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CENTRE SYMMETRY SETS OF FAMILIES OF PLANE CURVES

This article is dedicated to the memory of Vladimir M. Zakalyukin

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Abstract. We study centre symmetry sets and equidistants for a 1-parameter family of plane curves where, for a special member of the family, there exist two inflexions with parallel tangents. Some results can be obtained by reducing a generating family to normal forms, but others require direct calculation from the generating family.

1. Introduction

The *centre symmetry set* (CSS) of a hypersurface M in $\mathbb{R}^{k+1}$ is the envelope of (infinite) straight lines joining pairs of points of M with parallel tangent hyperplanes, or "parallel tangent chords" as we shall call them. The CSS, which is invariant under affine transformations of $\mathbb{R}^{k+1}$, has been studied in detail for many cases in, for example, [10, 7, 8, 9, 6]. In this article, we are principally concerned with $k = 1$, that is a plane curve M, but allowing the curve to vary in a generic 1-parameter family. For a generic smooth closed plane curve, the *inflexion points* (where the tangent line has at least 3-point contact) will all be ordinary (the contact is exactly 3-point) and no two will have parallel tangent lines. However, for a special member of a generic 1-parameter family, there can exist two inflexion points of M with parallel tangent lines. This introduces some features of the CSS which are not present for a generic curve, such as a *supercaustic*, introduced in [11], which we define and investigate in §2. We are interested in how the CSS evolves in such a family, and also in how the *equidistants* evolve – an equidistant is the set of points of the form $(1 \quad \lambda)a + \lambda b$ where λ is fixed and a, b are distinct points of M at which the tangent hyperplanes are parallel.

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The generating function method for investigating the CSS was introduced in [8]. In the context of a smooth parametrized plane curve $\gamma : S^1 \to \mathbb{R}^2$ it is as follows. Consider the function

$$\begin{aligned} &F : S^1 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}, \\ &F(\boldsymbol{n}, s, t, \lambda, \boldsymbol{x}) = (1 \quad \lambda)\langle \gamma(s) \quad \boldsymbol{x}, \boldsymbol{n}\rangle + \lambda\langle \gamma(t) \quad \boldsymbol{x}, \boldsymbol{n}\rangle. \end{aligned} \tag{1}$$

Here s, t are parameter values for γ, $\boldsymbol{n}$ is a unit vector in $\mathbb{R}^2$ and $\langle \ , \ \rangle$ can be interpreted as scalar product of vectors. We shall use here only a multi-local form of F: we choose two base values $s_0 \neq t_0$ where the tangents $\gamma'(s_0), \gamma'(t_0)$ are parallel, and s, t will be close to these base values. We denote by M a neighbourhood of $\gamma(s_0)$ on the curve, and by N a neighbourhood of $\gamma(t_0)$. Then, using subscripts to denote partial derivatives, the set of points

$$\Sigma_F = \{(\lambda, \boldsymbol{x}) : \exists\ (\boldsymbol{n}, s, t) \text{ with } F = F_{\boldsymbol{n}} = F_s = F_t = 0\}$$

consists of (i) points of the form $(0, \gamma(s))$ and $(1, \gamma(t))$, and (ii) points $(\lambda, (1 \quad \lambda)\gamma(s) + \lambda\gamma(t))$ where the tangents $\gamma'(s), \gamma'(t)$ are parallel (both perpendicular to $\boldsymbol{n}$). Thus, Σ_F is the union of all parallel tangent chords of γ close to the base pair, spread out in the λ-direction, together with copies of M and N.

Let $\lambda = \lambda_0 + \alpha$ where α is small and $\lambda_0 \neq 0, 1$. Setting up coordinates as in Figure 1, right, with $s_0 = t_0 = 0$, and two curve pieces given by $M : (s, f_2 s^2 + f_3 s^3 + \ldots)$ and $N : (t, 1 + g_2 t^2 + g_3 t^3 + \ldots)$, with $\boldsymbol{n} = (n, 1)$, we find that Σ_F is smooth at $(\lambda_0, (0, \lambda_0))$ unless $(1 \quad \lambda_0)g_2 + \lambda_0 f_2 = 0$. If f_2 and g_2 are nonzero and distinct then this gives the unique CSS point $(0, \lambda_0) = (0, g_2/(g_2 \quad f_2))$ on the chord joining the two basepoints (compare for example [7, Th.4]; if $f_2 = g_2 \neq 0$ the CSS point is at infinity). See Figure 1. In general, setting aside the values $\lambda_0 = 0, 1$, the CSS can be computed as a caustic: the set of critical values of the projection of Σ_F to $\boldsymbol{x}$.

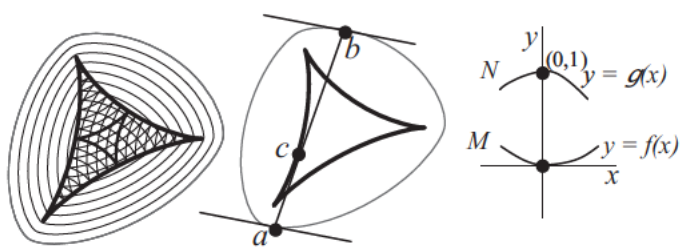

Fig. 1. Left: a closed curve, several equidistants, with the "half-way equidistant" shown by a heavier line, and the CSS, the outer 3-cusped curve passing through the cusps of the equidistants. Centre: two parallel tangents, at a and b, with the chord joining them, tangent to the CSS at c, the position being determined by the ratio of (euclidean) curvatures at a and b. Right: the standard setup for studying the CSS or the equidistants close to a particular parallel tangent chord.

The case of interest to us in this article, however, is $f_2 = g_2 = 0$, that is, both basepoints are inflexion points; then the above equation suggests that every point of the chord joining them "contributes to the CSS". We trace this back in §2 to the existence of supercaustics, and give a more general exposition.

It is notable that for the case $f_2 = g_2 = 0$, as a member of a generic family of curves, we have found that some arguments work well in analogy with those in the earlier works cited above, by reduction of the generating family to an appropriate "normal form", while for others a much more "hands-on" approach appears to be needed. In particular, we shall encounter some very degenerate situations where normal forms do not appear to help.

The remainder of the article is organized as follows. In §2, we study supercaustics in more detail than is needed for our main application. In §3, we study the CSS of a family of curves γ_ε, parametrized by ε, which contains a member γ_0 with parallel but distinct tangents at inflexion points. In particular, we show that the union of the CSS for all small ε – the "big CSS" – is a cuspidal edge surface, but with the function ε, whose level sets are the separate CSS, being very degenerate. In §4, we study families of equidistants associated with a fixed γ_ε and close to certain special values of λ . In §5, we show how, in some situations, it is possible to reduce the generating family to a normal form. These allow us to recognize the big CSS and the evolution of the momentary CSS as ε changes, but unfortunately *not* the momentary CSS in the parallel inflexional tangents case (Propositions 5.8 and 5.9). We also identify the "big equidistant" and evolution as ε changes of the momentary equidistants for a fixed value of λ away from the special values (Proposition 5.10).

