is

$$j^2 \frac{\partial G}{\partial x_i}(s_0, \boldsymbol{x}_0) = n_i(s_0) + n_i'(s_0)(s \quad s_0) + \frac{1}{2} n_i''(s_0)(s \quad s_0)^2.$$

We consider the following matrix:

$$A = \begin{pmatrix} n_1(s_0) & n_2(s_0) & n_3(s_0) \\ n_1'(s_0) & n_2'(s_0) & n_3'(s_0) \\ n_1''(s_0) & n_2''(s_0) & n_3''(s_0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{n_\gamma}(s_0) \\ \boldsymbol{n_\gamma}'(s_0) \\ \boldsymbol{n_\gamma}''(s_0) \end{pmatrix}.$$

By the Frenet–Serret type formulae, we have

$$\boldsymbol{n}_{\boldsymbol{\gamma}}' = \quad \kappa_n \boldsymbol{t} \quad \tau_g \boldsymbol{b} \text{ and } \boldsymbol{n}_{\boldsymbol{\gamma}}'' = (\kappa_g \tau_g \quad \kappa_n')\boldsymbol{t} \quad (\kappa_g \kappa_n + \tau_g')\boldsymbol{b} \quad (\kappa_n^2 + \tau_g^2)\boldsymbol{n_\gamma}.$$

Since $\{\boldsymbol{t}, \boldsymbol{b}, \boldsymbol{n_\gamma}\}$ is an orthonormal basis of $\mathbb{R}^3$, the rank of

$$A = \begin{pmatrix} \boldsymbol{n_\gamma}(s_0) \\ \kappa_n(s_0)\boldsymbol{t}(s_0) \quad \tau_g(s_0)\boldsymbol{b}(s_0) \\ (\kappa_g(s_0)\tau_g(s_0) \quad \kappa_n'(s_0))\boldsymbol{t}(s_0) \quad (\kappa_g(s_0)\kappa_n(s_0) + \tau_g'(s_0))\boldsymbol{b}(s_0) \quad (\kappa_n^2(s_0) + \tau_g^2(s_0))\boldsymbol{n_\gamma}(s_0) \end{pmatrix}$$

is equal to the rank of

$$\begin{pmatrix} 0 & 0 & 1 \\ \kappa_n(s_0) & \tau_g(s_0) & 0 \\ (\kappa_g(s_0)\tau_g(s_0) \quad \kappa_n'(s_0)) & (\kappa_g(s_0)\kappa_n(s_0)+\tau_g'(s_0)) & (\kappa_n^2(s_0)+\tau_g^2(s_0)) \end{pmatrix}.$$

Therefore, $\operatorname{rank} A = 3$ if and only if

$$0 \neq \kappa_n(\kappa_g\kappa_n+\tau_g')+\tau_g(\kappa_n\tau_g+\kappa_m') = \kappa_g(\kappa_n^2+\tau_g^2)+(\kappa_n\tau_g'+\kappa_n'\tau_g)$$

st $s = s_0$. The last condition is equivalent to the condition $\delta(s_0) \neq 0$. Moreover, the rank of

$$\begin{pmatrix} \boldsymbol{n}_\gamma(s_0) \\ \boldsymbol{n}_\gamma'(s_0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{n}_\gamma(s_0) \\ \kappa_n(s_0)\boldsymbol{t}(s_0) \quad \tau_g(s_0)\boldsymbol{b}(s_0) \end{pmatrix}$$

is always two.

If $g_{\boldsymbol{x}_0}$ has an A_k-singularity (k = 2, 3) at s_0, then G is $\mathcal{R}$-versal unfolding of $g_{\boldsymbol{x}_0}$. This completes the proof. ∎

Proof of Theorem 3.3. By a straightforward calculation, we have

$$\frac{\partial OD_\gamma}{\partial s} \times \frac{\partial OD_\gamma}{\partial u} = \left(\frac{\kappa_n}{\sqrt{\kappa_n^2+\tau_g^2}}+u\delta\right)\boldsymbol{n}_\gamma.$$

Therefore, (s_0, u_0) is non-singular if and only if

$$\frac{\partial OD_\gamma}{\partial s} \times \frac{\partial OD_\gamma}{\partial u} \neq \mathbf{0}.$$

This condition is equivalent to

$$\frac{\kappa_n(s_0)}{\sqrt{\kappa_n^2(s_0)+\tau_g^2(s_0)}}+u_0\delta(s_0) \neq 0.$$

This completes the proof of the assertion (1).

By Proposition 4.1, (2), the discriminant set $\mathcal{D}_G$ of the support function G of $\boldsymbol{\gamma}$ with respect to $\boldsymbol{n}_\gamma$ is the image of the osculating developable surface of M along $\boldsymbol{\gamma}$.

Suppose that $\delta(s_0) \neq 0$. It follows from Proposition 4.1, A, (3), (4) and (5) that $g_{\boldsymbol{x}_0}$ has the A_3-type singularity (respectively, the A_4-type singularity) at $s = s_0$ if and only if

$$u_0 = \frac{\kappa_n(s_0)}{\delta(s_0)\sqrt{\kappa_n^2(s_0)+\tau_g^2(s_0)}}$$

and $\sigma(s_0) \neq 0$ (respectively, $\sigma(s_0) = 0$ and $\sigma'(s_0) \neq 0$). By Theorem 4.2 and Proposition 3.3, we have the assertions (2), (α) and (3).