2. Supercaustics

When we investigate the CSS of two parametrized hypersurfaces $M = \{(\mathbf{s}, f(\mathbf{s}))\}$ and $N = \{(\mathbf{t}, g(\mathbf{t}))\}$ in $\mathbb{R}^{k+1}$ by means of the generating function

$$F(\boldsymbol{n}, \mathbf{s}, \mathbf{t}, \lambda, \boldsymbol{x}) = (1 \quad \lambda)\langle(\mathbf{s}, f(\mathbf{s})) \quad \boldsymbol{x}, \boldsymbol{n}\rangle + \lambda\langle(\mathbf{t}, g(\mathbf{t})) \quad \boldsymbol{x}, \boldsymbol{n}\rangle,$$

we consider the set $\mathcal{F}^{-1}(\mathbf{0})$, or its projection to $(\lambda, \boldsymbol{x})$-space, where

$$\mathcal{F} : \mathbb{R}^{4k+2} \to \mathbb{R}^{3k+1}, \quad \mathcal{F}(\boldsymbol{n}, \mathbf{s}, \mathbf{t}, \lambda, \boldsymbol{x}) = (F, F_{\boldsymbol{n}}, F_{\mathbf{s}}, F_{\mathbf{t}}). \tag{2}$$

It can happen that $\mathcal{F}^{-1}(\mathbf{0})$ is itself singular. This will occur when the rank of the Jacobian of $\mathcal{F}$ is less than $3k + 1$.

Definition 2.1. (See [11].) The *supercaustic* of the pair (M, N) is the projection to $(\lambda, \boldsymbol{x})$-space of the set of singular points of $\mathcal{F}^{-1}(\mathbf{0})$. This always includes $\lambda = 0, \boldsymbol{x} \in M$ and $\lambda = 1, \boldsymbol{x} \in N$, so we regard these as "trivial" parts and we are interested in the rest of the supercaustic, when this exists.

We shall see that, for generic M and N, the non-trivial supercaustic is empty, but that it can be non-empty in a generic 1-parameter family. In this article, we are principally concerned with the case $k = 1$, that is plane curves, but we shall state a more general version of the condition for the supercaustic to be non-empty. We write $\boldsymbol{n} = (n_1, n_2, \ldots, n_k, 1)$.

To prepare for the statement, we consider the case $k = 2$. We use the parametrizations

$$f(s_1, s_2) = f_{20}s_1^2 + f_{11}s_1s_2 + f_{02}s_2^2 + \ldots,$$
$$g(t_1, t_2) = 1 + g_{20}t_1^2 + g_{11}t_1t_2 + g_{02}t_2^2 + \ldots.$$

Writing down the Jacobian of $\mathcal{F}$ and evaluating at the basepoints $n_1 = n_2 = 0, s_1 = s_2 = 0, t_1 = t_2 = 0$, we obtain the 7×10 matrix

$$J_2 = \begin{pmatrix} x & y & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & \lambda & 1 & 0 & \lambda & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 & \lambda & 0 & 0 & 1 & 0 \\ \lambda & 1 & 0 & 2(\lambda & 1)f_{20} & (\lambda & 1)f_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & (\lambda & 1)f_{11} & 2(\lambda & 1)f_{02} & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 & 0 & 2\lambda g_{20} & \lambda g_{11} & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & \lambda g_{11} & 2\lambda g_{02} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is clear that, for any k, the only nonzero entries in the last $k + 1$ columns will occur, as in the case J_2, in the positions corresponding to $\mathcal{F}_{n_1x_1}, \mathcal{F}_{n_2x_2}, \ldots \mathcal{F}_{n_kx_k}, \mathcal{F}_{x_{k+1}}$. The last $k+1$ columns and the first $k+1$ rows can therefore be deleted, reducing the rank by $k + 1$, and column $3k + 1$ now consists of zeros and can be deleted without changing the rank. After performing row operations on the reduced $2k \times 3k$ we obtain, for the above case $k = 2$, and assuming $\lambda \neq 0, \lambda \neq 1$,

$$\begin{pmatrix} 0 & 0 & 2f_{20} & f_{11} & 2g_{20} & g_{11} \\ 0 & 0 & f_{11} & 2f_{02} & g_{11} & 2g_{02} \\ 1 & 0 & 0 & 0 & 2g_{20} & g_{11} \\ 0 & 1 & 0 & 0 & g_{11} & 2g_{02} \end{pmatrix},$$

from which the first two columns and the last two rows can be removed, reducing the rank by 2 (in general by k). The final matrix is $k \times 2k$ and has the form, removing the minus signs and factors of 2, $(A|B)$ where A is the symmetric matrix of the quadratic form of f and B is that of g. This has rank $< k$, and therefore the original Jacobian has rank $< k + k + k + 1 = 3k + 1$ if and only if $\lambda = 0$, $\lambda = 1$ or every $k \times k$ minor of $(A|B)$ is zero.

For the surface case $k = 2$, this implies (taking $\lambda \neq 0, 1$) that the basepoints $(0, 0, 0)$ on M and $(0, 0, 1)$ on N are both parabolic. Then we may choose the unique asymptotic direction on M at $(0, 0, 0)$ to be $(1, 0, 0)$

and considering the other 2×2 minors, it follows that this is also the unique asymptotic direction on N at $(0, 0, 1)$.

For the curve case $k = 1$, we deduce similarly that both points $(0, 0)$ on M and $(0, 1)$ on N are inflexions. For $k = 2$ the existence of parabolic points with parallel tangent planes is a generic condition, requiring four conditions with four degrees of freedom, but requiring also that the asymptotic directions are parallel is an additional condition which requires, in general, a 1-parameter family of surfaces to realize. Likewise, for $k = 1$, parallel tangents at inflexions occur only in a 1-parameter family of curves. The general case can be stated as follows.

Theorem 2.2. *For generic M and N, and away from $\lambda = 0, \lambda = 1$, the supercaustic is empty but can be nonempty for a 1-parameter family of k-manifolds in $\mathbb{R}^{k+1}$.*

For $k = 1$, the condition for the rank of $\mathcal{F}$ to drop below its maximum is that the basepoints on M and N are both inflexions (and have parallel tangent lines);

For $k = 2$, the condition is that the basepoints are parabolic points with parallel asymptotic directions (and parallel tangent planes).

For general k, the condition is that the $k \times 2k$ matrix $(A|B)$ should have rank $< k$, where A, B are the $k \times k$ matrices of the quadratic forms of M and N at the basepoints (that is, the quadratic forms of the parametrizing functions f and g). This can be expressed by saying that the second fundamental forms share a common kernel vector.

The supercaustic itself, in $(\lambda, \boldsymbol{x})$-space, then consists locally of all points of the form $(0, \boldsymbol{x}), \boldsymbol{x} \in M$ or $(1, \boldsymbol{x}), \boldsymbol{x} \in N$ (the trivial parts) or $(\lambda, (0, 0, \ldots, 0, \lambda))$.

Projecting to $\boldsymbol{x}$-space, we obtain $M \cup N \cup \{(0, \ldots, 0, \lambda)\}$. □

Remark 2.3. The same result holds for the case where M, N share the same tangent (hyper)-plane $x_{k+1} = 0$, being tangent to it at distinct points, say $(0, 0, \ldots, 0)$ and $(1, 0, \ldots, 0)$.

When we investigate the centre symmetry set of a pair of curves having a supercaustic, both for itself and as part of a 1-parameter family, we shall need the pairs of parallel tangent pairs close to those at the inflexion points. We pause here to describe these pairs, and extend the description to the case $k = 2$ of surfaces. Thus we ask the following.

> $k = 1$. Suppose that the basepoints on curves M and N are inflexions with parallel tangent lines. What are the nearby points on M and N with parallel tangent lines?
> $k = 2$. Suppose that the basepoints on surfaces M and N are parabolic with parallel asymptotic directions. What are the

nearby pairs of points on M and N which have parallel tangent planes?

For the case $k = 1$, and curves $y = f(x), y = g(x)$ with parallel inflexional tangents at $(0,0)$ and $(0,1)$, it is easy to see that the signs of $f''(0)$ and $g''(0)$ determine the nature of the nearby parallel tangents, as in Figure 2.

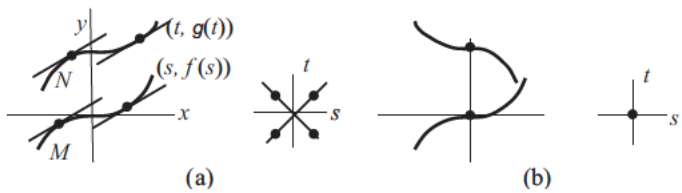

Fig. 2. Two inflexions with parallel tangents, (a) with the same orientations, that is $f''(0)g''(0) > 0$, and (b) with opposite orientations, that is $f''(0)g''(0) < 0$. In (a) there are sets of four nearby points all with parallel tangents, while in (b) there is no pair with parallel tangents apart from those at the inflexions. The (s,t) diagrams represent pairs (s,t) giving parallel tangents.

For surfaces ($k = 2$) the situation is more interesting. Consider two surfaces M, N with parabolic points at $(0,0,0)$ and $(0,0,1)$, having the same asymptotic direction $(0,1,0)$ there, so that the surfaces have the form $z = f_{20}x^2 +$ h.o.t. and $z = 1 + g_{20}x^2 +$ h.o.t. Consider next the (modified) Gauss maps of these surfaces, $(x,y) \mapsto (f_x, f_y)$ and $(x,y) \mapsto (g_x, g_y)$, defined close to the basepoints $(0,0)$ and $(0,1)$. The images of these, that is the fold lines of the Gauss map, are tangent at the origin as shown in Figure 3, where the arrows indicate the direction in which the Gauss map is a double cover. In all these diagrams, the image of the parabolic curve could be curved upwards or downwards; what matters is which image is "above" the other and the directions of the arrows. Some straightforward calculations show the following. We shall assume that $f_{20}, g_{20}, f_{03}, g_{03}$ are all nonzero, the last two conditions being those to avoid a cusp of Gauss on M or N (a further degeneracy). We also define, for M at $(0,0,0)$,

$$A = \frac{3f_{21}f_{03} \quad f_{12}^2}{f_{20}^2 f_{03}},$$

and similarly B for N at $(0,0,1)$. We then assume $A \neq B$ since if this fails, the two images in the Gauss sphere have inflexional contact. In fact, $A > B$ is the condition for the image for M to be "above" that for N.

Proposition 2.4.

(i) *When $f_{03}g_{03} > 0$ the situation is as in Figure 3(a),(c) with a locally connected region of the Gauss sphere occupied by parallel normals (that is parallel tangent planes) to M and N.*

(ii) *When $f_{03}g_{03} < 0$ and A B has the same sign as g_{03} then the situation is as in Figure 3(b), with two local regions of the Gauss sphere occupied by parallel normals to M and N other than those at the base points.*

(iii) *When $f_{03}g_{03} < 0$ and A B has the same sign as f_{03}, the situation is as in Figure 3(d), with no pairs of parallel normals apart from those at the base points.* □

Fig. 3. For two surfaces M and N, having (ordinary) parabolic points at the base points $(0,0,0)$ and $(0,0,1)$ with parallel tangent planes there, and parallel asymptotic directions, these show the images of the parabolic curves under the Gauss map; M could give the upper or the lower curve. The arrows point into the folds of the Gauss map and the images for M and N have ordinary contact. In cases (a), (b), (c) there will be pairs of points from M and N, close to the basepoints, with parallel tangent planes, while in case (d) there will not.

3. The CSS

The main focus of this article is on the centre symmetry sets of a generic 1-parameter family of curves, containing a special curve having two inflexions at which the tangents are distinct and parallel. We represent the special curve by a pair $M : y = f_0(x) = f_{30}x^3 + f_{40}x^4 + \ldots$, and $N : y = g_0(x) = 1 + g_{30}x^3 + g_{40}x^4 + \ldots$. On M, the parameter will be $x = s$ and on N, it will be $x = t$ while the family of curves will be parametrized by ε. Since nonsingular affine maps do not affect any of our results, we may use a 1-parameter family of such maps to reduce to the following.

Property 3.1. *For every ε close to 0, the curve $y = f(x,\varepsilon)$ has an inflexion at the origin with horizontal tangent there, and for every ε close to 0, the curve $y = g(x,\varepsilon)$ has an inflexion at $(0,1)$.*

In fact, we could impose a further condition, such as $f_{30} = 1$, but prefer to keep the symmetry of representation of f and g. The above allow us to write, up to order 5 in f and order 4 in g,

$$\begin{aligned} f(x,\varepsilon) &= x^3 f_1(x,\varepsilon) \\ &= f_{30}x^3 + f_{40}x^4 + f_{31}x^3\varepsilon + f_{50}x^5 + f_{41}x^4\varepsilon + f_{32}x^3\varepsilon^2 + \ldots, \\ g(x,\varepsilon) &= 1 + xg_1(\varepsilon) + x^3 g_2(x,\varepsilon) \\ &= 1 + g_{11}x\varepsilon + g_{30}x^3 + g_{12}x\varepsilon^2 + g_{40}x^4 + g_{31}x^3\varepsilon + g_{13}x\varepsilon^3 + \ldots, \end{aligned} \tag{3}$$

since $g_1(0) = 0$. When it is necessary to do calculations directly from the parametrizations, we shall use (3). We shall always assume that f_{30}, g_{30} and

g_{11} are nonzero; the last says in effect that the slope of the curve through $(0,1)$ is not stationary with respect to ε at $\varepsilon = 0$, while the first two say that the inflexions on M, N for $\varepsilon = 0$ are ordinary.

3.1. CSS of the base curves given by $\varepsilon = 0$. Note that we are concerned here with chords joining a point of $M = \{(s, f_0(s))\}$ and a point of $N = \{(t, g_0(t))\}$, where there are parallel tangent lines. We do not consider the contribution of chords joining two points of M (or of N) with parallel tangent lines, as in the left-hand diagram of Figure 2(a). This contribution is well-known and is described in, for example, [7, Sec. 4].

The CSS is the image in the (x, y)-plane of the critical set of the projection $\mathcal{F}^{-1}(\mathbf{0})$ to the (x, y)-plane. Since $\mathcal{F}^{-1}(\mathbf{0})$ is itself singular, the image of the singular set is included in the CSS and this is the y-axis together with the curves M ($\lambda = 0$) and N ($\lambda = 1$). For the rest of the CSS, the set of points (x, y) is obtained from the Jacobian matrix of $\mathcal{F}$ and comes to the following, where suffix s or t denotes differentiation.

$$(4) \qquad (x, y) = (1 - \lambda)(s, f_0(s)) + \lambda(t, g_0(t)) \qquad \text{where } f_{0s} = g_{0t} \text{ and } \lambda f_{0ss} + (1 - \lambda) g_{0tt} = 0.$$

Remark 3.2. This is nearly identical with the envelope of lines as obtained by the more traditional route, that is writing $L = 0$ for the equation of the line joining $(s, f_0(s))$ and $(t, g_0(t))$, $G = 0$ for the condition $f_{0s} - g_{0t} = 0$ and adding the "envelope" condition $L_s G_t - L_t G_s = 0$. But the latter definition does not automatically include M and N themselves.

For the case where $f_{30} g_{30} < 0$, there are no parallel tangents apart from those at the inflexion points (see Figure 2(b)), so the CSS in that case consists only of M, N and the y-axis.