Suppose that $\delta(s_0) = 0$. It follows from Proposition 4.1, B, (6) and (7) that $g_{\boldsymbol{x}_0}$ has the A_3-type singularity if and only if $\delta(s_0) = 0, \kappa_n(s_0) = 0$ and

$$\kappa_n'(s_0) \neq 0 \text{ or } \kappa_n'(s_0) + u_0 \left(2\kappa_g(s_0)\tau_g'(s_0) + \kappa_g'(s_0)\tau_g(s_0) \quad \kappa_n''(s_0)\right) \neq 0.$$

By Theorem 4.3 and Proposition 4.3, we have the assertion (2), (β). This completes the proof. ∎

5. Invariants of curves on surfaces

In this section, we consider geometric meanings of the invariant σ. Let $\boldsymbol{\Gamma} : I \to \mathbb{R}^3 \times S^2$ be a regular curve and $F : \mathbb{R}^3 \times S^2 \to \mathbb{R}$ a submersion. We say that $\boldsymbol{\Gamma}$ and $F^{\ 1}(0)$ have contact of *at least order* k for $t = t_0$ if the function $g(t) = F \circ \boldsymbol{\Gamma}(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k)}(t_0) = 0$. If $\boldsymbol{\gamma}$ and $F^{\ 1}(0)$ have contact of at least order k for $t = t_0$ and satisfies the condition that $g^{(k+1)}(t_0) \neq 0$, then we say that $\boldsymbol{\Gamma}$ and $F^{\ 1}(0)$ have *contact of order* k for $t = t_0$. For any $\boldsymbol{x} \in \mathbb{R}^3$, we define a function $\mathfrak{g}_{\boldsymbol{x}} : \mathbb{R}^3 \times S^2 \to \mathbb{R}$ by $\mathfrak{g}_{\boldsymbol{x}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{x} \quad \boldsymbol{u}, \boldsymbol{v}\rangle$. Then we have

$$\mathfrak{g}_{\boldsymbol{x}}{}^{\ 1}(0) = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^3 \times S^2 \mid \langle \boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{x}, \boldsymbol{v}\rangle\}.$$

If we fix $\boldsymbol{v} \in S^2$, then $\mathfrak{g}_{\boldsymbol{x}}{}^{\ 1}(0)|\mathbb{R}^3 \times \{\boldsymbol{v}\}$ is an affine plane defined by $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = c$, where $c = \langle \boldsymbol{x}, \boldsymbol{v}\rangle$. Since this plane is orthogonal to $\boldsymbol{v}$, it is parallel to the tangent plane $T_{\boldsymbol{v}}S^2$ at $\boldsymbol{v}$. Here we have a representation of the tangent bundle of S^2 as follows:

$$TS^2 = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^3 \times S^2 \mid \langle \boldsymbol{u}, \boldsymbol{v}\rangle = 1\}.$$

We consider the canonical projection $\pi_2|\mathfrak{g}_{\boldsymbol{x}}{}^{\ 1}(0) : \mathfrak{g}_{\boldsymbol{x}}{}^{\ 1}(0) \to S^2$, where $\pi_2 : \mathbb{R}^3 \times S^2 \to S^2$. Then $\pi_2|\mathfrak{g}_{\boldsymbol{x}}{}^{\ 1}(0) : \mathfrak{g}_{\boldsymbol{x}}{}^{\ 1}(0) \to S^2$ is a plane bundle over S^2. Moreover, we define a map $\Psi : \mathfrak{g}_{\boldsymbol{x}}{}^{\ 1}(0) \to TS^2$ by $\Phi(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}/\langle \boldsymbol{x}, \boldsymbol{v}\rangle, \boldsymbol{v})$. Then Φ is a bundle isomorphism. Therefore, we denote that $TS^2(\boldsymbol{x}) = \mathfrak{g}_{\boldsymbol{x}}{}^{\ 1}(0)$ and call it a *affine tangent bundle over* S^2 *through* $\boldsymbol{x}$. Let $\gamma : I \to M \subset \mathbb{R}^3$ be a unit speed curves on M with $\kappa_n^2(s) + \tau_g^2(s) \neq 0$. Suppose that $\delta(s) \neq 0$. By the proof of the assertion (B) of Theorem 3.1, the derivative of the vector valued function $\boldsymbol{f}$ of $OD_{\boldsymbol{\gamma}}$ is $\boldsymbol{f}'(s) = \sigma(s)\overline{D}_o(s)$. If we assume that $\sigma(s) \equiv 0$, then $\boldsymbol{f}$ is a constant vector $\boldsymbol{x}_0$. Then

$$\boldsymbol{\gamma}(s) \quad \boldsymbol{x}_0 = \frac{\kappa_n(s)}{\delta(s)\sqrt{\kappa_n^2(s) + \tau_g^2(s)}}\overline{D}_o(s).$$

Therefore

$$\mathfrak{g}_{\boldsymbol{x}_0}(\boldsymbol{\gamma}(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s)) = g_{\boldsymbol{x}_0}(s) = \langle \boldsymbol{\gamma}(s) \quad \boldsymbol{x}_0, \boldsymbol{n}_{\boldsymbol{\gamma}}(s)\rangle = 0.$$

If there exists $\boldsymbol{x}_0 \in \mathbb{R}^3$ such that $\mathfrak{g}_{\boldsymbol{x}_0}(\boldsymbol{\gamma}(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s)) = 0$, then we have

$$\boldsymbol{\gamma}(s) \quad \boldsymbol{x}_0 = \frac{\kappa_n(s)}{\delta(s)\sqrt{\kappa_n^2(s)+\tau_g^2(s)}}\overline{D}_o(s),$$

and $\sigma(s) \equiv 0$. We consider a regular curve $(\boldsymbol{\gamma}, \boldsymbol{n}_{\boldsymbol{\gamma}}) : I \to \mathbb{R}^3 \times S^2$. Then we have the following proposition.

Proposition 5.1. *Let $\gamma : I \to M \subset \mathbb{R}^3$ be a unit speed curve on M with $\kappa_n^2(s)+\tau_g^2(s) \neq 0$ and $\delta(s) \neq 0$. Then there exists $\boldsymbol{x}_0 \in \mathbb{R}^3$ such that $(\boldsymbol{\gamma}, \boldsymbol{n}_{\boldsymbol{\gamma}})(I) \subset TS^2(\boldsymbol{x}_0)$ if and only if $\sigma(s) \equiv 0$.*

The result of the above proposition explains that the geometric meaning of the singularities of $OD_{\boldsymbol{\gamma}}$ is related not only to the curve but also to the shape of the surface along the curve. Let $\boldsymbol{\gamma} : I \to M \subset \mathbb{R}^3$ be a unit speed curve with $\kappa_n^2(s)+\tau_g^2(s) \neq 0$. Then we consider the support function $g_{\boldsymbol{x}_0}(s) = \mathfrak{g}_{\boldsymbol{x}_0}(\boldsymbol{\gamma}(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s))$. By the assertion (2) of Proposition 4.1, $(\boldsymbol{\gamma}, \boldsymbol{n}_{\boldsymbol{\gamma}})$ is tangent to $TS^2(\boldsymbol{x}_0)$ at $s = s_0$ if and only if $\boldsymbol{x}_0 = OD_{\boldsymbol{\gamma}}(s_0, u_0)$ for some $u_0 \in \mathbb{R}$. Then we have the following proposition.