Notation. For the case $f_{30} g_{30} > 0$, we may assume both are positive and write

$$(5) \qquad f_{30} = a_3^2, \ g_{30} = b_3^2 \text{ for some numbers } a_3 > 0, b_3 > 0, a_3 - b_3 \neq 0.$$

From (4), we find that the branches of the set $f_{0s} = g_{0t}$ are (compare the right–hand diagram of Figure 2(a))

$$t = \pm \frac{a_3}{b_3} s - \frac{2}{3} \frac{a_3^3 g_{40} \pm b_3^3 f_{40}}{a_3 b_3^4} s^2 + \dots,$$

and that these give branches of the CSS tangent to the y-axis:

$$(6) \qquad x = \frac{3}{8} \frac{a_3 b_3 (b_3 \mp a_3)^3}{a_3^3 g_{40} \mp b_3^3 f_{40}} \left(y - \frac{b_3}{b_3 \mp a_3} \right)^2 + \dots.$$

Definition 3.3. The two points on the y-axis at which these branches are tangent, namely

$$\left(0, \frac{b_3}{b_3 \quad a_3}\right) \text{ and } \left(0, \frac{b_3}{b_3 + a_3}\right)$$

are called *special points* and their y coordinates the *special values* of y, or of λ, since the structure of the CSS is different at these points.

Note that the special points can never coincide with $(0,0)$ or $(0,1)$; in fact the second special point lies between $(0,0)$ and $(0,1)$ and the first does not.

Hence:

Proposition 3.4. *The CSS of the base curves given by $\varepsilon = 0$ consists of the curves M and N (the "trivial" part), together with the y-axis and the two "parabolic" curves (6) tangent to the y-axis at the special points. The curves can be independently on either side of the y-axis.* □

Remark 3.5. The points of the envelope of a family of lines can "usually" be thought of as limits of intersections of line pairs of the family (see for example [5, Sec. 5.8]). So it is of interest to ask whether all the points of the above envelope are obtained in this way, as limits of intersections of pairs of parallel tangent chords. In fact, all of the envelope apart from the "trivial" components $M \cup N$ is obtained by such a limiting process.

Each small value of t gives two values of s close to 0 for which the tangents are parallel, as in Figure 2(a); let us take two such values of t, say t_1 and t_2, where $t_2 = kt_1$ and k is to be determined. Then t_1 has two corresponding s, say s_{11} and s_{12}, where $s_{12} < 0 < s_{11}$, and similarly t_2 has s_{21} and s_{22}, where $s_{21} < 0 < s_{22}$. Some calculation shows the following.

The limit of intersections of chords $t_1 s_{11}$ and $t_2 s_{21}$

$$\text{as } t_1 \to 0 \text{ is } \left(0, \frac{b_3}{b_3 \quad a_3}\right).$$

The limit of intersections of chords $t_1 s_{11}$ and $t_2 s_{22}$

$$\text{is } \left(0, \frac{b_3(k+1)}{b_3(k+1) + a_3(k \quad 1)}\right).$$

We can make the last expression equal to any value y_0 by taking

$$k = \frac{y_0(a_3 \quad b_3) + b_3}{y_0(a_3 + b_3) \quad a_3}.$$

For example, $y_0 = 0$ requires $k = \quad 1$, $y_0 = 1$ requires $k = 1$ and y_0 equal to one of the special values above requires $k = 0$ or $k = \infty$, the latter being interpreted as $t_1 = 0$.

Thus, every point of the y-axis is a limit of intersections of "nearby parallel tangent chords". The limits of the other two intersections, namely $t_1 s_{12}$, $t_2 s_{21}$ and $t_1 s_{12}, t_2 s_{22}$ trace out the remaining parts of the envelope, namely the smooth curves tangent to the y-axis at the special points, one of which is drawn as a solid line CSS_0 in Figure 5(a).

3.2. CSS of the family of curves. The CSS of the various curves of the family (3) is described by a surface in (x, y, ε)-space, the "big CSS", whose plane sections $\varepsilon =$ constant give the CSS of the individual curves. The CSS for $\varepsilon = 0$ was examined in the last section. We now consider the augmented function and map

$$\widetilde{F}(\boldsymbol{n}, s, t, \lambda, x, y, \varepsilon) = (1 \quad \lambda)\langle (s, f(s,\varepsilon)) \quad (x, y), \boldsymbol{n}\rangle + \lambda\langle (t, g(t,\varepsilon)) \quad (x, y), \boldsymbol{n}\rangle, \tag{7}$$

$$\widetilde{\mathcal{F}}(\boldsymbol{n}, s, t, \lambda, x, y, \varepsilon) = (\widetilde{F}, \widetilde{F}_{\boldsymbol{n}}, \widetilde{F}_s, \widetilde{F}_t). \tag{8}$$

We write $\boldsymbol{n} = (n, 1)$; then the Jacobian matrix of $\widetilde{\mathcal{F}}$ at $n = s = t = \varepsilon = 0$ is

$$\begin{pmatrix} x & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & \lambda & \lambda & 0 & 1 & 0 & 0 \\ 1 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 & 0 & \lambda g_{11} \end{pmatrix}. \tag{9}$$

Since $g_{11} \neq 0$ this has rank 4, provided $\lambda \neq 0, 1$. Hence $\widetilde{\mathcal{F}}^{\,1}(\mathbf{0})$ is a smooth 3-manifold in the source space in a neighbourhood of any point $((0,1), 0, 0, \lambda, x, y, 0)$ where $\lambda \neq 0, 1$. The critical set of the projection of this 3-manifold to (x, y, ε)-space requires the additional condition $\lambda f_{ss} + (1 \quad \lambda) g_{tt} = 0$, so that, as in (4), the "big CSS" is given by

$$(x, y) = (1 \quad \lambda)(s, f(s,\varepsilon)) + \lambda(t, g(t,\varepsilon)) \quad \text{where } f_s = g_t \text{ and } \lambda f_{ss} + (1 \quad \lambda) g_{tt} = 0. \tag{10}$$

Let us write $\lambda = \lambda_0 + \alpha$ where α is small. The set $\widetilde{\mathcal{F}}^{\,1}(\mathbf{0})$ can be locally parametrized by s, t, α, and, on the critical set of the projection $\widetilde{\mathcal{F}}^{\,1}(\mathbf{0})$ to x, y, ε-space, t can be expressed as a smooth function of s, α. Furthermore, the image of the critical set of this projection (the big CSS) is smooth, provided λ_0 does not take either of the special values of λ as in Definition 3.3. Assuming this, the equation of the big CSS can be written as

$$\varepsilon = \frac{3a_3^2 b_3^2}{(b_3 \quad \lambda_0(a_3 + b_3))(b_3 + \lambda_0(a_3 \quad b_3)) g_{11}} x^2 + \text{ h.o.t. in } x \text{ and } y. \tag{11}$$

We therefore have the following.

Proposition 3.6. *Locally to* $(0, \lambda_0)$ *on the* y*-axis, where* λ_0 *is not a special value, the CSS given by* $\varepsilon =$ *constant comprises two smooth curves, one on each side of the* y*-axis. As* $\varepsilon \to 0$ *these two curves move into coincidence along the* y*-axis.* □

Remark 3.7. In the case of two "opposite inflexions", as in Figure 2(b), the denominator of (11) becomes $(a_3^2\lambda_0^2 + b_3^2(1 \quad \lambda_0)^2)g_{11}$, which is never zero. In this case, the conclusion of the above proposition always holds (away from $\lambda_0 = 0, 1$).

The situation at a special point on the y-axis must be different, since there, the local picture of the CSS for $\varepsilon = 0$ is a line (the y-axis) and a parabolic curve by Proposition 3.4, and in fact, the big CSS is singular. We find that the big CSS is (for a generic family of curves) locally diffeomorphic to a cuspidal edge surface. There are several ways to see this. The most immediate way is to use the projection from $\widetilde{\mathcal{F}}^{\ 1}(\mathbf{0})$ to (x, y, ε). From (9), the first, third, fourth and seventh columns are independent since $\lambda \neq 0, 1$ at a special value (and $g_{11} \neq 0$, as assumed throughout). Therefore, we can use s, x, y as parameters on the smooth manifold $\widetilde{\mathcal{F}}^{\ 1}(\mathbf{0})$ close to $n = s = t = \varepsilon = 0$. The base values of s, x are zero but that of y is λ_0 so we need to write $Y = y \quad \lambda_0$ and expand as a function of s, x, Y. Expressed using these parameters, and using the special value $\lambda_0 = b_3/(a_3 + b_3)$, the map to (x, Y, ε) takes the form

$$(s, x, Y) \mapsto$$
$$\left(x,\ Y,\ \frac{6a_3(a_3+b_3)}{g_{11}}sx + \frac{4(a_3^3 g_{40} + b_3^3 f_{40})}{b_3^3 g_{11}}s^3 + \frac{6a_3(a_3+b_3)^2}{b_3 g_{11}}s^2 Y + \dots\right),$$

where there is also a quadratic term in x^2 and other cubic terms in s, x, Y, besides terms of degree > 3. Provided the displayed coefficients are nonzero, this is enough to recognize the germ at $s = x = Y = 0$, up to left-right, that is $\mathcal{A}$-equivalence, using the classification in [4]. In fact, the germ is then $\mathcal{A}$-equivalent to $(s, x, y) \mapsto (x, y, sx + x^3)$ and the set of critical values of this germ, that is the big CSS, is therefore a cuspidal edge. At the other special value the conclusion is similar.

Proposition 3.8. *In addition to the usual assumptions that all of* $f_{30} = a_3^2, g_{30} = b_3^2, a_3 \quad b_3$ *and* g_{11} *are nonzero, assume that*

$$\frac{f_{40}}{a_3^3} \neq \mp\frac{g_{40}}{b_3^3}.$$

Then the big CSS in x, y, ε*-space, close to the point* $(0, \lambda_0, 0)$ *where* λ_0 *is one of the special values* $b_3/(b_3 \pm a_3)$*, is locally diffeomorphic to a cuspidal edge. See Figure 4.* □

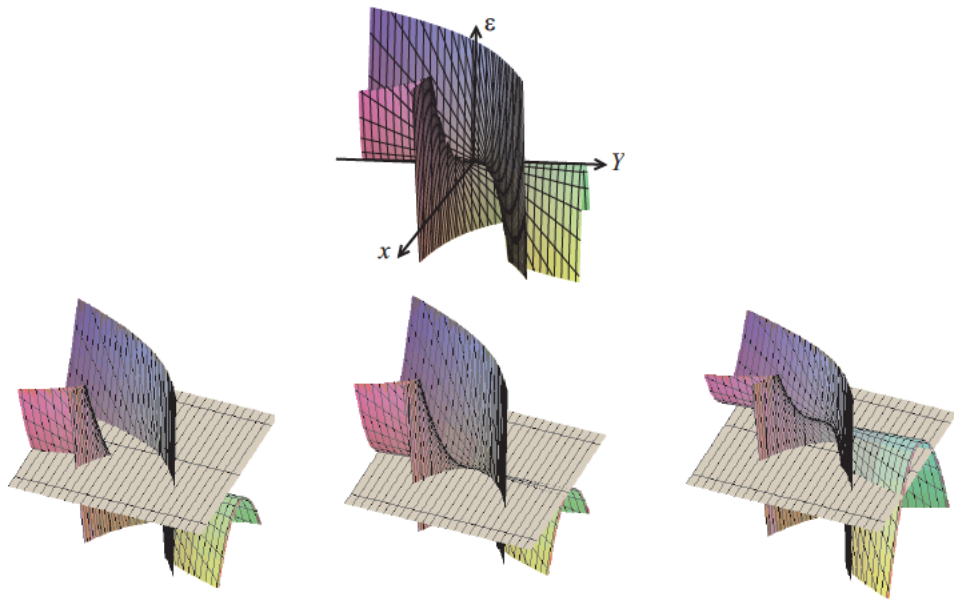

Fig. 4. Left: the "big CSS" or union of the CSS for ε close to 0, in (x, Y, ε)-space, for λ_0 equal to one of the special values. This requires $a_3 b_3 > 0$, as in Figure 2(a). The ε axis is vertical and the surface contains a line, the Y-axis, and a cuspidal edge surface which osculates the plane $\varepsilon = 0$. The other three diagrams show horizontal plane sections $\varepsilon =$ constant of this surface.

Remark 3.9. There is an interesting geometrical interpretation of the condition in Proposition 3.8. Consider the "reflexion" of the curve $y = f_0(x)$ in the point $(0, \lambda_0) = (0, b_3/(b_3 \pm a_3))$, but scaled so that $(0, 0)$ is sent to $(0, 1)$. This amounts to the affine map $(x, y) \mapsto \left(\dfrac{x(\lambda_0 \quad 1)}{\lambda_0}, \dfrac{y(\lambda_0 \quad 1)}{\lambda_0} + 1 \right)$. Then the "reflected" curve M^* is $y = 1 + b_3^2 x^3 \mp (b_3^3/a_3^3) f_{40} x^4 + (b_3^4/a_3^4) f_{50} x^5 + \ldots$, to be compared with the curve N with equation $y = g_0(x) = 1 + b_3^2 x^3 + g_{40} x^4 + g_{50} x^5 + \ldots$. The curves M^* and N have at least 4-point contact, and at least 5-point contact if and only if the condition of the proposition is violated.

It is clear that the function ε on this cuspidal edge surface is a very degenerate function since the plane $\varepsilon = 0$ is tangent to the surface along an entire line $x = \varepsilon = 0$. This makes it unlikely that we can describe the family of CSS by reducing ε to a normal form. Instead, by carefully parametrizing the surface and considering small values of ε, we find that the level sets, that is the individual CSS, are as in Figure 5, where there is one cusp on each side of $\varepsilon = 0$ and each half of the y (or Y) axis is a limit from only one side as $\varepsilon \to 0$. The details are in the following proposition, where we use the

abbreviation $A^{\pm} = a_3^3 g_{40} \pm b_3^3 f_{40}$ $(\neq 0)$ and $y_0^{\pm} = b_3/(b_3 \pm a_3)$. The CSS for $\varepsilon = 0$ is given in Proposition 3.4.

Proposition 3.10. *The cuspidal edge of the big CSS in (x, y, ε)-space has the parametrization, close to $(0, y_0^{\pm}, 0)$*

$$\left(\frac{a_3 b_3 (b_3 \pm a_3)^3}{2A^{\pm}} (y \quad y_0^{\pm})^2 + \ldots,\ y,\ \mp \frac{(b_3 \pm a_3)^6 a_3^3 b_3^3}{2(A^{\pm})^2 g_{11}} (y \quad y_0^{\pm})^3 + \ldots \right).$$

The locus of cusps in the (x, y) plane, for varying ε, is therefore given by the first two coordinates, and since $\frac{1}{2} > \frac{3}{8}$, the CSS for $\varepsilon = 0$ is locally between this locus and the y-axis. See Figure 5. □

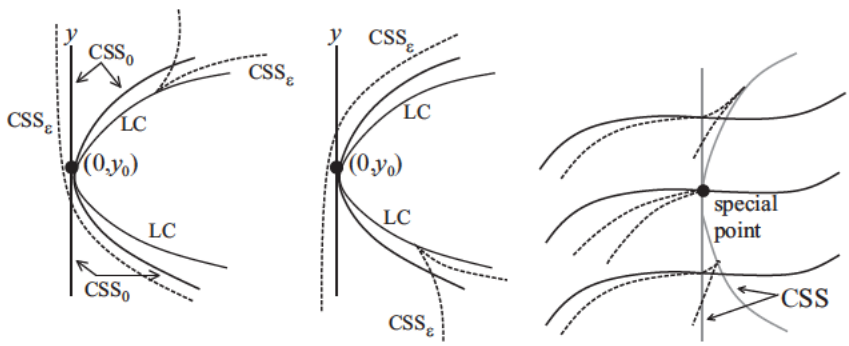

Fig. 5. The case $\varepsilon = 0$. (a) The thick solid lines, marked CSS_0, are the part close to a special point $(0, y_0)$ of the CSS for $\varepsilon = 0$, that is for two curves with parallel tangents at inflexion points. The thin solid line marked LC is the locus of cusps on the CSS for ε close to 0. The dashed line, marked CSS_ε, is the CSS itself for ε close to 0; one diagram will be for $\varepsilon > 0$ and the other for $\varepsilon < 0$. Compare Figure 4. In (b) equidistants are drawn for λ at, and close to, a special point. The CSS is drawn in a lighter colour and the two "branches" of the equidistants are solid and dashed lines. At the special value one branch has a rhamphoid cusp.