Proposition 5.2. *Let $\boldsymbol{\gamma} : I \to M \subset \mathbb{R}^3$ be a unit speed curve with $\kappa_n^2(s)+\tau_g^2(s) \neq 0$ and $\delta(s) \neq 0$. For $\boldsymbol{x}_0 = OD_{\boldsymbol{\gamma}}(s_0, u_0)$, we have the following:*

(1) *The order of contact of $(\boldsymbol{\gamma}, \boldsymbol{n}_{\boldsymbol{\gamma}})$ with $TS^2(\boldsymbol{x}_0)$ at $s = s_0$ is two if and only if*

$$(**) \qquad u_0 = \frac{\kappa_n(s_0)}{\delta(s_0)\sqrt{\kappa_n^2(s_0)+\tau_g^2(s_0)}},$$

and $\sigma(s_0) \neq 0$.

(2) *The order of contact of $(\boldsymbol{\gamma}, \boldsymbol{n}_{\boldsymbol{\gamma}})$ with $TS^2(\boldsymbol{x}_0)$ at $s = s_0$ is three if and only if (**) and $\sigma(s_0) = 0$ and $\sigma'(s_0) \neq 0$.*

Proof. By the assertions (3), (4) of Proposition 4.1, the conditions $g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = g''_{\boldsymbol{x}_0}(s_0) = 0$ and $g^{(3)}_{\boldsymbol{x}_0}(s_0) \neq 0$ if and only if (**) and $\sigma(s_0) \neq 0$. Since $g_{\boldsymbol{x}_0} = \mathfrak{g}_{\boldsymbol{x}_0} \circ (\boldsymbol{\gamma}, \boldsymbol{n}_{\boldsymbol{\gamma}})$, the above condition means that $(\boldsymbol{\gamma}, \boldsymbol{n}_{\boldsymbol{\gamma}})$ and $TS^2(\boldsymbol{x}_0)$ have contact of order two at $s = s_0$. For the proof the assertion (2), we use the assertions (4), (5) of Proposition 4.1 exactly the same way as the above case. ∎

Therefore, the geometric meaning of the classification results of Theorem 3.3 are given as follows.

Theorem 5.3. *Let $\boldsymbol{\gamma} : I \to M \subset \mathbb{R}^3$ be a unit speed curve with $\kappa_n^2(s)+\tau_g^2(s) \neq 0$ and $\delta(s) \neq 0$.*

(1) *The image of osculating developable surface $OD_{\boldsymbol{\gamma}}$ of M along $\boldsymbol{\gamma}$ is locally diffeomorphic to the cuspidaledge $C \times \mathbb{R}$ at (s_0, u_0) if*

$$u_0 = \frac{\kappa_n(s_0)}{\delta(s_0)\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)}}$$

and the order of contact of $(\boldsymbol{\gamma}, \boldsymbol{n}_{\boldsymbol{\gamma}})$ with $TS^2(\boldsymbol{x}_0)$ at $s = s_0$ is two.

(2) *The image of osculating developable surface $OD_{\boldsymbol{\gamma}}$ of M along $\boldsymbol{\gamma}$ is locally diffeomorphic to the swallowtail SW at (s_0, u_0) if*

$$u_0 = \frac{\kappa_n(s_0)}{\delta(s_0)\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)}}$$

and the order of contact of $(\boldsymbol{\gamma}, \boldsymbol{n}_{\boldsymbol{\gamma}})$ with $TS^2(\boldsymbol{x}_0)$ at $s = s_0$ is three.

6. Curves on special surfaces

In this section, we consider curves on special surfaces.

6.1. Curves on developable surfaces. In this subsection, we consider the case when the surface itself is a developable surface where the curve is located on.

Theorem 6.1. *Suppose that $M \subset \mathbb{R}^3$ is a developable surface. Let $\gamma : I \to M \subset \mathbb{R}^3$ be a unit speed curve in the regular part of M with $(\kappa_n(s), \tau_g(s)) \neq (0,0)$. Then $OD_\gamma(I \times \mathbb{R}) \subset M$.*

Proof. We assume that the developable surface M is the image of

$$F_{(\boldsymbol{c},\boldsymbol{\xi})}(t,u) = \boldsymbol{c}(t) + u\boldsymbol{\xi}(t),$$

where $\boldsymbol{c}(t)$ is the base curve and $\boldsymbol{\xi}(t)$ is the director curve. Then we have $\det\big(\dot{\boldsymbol{c}}(t), \boldsymbol{\xi}(t), \dot{\boldsymbol{\xi}}(t)\big) = 0$. We now consider a curve on M parametrized by

$$\boldsymbol{\gamma}(s) = \boldsymbol{c}\big(t(s)\big) + u(s)\boldsymbol{\xi}\big(t(s)\big),$$

where s is the arc-length parameter of $\boldsymbol{\gamma}$. We now try to get $OD_{\boldsymbol{\gamma}}$. Since

$$\frac{\partial F_{(\boldsymbol{c},\boldsymbol{\xi})}}{\partial t}(t,u) = \dot{\boldsymbol{c}}(t) + u\dot{\boldsymbol{\xi}}(t), \quad \frac{\partial F_{(\boldsymbol{c},\boldsymbol{\xi})}}{\partial u}(t,u) = \boldsymbol{\xi}(t),$$

the unit normal vector along $\boldsymbol{\gamma}$ is

$$\boldsymbol{n}_{\boldsymbol{\gamma}} = \frac{1}{l}\Big((\dot{\boldsymbol{c}} + u\dot{\boldsymbol{\xi}}) \times \boldsymbol{\xi}\Big) = \frac{1}{l}\Big((\dot{\boldsymbol{c}} \times \boldsymbol{\xi}) + u(\dot{\boldsymbol{\xi}} \times \boldsymbol{\xi})\Big),$$