4. Equidistants

The λ-equidistant is the set of points $(1 \quad \lambda)a + \lambda b$ for a fixed λ where the tangents to M at a and to N at b are parallel. The singular points of equidistants, for a fixed ε, sweep out the centre symmetry set.

4.1. Fixed λ equidistants. In this subsection, we consider λ to be fixed at say λ_0. When λ_0 is not a special value as in Definition 3.3, the normal form technique of §5 can be applied; see Proposition 5.10. Suppose now that λ_0 is one of the special values – hence the inflexions satisfy $f_{30} = a_3^2$, $g_{30} = b_3^2$ where $a_3 > 0, b_3 > 0$ – and we ask how the equidistants vary as ε passes through 0. Thus the "big equidistant" in this context lies in (x, y, ε)-space.

An option here is to regard $\widetilde{F}$, for fixed λ, as an unfolding of a function in the variables n, s, t with unfolding parameters x, y, ε. Then we ask

(i) what is the singularity of the function $F_0(n,s,t) = \widetilde{F}((n,1),s,t,\lambda_0,0,y_0,0)$, at $n = s = t = 0$, where $y_0 = \lambda_0$ is a special value?
(ii) writing $y = y_0 + Y$, is this function versally unfolded by the parameters x, Y, ε?

In fact, using the special value $b_3/(a_3+b_3)$, the expansion of F_0 is

$$\left(\frac{a_3}{a_3+b_3}\right) ns + \left(\frac{b_3}{a_3+b_3}\right) nt + \left(\frac{a_3^3}{a_3+b_3}\right) s^3 + \left(\frac{b_3^3}{a_3+b_3}\right) t^3 + \text{ h.o.t..}$$

Using the substitution $n = u + v$, $s = \frac{1}{a_3}((u \quad v)(a_3+b_3) \quad b_3 t)$ reduces F_0 to

$$u^2 \quad v^2 + \frac{b_3(a_3^3 g_{40} + b_3^3 f_{40})}{a_3^3(a_3+b_3)} t^4 + \text{ h.o.t.,}$$

which is of type A_3 at $u = v = t = 0$, provided $a_3^3 g_{40} + b_3^3 f_{40} \neq 0$, the same condition that occurred in Proposition 3.8 above. It is then a routine matter to check that, using only $b_3 \neq 0$, $\widetilde{F}$ is a versal unfolding of this A_3 singularity. Hence we have the following.

Proposition 4.1. *Consider the fixed, special value $b_3/(b_3 \pm a_3)$ of λ. Assume that $a_3, b_3, a_3 \quad b_3, g_{11}$ and $a_3^3 g_{40} \pm b_3^3 f_{40} \neq 0$ are all nonzero, for the corresponding sign $\pm$. Then the big equidistant, that is the set in (x,y,ε)-space consisting of all the equidistants for ε close to 0, is locally diffeomorphic to a swallowtail surface.* □

The function ε on this big equidistant has level sets $\varepsilon =$ constant which are the individual equidistants. Unlike the case above (Proposition 3.8) where ε is a highly non-generic function, in the present situation we can identify ε in a standard list of functions on a swallowtail (see e.g. [1, p. 565]).

For the standard swallowtail, that is the discriminant surface of the monic reduced quartic polynomial $w^4 + p + qw + rw^2$, the stable function on (p,q,r)-space preserving the swallowtail is the function r. The key property for recognizing this function is that the level set $r = 0$ is transverse to the limiting tangent line to the cusp edge and self-intersection curve on the swallowtail surface (on the standard swallowtail this limiting tangent line is the r-axis). (It is then automatically transverse to the limiting tangent plane to the smooth 2-dimensional strata through the origin.)

The "next" function on the standard swallowtail is $q + r^2$, whose level set $q + r^2 = 0$ is not transverse to the r-axis. We can distinguish this function from more degenerate ones by considering the contact of the level set with

the *self-intersection* curve on the swallowtail[1]. The self-intersection curve has parametrization $(w^4, 0, \quad 2w^2)$, having 4-point contact with the level set $q + r^2 = 0$. Examining the self-intersection curve of the swallowtail surface arising as a big equidistant, we find the following.

Proposition 4.2. *The function ε on the swallowtail surface in Proposition 4.1 is equivalent to the function $q + r^2$ on the standard swallowtail provided the additional condition*

$$\frac{4f_{40}^2}{a_3^6} \quad \frac{3f_{50}}{a_3^4} \neq \frac{4g_{40}^2}{b_3^6} \quad \frac{3g_{50}}{b_3^4}$$

holds. The resulting transition on equidistants is illustrated in Figure 6(a). □

Remark 4.3. The geometrical meaning of the condition in the proposition is not clear to us; however, if the curves M^* and N in Remark 3.9 have at least 6-point contact then *both* the conditions in Propositions 3.8 and 4.2 are violated.

4.2. $\varepsilon = 0$ equidistants. We can also fix ε at 0 and ask how the equidistants evolve as λ moves through a value λ_0. The case where λ_0 is not special, as in Definition 3.3, is easy (again we assume $a_3 b_3 > 0$). Recall from Proposition 3.4 that the CSS for $\varepsilon = 0$ and away from special points consists of just the y-axis. A direct calculation shows that the two branches of the parallel tangent set (Figure 2(a)) give rise to two smooth branches of the equidistant through $(0, \lambda_0 + \alpha)$, where λ_0 is not special and α is small, given by

$$y = \lambda_0 + \alpha + \frac{a_3^2 b_3^2}{(b_3(1 \quad \lambda_0) \pm a_3\lambda_0)^2} x^3 + \text{ h.o.t},$$

where the higher terms depend on α as well as x (and λ_0). These are two curves having inflexions parallel to those of the curves M and N, and having exactly 3-point contact. See Figure 6(b). Together, these two curves exhibit an A_5 singularity (equivalent to $y^2 \quad x^6$); this singularity conserves the idea that "the CSS is swept out by the singularities of equidistants". As λ_0 varies locally, the two curves move vertically but are unchanged to third order.

Determining the structure of the family of equidistants for λ close to one of the special values, and $\varepsilon = 0$, is more problematic, and we have not identified the big equidistant in this case. However, explicit calculations can be done and Figure 5(b) shows a typical way in which the $\varepsilon = 0$ equidistants evolve for varying λ close to a special point of the y-axis.

[1] Contact with the cuspidal edge curve does not work; the authors thank J. W. Bruce for this crucial insight.

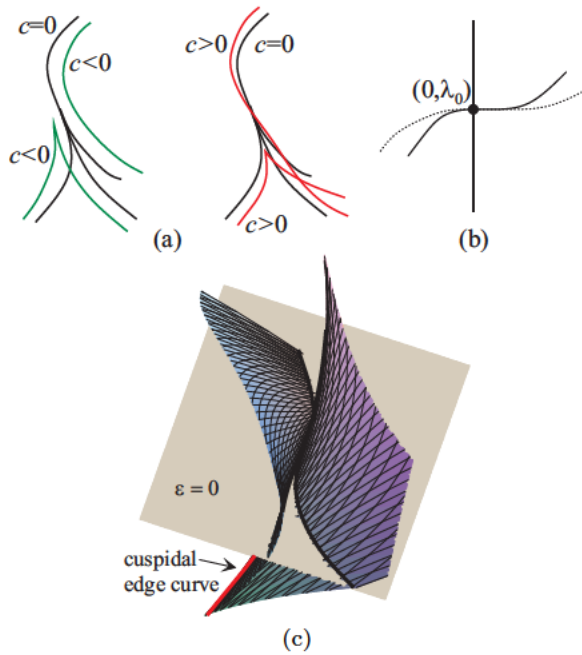

Fig. 6. (a): Sections $q + r^2 = c$ of the standard swallowtail surface (see text). On the left, $c < 0$ and $c = 0$; on the right $c = 0$ and $c > 0$. This transition is the same as that on the equidistants for a fixed, special value of λ, as ε passes through 0. The section $c = 0$ consists of a smooth branch and another with a rhamphoid cusp ($\mathcal{A}$-equivalent to (u^2, u^5)). (b): An equidistant for $\varepsilon = 0$ through a point of the y-axis away from a special point; see §4.2. There are two branches and one is drawn dashed. (c): The equidistant in (b) is the section $\varepsilon = 0$ of the *big equidistant* which is a cuspidal edge surface, the transition through $\varepsilon = 0$ being a beaks transition. See Proposition 5.10.

5. Reductions to a normal form

In this section, we study the same problem using a slightly different approach, starting from the same generating function (1) but attempting to reduce to normal forms under appropriate equivalences. This method has been used extensively to study the CSS. See for example [12, 13]; also [15, 3] for background details. However, there are some situations where a "direct" approach such as we have adopted above seems to be the only option, for example the evolution of CSS close to $\varepsilon = 0$ described in Figure 5. Various difficulties have arisen in applying the "normal form" method to these situations.

The situations where the reduction method is successful are:

(i) the study, up to local diffeomorphism, of the *big equidistant*, namely the union in $(\lambda, x, y, \varepsilon)$-space of the λ-equidistants for an arbitrary curve of the family;

(ii) the evolution of the equidistants in the family of curves, for a fixed λ away from special values of Definition 3.3;
(iii) the *big CSS*, that is the union in (x, y, ε)-space of the CSS for curves in the family; see Figure 4 and Proposition 3.8.

The key first step in finding normal forms is the following reduction in the number of variables using *stabilization* (see [2]), that is writing the family $\widetilde{F}$ in equation (7) as the sum of two terms, where the first is a nondegenerate quadratic form in variables not occurring in the second term. The constructions we want, that is the CSS and equidistants, remain the same up to local diffeomorphism by such a stabilization.

The proposition holds without assuming the basepoints $(0, 0)$ and $(0, 1)$ on the curves M, N are inflexions, merely that the tangents there are parallel, so that we can write $M : y = f(x, \varepsilon),\ f(0, 0) = f_x(0, 0) = 0$ and $N : y = g(x, \varepsilon),\ g(0, 0) = 1,\ g_x(0, 0) = 0$. The "base chord" is as before the y-axis.

Proposition 5.1. *The germ of the family $\widetilde{F}$ at a point $(n, s, t, \lambda, x, y, \varepsilon) = (0, 0, 0, \lambda_0, 0, y_0, 0)$, where $\lambda_0 \neq 1$, is stably equivalent to the family germ*

$$\Phi(t, \lambda, x, y, \varepsilon) = \begin{pmatrix} 1 & \lambda \end{pmatrix} f\left(\frac{\begin{matrix} x & \lambda t \end{matrix}}{\begin{matrix} 1 & \lambda \end{matrix}}, \varepsilon\right) + \lambda g(t, \varepsilon) \quad y \tag{12}$$

in the variable $t \in \mathbb{R}$ and parameters $(\lambda, x, y, \varepsilon) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$ at $t = 0, \lambda = \lambda_0, x = 0, y = y_0, \varepsilon = 0$.

Proof. We have $\widetilde{F} = An + B$ where $A = \begin{pmatrix} 1 & \lambda \end{pmatrix} s + \lambda t \quad x$ and $B = \begin{pmatrix} 1 & \lambda \end{pmatrix} f(s, \varepsilon) + \lambda g(t, \varepsilon) \quad y$.

For $\lambda \neq 1$, we can write $s = \dfrac{\begin{matrix} A + x & \lambda t \end{matrix}}{\begin{matrix} 1 & \lambda \end{matrix}}$. In the new coordinates, we have $\widetilde{F} = An + B(A, t, \lambda, x, y, \varepsilon)$ where the function B does not depend on n. Applying Hadamard's lemma to the function B, we have $B(A, t, \lambda, x, y, \varepsilon) = B(0, t, \lambda, x, y, \varepsilon) + A\phi$ where ϕ is a smooth function of $A, t, \lambda, x, y, \varepsilon$ which in fact vanishes at $(0, 0, \lambda_0, 0, y_0, 0)$. Now the function $\widetilde{F}$ takes the form $A(n + \phi) + B(0, t, \lambda, x, y, \varepsilon)$, and since n does not appear in B, we can replace $n + \phi$ by n, so that the first term An is a nondegenerate quadratic form in variables not appearing in B. Therefore, the function $\widetilde{F}$ is stably equivalent to the function $\Phi = B(0, t, \lambda, x, y, \varepsilon)$, being the restriction of the function B to the subspace $A = 0$. This completes the proof of Proposition 5.1. □

Let M be the set $\{(s, f(s, \varepsilon))\}$ and let N be the set $\{(t, g(t, \varepsilon)\}$. We can assume that, for all ε, M passes through the origin with horizontal tangent, and that N passes through the point $(0, 1)$.

$$f(s, \varepsilon) = s^2 f_1(s, \varepsilon) = f_{20}s^2 + f_{30}s^3 + f_{21}s^2\varepsilon + f_{40}s^4 + \dots$$
$$g(t, \varepsilon) = 1 + tg_1(\varepsilon) = 1 + g_{11}t\varepsilon + g_{20}t^2 + g_{30}t^3 + g_{21}t^2\varepsilon + f_{40}t^4.$$

Expanding in terms of t, x, y, ε gives:

$$\begin{aligned}\Phi = \lambda \quad & y + \lambda\left(g_{11}\varepsilon \quad 2\frac{f_{20}x}{1 \quad \lambda}\right)t \\ & + \lambda\left(\frac{f_{20}\lambda}{1 \quad \lambda} + g_{20} + 3\frac{f_{30}\lambda x}{(1 \quad \lambda)^2} + \left(\frac{f_{21}\lambda}{1 \quad \lambda} + g_{21}\right)\varepsilon\right)t^2 \\ & + \lambda\left(g_{30} \quad \frac{f_{30}\lambda^2}{(1 \quad \lambda)^2} \quad 4\frac{f_{40}\lambda^2 x}{(1 \quad \lambda)^3} + \left(\quad \frac{f_{31}\lambda^2}{(1 \quad \lambda)^2} + g_{31}\right)\varepsilon\right)t^3 \\ & + \lambda\left(\frac{f_{40}\lambda^3}{(1 \quad \lambda)^3} + g_{40} + 5\frac{f_{50}\lambda^3 x}{(1 \quad \lambda)^4} + \left(\frac{f_{41}\lambda^3}{(1 \quad \lambda)^3} + g_{41}\right)\varepsilon\right)t^4 + \cdots\end{aligned}$$

where terms of order greater than 1 in ε, x and y are denoted by dots.