where $l(s) = \|\partial F_{(\boldsymbol{c},\boldsymbol{\xi})}/\partial t \times \partial F_{(\boldsymbol{c},\boldsymbol{\xi})}/\partial u\| \, (t(s), u(s))$. We also have

$$\begin{aligned}
\boldsymbol{t} &= u'\boldsymbol{\xi} + t'(\dot{\boldsymbol{c}} + u\dot{\boldsymbol{\xi}}), \\
\boldsymbol{b} &= \frac{1}{l}\left(\left\{(\dot{\boldsymbol{c}} + u\dot{\boldsymbol{\xi}}) \times \boldsymbol{\xi}\right\} \times \boldsymbol{t}\right) \\
&= \frac{1}{l}\left(\langle \dot{\boldsymbol{c}} + u\dot{\boldsymbol{\xi}}, \boldsymbol{t}\rangle\, \boldsymbol{\xi} \quad \langle \boldsymbol{\xi}, \boldsymbol{t}\rangle (\dot{\boldsymbol{c}} + u\dot{\boldsymbol{\xi}})\right).
\end{aligned}$$

Moreover, we have

$$\boldsymbol{n}' = \frac{t'}{l}\left(\ddot{\boldsymbol{c}} \times \boldsymbol{\xi} + \dot{\boldsymbol{c}} \times \dot{\boldsymbol{\xi}}\right) + \left(\frac{1}{l}\right)' \dot{\boldsymbol{c}} \times \boldsymbol{\xi} + \frac{t'u}{l}\, \ddot{\boldsymbol{\xi}} \times \boldsymbol{\xi} + \left(\frac{u}{l}\right)' \dot{\boldsymbol{\xi}} \times \boldsymbol{\xi}.$$

Therefore, we have

$$\kappa_n(s) = \quad \frac{t'^2(s)d(s)}{l(s)}, \tau_g(s) = \frac{t'(s)d(s)}{l^2(s)} \langle \boldsymbol{\xi}(t(s)), \boldsymbol{t}(s)\rangle,$$

where

$$d(s) = \det\left(\dot{\boldsymbol{c}}(t(s)) + u(s)\dot{\boldsymbol{\xi}}(t(s)),\ \ddot{\boldsymbol{c}}(t(s)) + u(s)\ddot{\boldsymbol{\xi}}(t(s)),\ \boldsymbol{\xi}(t(s))\right).$$

Since $(\kappa_n(s), \tau_g(s)) \neq (0,0)$, $d(s) \neq 0$ and $t'(s) \neq 0$. It follows that

$$\begin{aligned}
\tau_g \boldsymbol{t} \quad \kappa_n \boldsymbol{b} &= \frac{t'd}{l^2}\left(\langle \boldsymbol{\xi}, \boldsymbol{t}\rangle\left(u'\boldsymbol{\xi} + t'(\dot{\boldsymbol{c}} + u\dot{\boldsymbol{\xi}})\right) + t'\left(\langle \dot{\boldsymbol{c}} + u\dot{\boldsymbol{\xi}}, \boldsymbol{t}\rangle\, \boldsymbol{\xi} \quad \langle \boldsymbol{\xi}, \boldsymbol{t}\rangle(\dot{\boldsymbol{c}} + u\dot{\boldsymbol{\xi}})\right)\right) \\
&= \frac{t'd}{l^2}\langle u'\boldsymbol{\xi} + t'(\dot{\boldsymbol{c}} + u\dot{\boldsymbol{\xi}}), \boldsymbol{t}\rangle\, \boldsymbol{\xi} \\
&= \frac{t'd}{l^2}\langle \boldsymbol{t}, \boldsymbol{t}\rangle\, \boldsymbol{\xi} = \frac{t'd}{l^2}\, \boldsymbol{\xi},
\end{aligned}$$

so that $D_o(s)$ is parallel to the director curve $\boldsymbol{\xi}(t(s))$. This means that $OD_\gamma(I \times \mathbb{R}) \subset M$. ∎

Corollary 6.2. *Let $M \subset \mathbb{R}^3$ be a regular surface and let $\gamma : I \quad \to M \subset \mathbb{R}^3$ be a unit speed curve with $\kappa_n^2(s) + \tau_g^2(s) \neq 0$. Then there exists a unique developable surface which tangent to M along γ.*

Proof. For the existence, we have the osculating developable surface $OD_{\boldsymbol{\gamma}}$ along $\boldsymbol{\gamma}$. On the other hand, let N be a developable surface tangent to M along $\boldsymbol{\gamma}$. Since Im $\boldsymbol{\gamma} \subset N$, $\boldsymbol{n}_{\boldsymbol{\gamma}}, \boldsymbol{t}$ are the common for N and M. Therefore, the Darboux frame $\{\boldsymbol{t}, \boldsymbol{b}, \boldsymbol{n}_{\boldsymbol{\gamma}}\}$ along $\boldsymbol{\gamma}$ are the common for N and M. By Theorem 6.1, $OD_{\boldsymbol{\gamma}} \subset N$. This means that the uniqueness holds. ∎

By the above corollary, the notion of osculating developable surfaces along curves on M with $\kappa_n^2(s) + \tau_g^2(s) \neq 0$ is well-defined. For a regular curve $\boldsymbol{\gamma}$ on M with $\kappa_n^2(s) + \tau_g^2(s) \neq 0$, we call the developable surface which is tangent to M along $\boldsymbol{\gamma}$ an *osculating developable surface* along $\boldsymbol{\gamma}$.

On the other hand, we have assumed that $\kappa_n^2(s)+\tau_g^2(s) \neq 0$. If $\kappa_n \equiv 0$ and $\tau_g \equiv 0$, then we have the following theorem.

Theorem 6.3. *Let M be a developable surface and let $\boldsymbol{\gamma} : I \to M \subset \mathbb{R}^3$ be a unit speed curve. Then $\kappa_n \equiv \tau_g \equiv 0$ if and only if $\boldsymbol{\gamma}$ is a ruling of M.*

Proof. In general, the torsion of the curve $\boldsymbol{\gamma}$ as a space curve is given by

$$\tau = \tau_g + \frac{\kappa_g \kappa_n' - \kappa_g' \kappa_n}{\kappa_g^2 + \kappa_n^2}.$$

Under the assumption that $\kappa_n \equiv 0$ and $\tau_g \equiv 0$, the torsion τ is constantly equal to 0. Thus, $\boldsymbol{\gamma}$ is a plane curve. In this case, the image of $\boldsymbol{\gamma}$ is the intersection of M with the tangent plane. Since M is a developable surface, it is a ruling. For the converse, we assume that $\boldsymbol{\gamma}$ is a ruling of M. Since $\boldsymbol{\gamma}$ is a line in $\mathbb{R}^3$, $\boldsymbol{t}$ is a constant vector. The assumption that M is a developable surface implies that $\boldsymbol{n}$ along a ruling is constant. By the the Frenet–Serret type formulae, $\kappa_n \equiv 0$ and $\tau_g \equiv 0$. ∎

Corollary 6.4. *Let $M \subset \mathbb{R}^3$ be a regular surface and $\boldsymbol{\gamma} : I \to M \subset \mathbb{R}^3$ be a unit speed curve on M. If there are two osculating developable surfaces along $\boldsymbol{\gamma}$, then $\boldsymbol{\gamma}$ is a straight line.*

Proof. Under the assumption of $\kappa_n^2 + \tau_g^2 \neq 0$, the osculating developable surface along $\boldsymbol{\gamma}$ is unique by Corollary 6.2. If $\kappa_n \equiv 0$ and $\tau_g \equiv 0$, $\boldsymbol{\gamma}$ is a plane curve. In this case, the tangent plane of M at $\boldsymbol{\gamma}(s_0)$ is an osculating developable surface along $\boldsymbol{\gamma}$. If there is another osculating developable surface along $\boldsymbol{\gamma}$, $\boldsymbol{\gamma}$ is a ruling of this developable surface by Theorem 6.3. If $\kappa_n = \tau_g = 0$ at an isolated point s_0, then the uniqueness of the osculating developable surface holds for a neighborhood of s_0 in I except at s_0. Passing to the limit $s \to s_0$, the uniqueness to the osculating developable surface holds at s_0. This completes the proof. ∎

Example 6.5. Let $T \subset \mathbb{R}^3$ be a torus of revolution of a circle. If $\boldsymbol{\gamma}$ is the circle consists of parabolic points (i.e., the Gaussian curvature vanishes along $\boldsymbol{\gamma}$), there exists the unique tangent plane along $\boldsymbol{\gamma}$. Since a plane is a developable surface, it is the unique osculating developable surface along $\boldsymbol{\gamma}$. A circle is a planar curve, so that $\kappa_n \equiv \tau_g \equiv 0$. However, the uniqueness of the osculating developable surface holds.

6.2. Curves on the unit sphere. In this subsection, we consider the case when M is the unit sphere $S^2 = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \|\boldsymbol{x}\| = 1\}$. Let $\gamma : I \to S^2 \subset \mathbb{R}^3$ be a unit speed curve. In this case, we have $\kappa_n(s) \equiv 1$ or $\kappa_n(s) \equiv -1$. The Darboux frame along $\boldsymbol{\gamma}$ is $\{\boldsymbol{\gamma}, \boldsymbol{t}, \boldsymbol{b}\}$ which is called the *Saban frame*. The

Frenet–Serret type formulae is as follows:

$$\begin{pmatrix}\boldsymbol{\gamma}' \\ \boldsymbol{t}' \\ \boldsymbol{b}'\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 \\ 1 & 0 & \kappa_g \\ 0 & \kappa_g & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma} \\ \boldsymbol{t} \\ \boldsymbol{b}\end{pmatrix}.$$

It follows that $D(s) = \mp\boldsymbol{b}(s)$ and $OD_{\boldsymbol{\gamma}}(s,u) = \boldsymbol{\gamma}(s) \mp u\boldsymbol{b}(s)$ Therefore, we have

$$\delta(s) = \kappa_g(s),\ \sigma(s) = \pm\frac{\kappa_g'(s)}{\kappa_g^2(s)}.$$

Then we have the following theorem as a corollary of Theorem 3.3.

Theorem 6.6. *Let $\boldsymbol{\gamma} : I \to S^2 \subset \mathbb{R}^3$ be a unit speed curve. Then we have the following:*

(1) *$(OD_{\boldsymbol{\gamma}}, (s_0, u_0))$ is regular if and only if $\pm 1 + u_0\kappa_g(s_0) \neq 0$.*
(2) *The image of $(OD_{\gamma}, (s_0, u_0))$ is locally diffeomorphic to the cuspidaledge $C \times \mathbb{R}$ if $\kappa_g(s_0) \neq 0, \kappa_g'(s_0) \neq 0$ and $u_0 = \mp 1/\kappa_g(s_0)$.*
(3) *The image of $(OD_{\gamma}, (s_0, u_0))$ is locally diffeomorphic to the swallowtail if $\kappa_g(s_0) \neq 0, \kappa_g'(s_0) = 0, \kappa_g''(s_0) \neq 0$ and $u_0 = \mp 1/\kappa_g(s_0)$.*

Proposition 6.7. *Let $\boldsymbol{\gamma} : I \to S^2 \subset \mathbb{R}^3$ be a unit speed curve.*

(1) *If $\boldsymbol{\gamma}$ is a great circle, then $OD_{\boldsymbol{\gamma}}$ is a circular cylinder.*
(2) *If $\boldsymbol{\gamma}$ is a small circle, then $OD_{\boldsymbol{\gamma}}$ is a circular cone.*

Proof. Suppose that $\boldsymbol{\gamma}$ is a great circle. Then $\kappa_g(s) \equiv 0$ and $\boldsymbol{b}(s)$ is constant. Therefore, $OD_{\boldsymbol{\gamma}}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{b}$ is a circular cylinder tangent to S^2 along $\boldsymbol{\gamma}$. Suppose that $\boldsymbol{\gamma}$ is a small circle. Then $\kappa_g'(s) \equiv 0$, so that $\sigma(s) \equiv 0$. It follows that $OD_{\boldsymbol{\gamma}}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{b}(s)$ is a cone tangent to S^2 along $\boldsymbol{\gamma}$. ■

We give some examples of curves on a unit sphere.

Example 6.8. We consider a space curve $\gamma : I \to S^2 \subset \mathbb{R}^3$ defined by

$$\boldsymbol{\gamma}(t) = \left(\frac{1}{\sqrt{3}}\cos t, \frac{1}{\sqrt{2}}\sin t, \sqrt{\frac{3+\cos^2 t}{6}}\right).$$

We remark that the image of $\boldsymbol{\gamma}$ is given by the intersection of S^2 with the elliptic cylinder parametrized by

$$\boldsymbol{p}(u,v) = \left(\frac{1}{\sqrt{3}}\cos u, \frac{1}{\sqrt{2}}\sin u, v\right).$$

By straightforward calculations, we have

$$\boldsymbol{t}(t) = \sqrt{\frac{7+\cos 2t}{3+\cos 2t}}\left(\frac{1}{\sqrt{3}}\sin t, \frac{1}{\sqrt{2}}\cos t, \frac{\cos t \sin t}{\sqrt{3(7+\cos 2t)}}\right),$$

$$\boldsymbol{b}(t) = \sqrt{\frac{1}{3+\cos 2t}}\left(\frac{4}{\sqrt{6}}\cos t, \sin t, \sqrt{\frac{7+\cos 2t}{6}}\right).$$

Moreover, the geodesic curvature of γ and it's derivative are

$$\kappa_g(t) = \frac{4\sqrt{2}}{(3+\cos 2t)^{\frac{3}{2}}}, \quad \kappa_g'(t) = \frac{12\sqrt{2}\sin 2t\sqrt{7+\cos 2t}}{(3+\cos 2t)^3}.$$

We have the swallowtail singularities at $t = \pi/2, 0, \pi/2, \pi$ (see, Fig. 3).

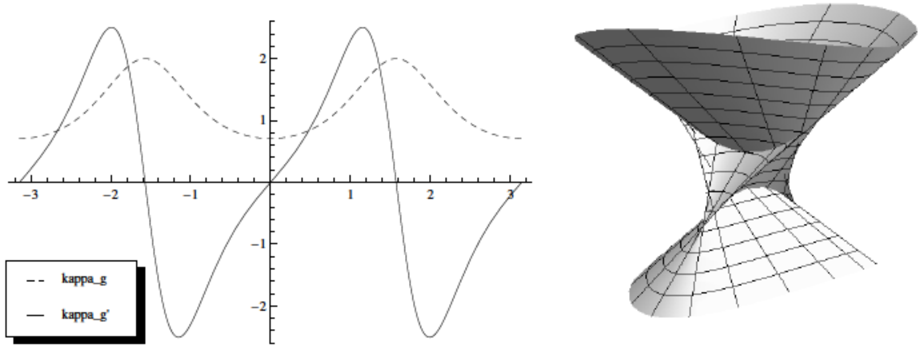

Fig. 3. The graphs of $\kappa_g(t), \kappa_g'(t)$ and the image of OD_γ for $0 \le u \le 2$

Example 6.9. We consider a space curve $\gamma : I \to S^2 \subset \mathbb{R}^3$ defined by

$$\gamma(t) = \left(\frac{8\sqrt{3}\cos t}{26+\sin^2 t}, \frac{12\sqrt{2}\sin t}{26+\sin^2 t}, \frac{22+\sin^2 t}{26+\sin^2 t}\right).$$

We remark that the image of γ is given by the inverse stereographic image of the ellipse

$$\left(\frac{1}{\sqrt{3}}\cos t, \frac{1}{\sqrt{2}}\sin t, 1\right)$$

on the plane $x_3 = 1$. Here, the center of the stereographic projection is $(0,0,1)$. The pictures of graphs of the geodesic curvature κ_g, it's derivative κ_g' and the image of OD_γ are drawn as follows:

We can show that there are swallowtail singularities at $t = \pi/2, 0, \pi/2, \pi$.

6.3. Curves on the graph of a function. In this subsection, we consider curves on the graph of a function. Let $\boldsymbol{X} : U \to \mathbb{R}^3$ be a surface defined by $\boldsymbol{X}(x,y) = (f(x,y), x, y)$ for a C^∞-function. In this case, the unit normal vector field is

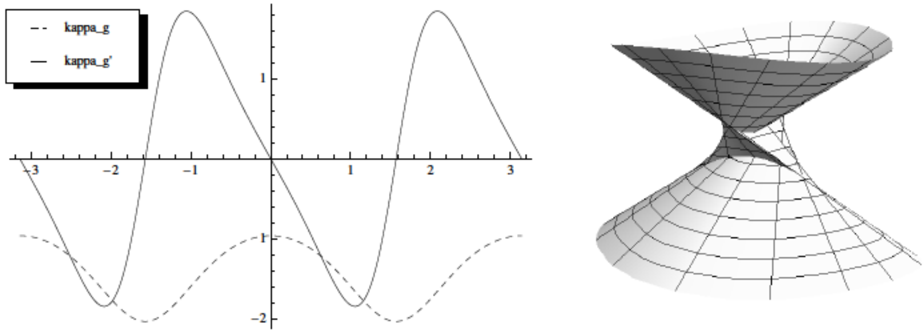

Fig. 4. The graphs of $\kappa_g(t), \kappa_g'(t)$ and the image of OD_γ for $\quad 3/2 \leq u \leq 0$

$$\boldsymbol{n} = \frac{1}{\sqrt{1+f_x^2+f_y^2}}(1, \quad f_x, \quad f_y).$$

We now consider a curve γ on the surface $M = \boldsymbol{X}(U)$ defined by $\gamma(y) = \boldsymbol{X}(0, y) = (f(0, y), 0, y)$. Then the Darboux frame consists of

$$\begin{aligned}
\boldsymbol{n}_\gamma(y) &= \frac{1}{\sqrt{1+f_x^2+f_y^2}}(1, \quad f_x, \quad f_y), \\
\boldsymbol{t}(y) &= \frac{1}{\sqrt{1+f_y^2}}(f_y, 0, 1), \\
\boldsymbol{b}(y) &= \boldsymbol{n}_\gamma(y) \times \boldsymbol{t}(y) = \frac{1}{\sqrt{1+f_y^2}\sqrt{1+f_x^2+f_y^2}}(\quad f_x, \quad f_y^2 \quad 1, f_x f_y).
\end{aligned}$$

We denote that $\boldsymbol{t}'(y) = (d\boldsymbol{t}/ds)(s(y))$, where s is the arc-length parameter. Then we have

$$\begin{aligned}
\boldsymbol{t}'(y) &= \frac{(f_{yy}, \ 0, \quad f_y f_{yy})}{(1+f_y^2)^2}, \\
\boldsymbol{n}_\gamma'(y) &= \frac{(\quad f_x f_{xy} \quad f_y f_{yy}, \ f_x f_y f_{yy} \quad f_{xy} \quad f_y^2 f_{xy}, \ f_x f_y f_{xy} \quad f_{yy} \quad f_x^2 f_{yy})}{(1+f_y^2)^{\frac{1}{2}}(1+f_x^2+f_y^2)^{\frac{3}{2}}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\kappa_g(y) &= \langle \boldsymbol{t}', \boldsymbol{b} \rangle = \quad \frac{f_x f_{yy}}{(1+f_y^2)^{\frac{3}{2}}(1+f_x^2+f_y^2)^{\frac{1}{2}}}, \\
\kappa_n(y) &= \langle \boldsymbol{t}', \boldsymbol{n}_\gamma \rangle = \frac{f_{yy}}{(1+f_y^2)(1+f_x^2+f_y^2)^{\frac{1}{2}}}, \\
\tau_g(y) &= \quad \langle \boldsymbol{n}_\gamma', \boldsymbol{b} \rangle = \frac{f_x f_y f_{yy} \quad f_{xy}(1+f_y^2)}{(1+f_y^2)(1+f_x^2+f_y^2)}.
\end{aligned}$$

Taking the derivatives of the above, the condition $f_x(0,0) = f_y(0,0) = 0$ induces

$$
\begin{aligned}
\kappa_g(0) &= 0,\\
\kappa_g'(0) &= \quad f_{yy}f_{xy},\\
\kappa_g''(0) &= \quad 2f_{yyy}f_{xy} \quad f_{yy}f_{xyy},\\
\kappa_g'''(0) &= \quad 3f_{yyy}f_{xyy} \quad f_{xyyy}f_{yy} + 3f_{yy}f_{xy}^3 \quad (3f_{yyyy} \quad 13f_{yy}^3)f_{xy},\\
\kappa_n(0) &= f_{yy},\\
\kappa_n'(0) &= f_{yyy},\\
\kappa_n''(0) &= f_{yyyy} \quad f_{yy}f_{xy}^2 \quad 3f_{yy}^3,\\
\kappa_n'''(0) &= f_{yyyyy} \quad 3f_{yy}f_{xy}f_{xyy} \quad f_{yyy}(19f_{yy}^2 + 3f_{xy}^2),\\
\tau_g(0) &= \quad f_{xy},\\
\tau_g'(0) &= \quad f_{xyy},\\
\tau_g''(0) &= 4f_{yy}^2f_{xy} \quad f_{xyyy} + 2f_{xy}^3,\\
\tau_g'''(0) &= 2f_{xyy}(5f_{yy}^2 + 6f_{xy}^2) + 15f_{yyy}f_{yy}f_{xy} \quad f_{xyyyy}.
\end{aligned}
$$

Moreover, we consider the derivatives of δ and σ, so that

$$
\begin{aligned}
\delta(0) &= \frac{f_{yyy}f_{xy} \quad f_{yy}f_{xyy}}{f_{yy}^2 + f_{xy}^2},\\
\delta'(0) &= \Big\{ f_{yyyy}f_{xy}^3 + f_{yy}^2(2f_{yyy}f_{xyy} \quad f_{xyyy}f_{yy})\\
&\qquad + f_{yy}f_{xy}(\quad 2f_{yyy}^2 + f_{yyyy}f_{yy} + 2f_{xyy}^2)\\
&\qquad\quad f_{xy}^2(2f_{yyy}f_{xyy} + f_{xyyy}f_{yy})\Big\}(f_{yy}^2 + f_{xy}^2)^{\quad 2},\\
\delta''(0) &= \Big\{ f_{yyy}f_{xy}^7 \quad f_{yy}f_{xy}^6f_{xyy} + f_{xy}^5(f_{yyyyy} + 2f_{yyy}f_{yy}^2)\\
&\qquad + f_{yy}^3\big(f_{yy}(3f_{yyy}f_{xyyy} \quad f_{xyyyy}f_{yy}) + f_{xyy}(3f_{yyyy}f_{yy} \quad 6f_{yyy}^2) + 2f_{xyy}^3\big)\\
&\qquad\quad f_{xy}^4\big(3f_{yyy}f_{xyyy} + f_{xyyyy}f_{yy} + f_{xyy}(3f_{yyyy} + 2f_{yy}^3)\big)\\
&\quad + f_{xy}^3\big(6f_{yyy}f_{xyy}^2 \quad 2f_{yyy}^3 + 6f_{yy}(f_{xyyy}f_{xyy} \quad f_{yyy}f_{yyyy}) + f_{yyy}f_{yy}^4 + 2f_{yyyyy}f_{yy}^2\big)\\
&\quad + f_{yy}^2f_{xy}\big(\quad 18f_{yyy}f_{xyy}^2 + 6f_{yyy}^3 + 6f_{xyyy}f_{yy}f_{xyy} \quad 6f_{yyyy}f_{yyy}f_{yy} + f_{yyyyy}f_{yy}^2\big)\\
&\qquad\quad f_{yy}f_{xy}^2\big(2f_{xyyyy}f_{yy}^2 + f_{xyy}(6f_{xyy}^2 \quad 18f_{yyy}^2 + f_{yy}^4)\big)\Big\}(f_{yy}^2 + f_{xy}^2)^{\quad 3},\\
\sigma(0) &= \frac{\big(\quad 2f_{yyy}^2f_{xy} \quad f_{xyyy}f_{yy}^2 + f_{yy}(2f_{yyy}f_{xyy} + f_{yyyy}f_{xy})\big)\sqrt{f_{yy}^2 + f_{xy}^2}}{(f_{yy}f_{xyy} \quad f_{yyy}f_{xy})^2},
\end{aligned}
$$

$$\begin{aligned}
\sigma'(0) = \Big\{ & f_{xy}^4\big(f_{yyyy}(f_{yyy}^2 \quad 2f_{yyyy}f_{yy}) + f_{yyy}f_{yyyyy}f_{yy}\big) \\
& + f_{yy}^3\big(\quad 2f_{xyyy}^2 f_{yy}^2 \quad f_{xyy}^2(2f_{yyy}^2 + 3f_{yyyy}f_{yy}) \\
& \qquad + f_{yy}f_{xyy}(5f_{yyy}f_{xyyy} + f_{xyyyy}f_{yy})\big) \\
& \quad f_{xy}^3\big(f_{yy}(4f_{xyyy}(f_{yyy}^2 \quad f_{yyyy}f_{yy}) + f_{yyy}f_{xyyyy}f_{yy}) \\
& \qquad + f_{xyy}(2f_{yyy}^3 \quad 3f_{yyyy}f_{yyy}f_{yy} + f_{yyyyy}f_{yy}^2)\big) \\
& + f_{yy}f_{xy}^2\big(\quad 2f_{yyy}^4 \quad 2f_{yy}^2(f_{yyyy}^2 + f_{xyyy}^2) + 2f_{yyyy}f_{yyy}^2 f_{yy} \\
& \quad + 4f_{xyy}^2(f_{yyy}^2 \quad f_{yyyy}f_{yy}) \\
& \quad + f_{yy}f_{xyy}(3f_{yyy}f_{xyyy} + f_{xyyyy}f_{yy}) + f_{yyy}f_{yyyyy}f_{yy}^2\big) \\
& + f_{yy}^2 f_{xy}\big(\quad 2f_{yyy}f_{xyy}^3 + f_{xyyy}f_{yy}f_{xyy}^2 + f_{yy}f_{xyyy}(4f_{yyyy}f_{yy} \quad 5f_{yyy}^2) \\
& \qquad f_{yyy}f_{xyyyy}f_{yy}^2 + f_{xyy}(4f_{yyy}^3 + f_{yyyy}f_{yyy}f_{yy} \quad f_{yyyyy}f_{yy}^2)\big)\Big\} \\
& \qquad \cdot (f_{yy}^2 + f_{xy}^2) \quad ^{\frac{1}{2}} (f_{yyy}f_{xy} \quad f_{yy}f_{xyy}) \quad ^{3}.
\end{aligned}$$

Example 6.10. We consider the case $f(x,y) = axy + by^4$. Then $f_{xy} = a, f_{yyyy} = b$, so that

$$\delta(0) = 0, \quad \delta'(0) = \frac{b}{a}.$$

If $a \neq 0, b \neq 0$, then $\delta(0) = 0, \delta'(0) \neq 0$. Since $\kappa_n(0) = 0$, it is the case (2)(ii) in Theorem 3.3. In the case $a = 2, b = 1$, we can draw the pictures as follows:

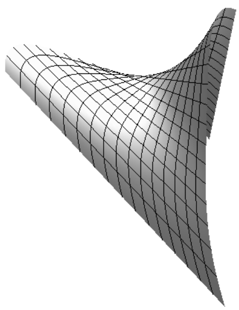

Fig. 5. The surface of 6.10

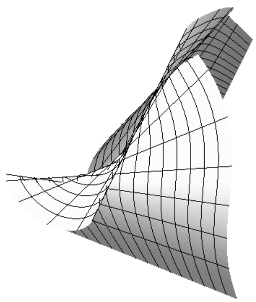

Fig. 6. The osculating developable of the surface 6.10 along $\boldsymbol{\gamma}$.

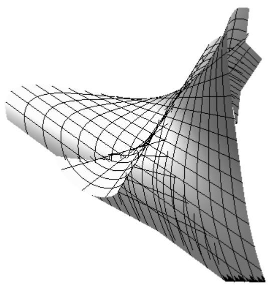

Fig. 7. The surface of 6.10 and its' osculating developable along $\boldsymbol{\gamma}$.

It seems that the osculating developable along $\boldsymbol{\gamma}$ is the Mond surface (i.e., the cuspidal beaks) (cf., [7]). Actually, we can show that it is the cuspidal beaks by using the criteria in [6].

Example 6.11. We consider the case $f(x,y) = axy + by^3$. Then $f_{xy} = a, f_{yyy} = b$, so that $\delta(0) = b/a$. Then $\delta(0) \neq 0$ if and only if $a \neq 0$ and $b \neq 0$. We also have $\sigma(0) = \quad 2|a|/a = \mp 2$.

In this case, we can draw the pictures as follows:

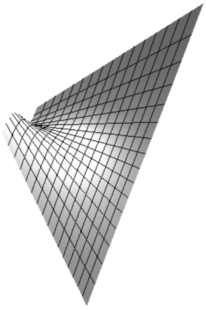

Fig. 8. The surface of Example 6.11

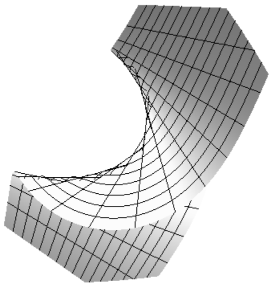

Fig. 9. The osculating developable of Example 6.11

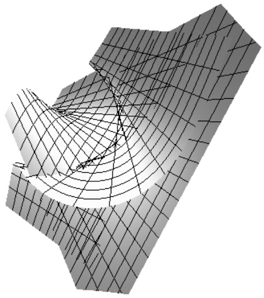

Fig. 10. The surface of Example 6.11 and the osculating developable along γ

The osculating developable along γ has the cuspidaledge.

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