Reducing from $\widetilde{F}$ to Φ gives the following, where $\boldsymbol{x}$ stands for (x, y).

- The big equidistant is the set of points in $(\lambda, \boldsymbol{x}, \varepsilon)$-space for which there exists t with $\Phi = \Phi_t = 0$.
- For a fixed ε, the big ε-equidistant is the intersection of the big equidistant with $\varepsilon =$ constant. Fixing both λ and ε, we obtain a particular equidistant for one curve of the family.
- The big CSS is the set of critical values of the projection from the big equidistant to $(\boldsymbol{x}, \varepsilon)$-space. This consists of the images of points where $\Phi_{tt} = 0$. Intersections with $\varepsilon =$ constant are the individual CSS.

In this section, we use these techniques to study, up to local diffeomorphism, the big equidistant, the big CSS and the metamorphoses of the ε-equidistants for a fixed λ away from special values.

We have the following theorems, where M and N are (germs of) generic smooth plane curves, varying in a generic family parametrized (as in the above sections) by values of ε close to 0.

Proposition 5.2. *The germ at any point of the big CSS away from M and N is diffeomorphic to one of the standard caustics of A_r type with $r = 2, 3$ or 4 (regular surface, cuspidal edge or swallowtail).*

The cuspidal edge case we have met in Proposition 3.8; the swallowtail case arises from the appearance or disappearance of two cusps on the CSS; it is described in [7, Th.7], and will not be investigated here, but see Proposition 5.9.

Remarks 5.3. There still remains the possibility of extending the current results to finding normal forms up to other equivalences that study, for example, how the caustic bifurcates as ε varies near 0 or the two parameter family of equidistants as λ and ε vary. Various difficulties arise whilst using

the current techniques and we have so far been unable reduce the generating family to normal forms in these instances.

Consider the following equivalence relations, where the base values of the variable and parameters $t, \lambda, \boldsymbol{x}, \varepsilon$ are, as usual, $(0, \lambda_0, (0, y_0), 0)$.

Definition 5.4. Two germs of families Φ_1 and Φ_2 of the variable t and with parameters $\lambda, \boldsymbol{x}, \varepsilon$ are called *contact equivalent* if there exists a nonzero function $\phi(t, \lambda, \boldsymbol{x}, \varepsilon)$ and diffeomorphism germ $\theta : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R} \times \mathbb{R}^4$, of the form $\theta : (t, \lambda, \boldsymbol{x}, \varepsilon) \mapsto (T(t, \lambda, \boldsymbol{x}, \varepsilon), \Lambda(\lambda, \boldsymbol{x}, \varepsilon), X(\lambda, \boldsymbol{x}, \varepsilon), E(\lambda, \boldsymbol{x}, \varepsilon))$ such that $\phi\Phi_1 = \Phi_2 \circ \theta$.

Definition 5.5. Two germs of families Φ_1 and Φ_2 of the variable t and with parameters $\lambda, \boldsymbol{x}, \varepsilon$ are called *space-time contact equivalent* if there exists a nonzero function $\phi(t, \lambda, \boldsymbol{x}, \varepsilon)$ and diffeomorphism germ $\theta : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R} \times \mathbb{R}^4$, of the form $\theta : (t, \lambda, \boldsymbol{x}, \varepsilon) \mapsto (T(t, \lambda, \boldsymbol{x}, \varepsilon), \Lambda(\lambda, \boldsymbol{x}, \varepsilon), X(\boldsymbol{x}, \varepsilon), E(\boldsymbol{x}, \varepsilon))$ such that $\phi\Phi_1 = \Phi_2 \circ \theta$.

Definition 5.6. Two germs of families Φ_1 and Φ_2 of the variable t and with parameters $\lambda, \boldsymbol{x}, \varepsilon$ are called $(\lambda, \boldsymbol{x}, \varepsilon)$*-contact equivalent* if there exists a nonzero function $\phi(t, \boldsymbol{x}, \lambda, \varepsilon)$ and diffeomorphism germ $\theta : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R} \times \mathbb{R}^4$, of the form $\theta : (t, \lambda, \boldsymbol{x}, \varepsilon) \mapsto (T(t, \lambda, \boldsymbol{x}, \varepsilon), \Lambda(\lambda, \boldsymbol{x}, \varepsilon), X(\boldsymbol{x}, \varepsilon), E(\varepsilon))$ such that $\phi\Phi_1 = \Phi_2 \circ \theta$.

Definition 5.7. Two germs of families Φ_1 and Φ_2 of the variable t and with parameters $\boldsymbol{x}, \varepsilon$ are called *time-space contact equivalent* if there exists a nonzero function $\phi(t, \boldsymbol{x}, \varepsilon)$ and diffeomorphism germ $\theta : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$, of the form $\theta : (t, \boldsymbol{x}, \varepsilon) \mapsto (T(t, \boldsymbol{x}, \varepsilon), X(\boldsymbol{x}, \varepsilon), E(\varepsilon))$ such that $\phi\Phi_1 = \Phi_2 \circ \theta$. Note that here λ is fixed.

The key properties are the following.

- Contact equivalence preserves the big equidistant up to local diffeomorphism, but not the level sets $\lambda =$ constant or $\varepsilon =$ constant.
- Space-time contact equivalence preserves the big CSS up to local diffeomorphism, but not the level sets $\varepsilon =$ constant.
- If two germs are (λ, $\boldsymbol{x}$, ε)-contact equivalent then their families of CSS as ε varies have equivalent bifurcations. We say that two families of CSS have equivalent bifurcations if there is a diffeomorphism germ mapping one big CSS to the other respecting the fibers of its projection to ε. So families have equivalent bifurcations if via some appropriate reparametrisation of the ε, each momentary CSS from one family is diffeomorphic to the respective momentary CSS of the other family.

- In this version of time-space contact equivalence 5.7, ε plays the role of time and the bifurcations of the individual equidistants for a fixed value of λ and varying ε are preserved up to local diffeomorphism.

Proposition 5.8. *For a generic pair of M and N, at any point $\boldsymbol{x}$ of a parallel tangent chord except the points of M, N themselves ($\lambda = 0$ or $\lambda = 1$), the germ of the respective generating family Φ is space-time contact equivalent to one of the standard versal deformations in parameters $(\lambda, \boldsymbol{x}, \varepsilon) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$ of the function germs at the origin in the variable t of the type A_k for $k = 1, ..., 4$ as follows:*

$$A_1 : \Phi = t^2 + \lambda, \qquad A_2 : \Phi = t^3 + xt + \lambda,$$
$$A_3 : \Phi = t^4 + yt^2 + xt + \lambda, \qquad A_4 : \Phi = t^5 + \varepsilon t^3 + yt^2 + xt + \lambda.$$

Proposition 5.8 gives the normal forms for the big equidistant in $(\lambda, \boldsymbol{x}, \varepsilon)$-space and the big CSS in $(\boldsymbol{x}, \varepsilon)$-space. In the case A_1, the big equidistant germ is smooth and the big CSS is empty. For A_2 the big equidistant is diffeomorphic to the product of a cusp with 2-dimensional space whilst the big CSS is a smooth surface. For the case A_3, the big equidistant is diffeomorphic to the product of a swallowtail with a line and the big CSS is diffeomorphic to a cuspidal edge surface, as in Proposition 3.8. For A_4, the big equidistant is diffeomorphic to a butterfly and the big CSS is diffeomorphic to a swallowtail.

Note that up to diffeomorphism, this does not tell us anything about the special way that the big CSS sits in the (x, ε)-space in the "parallel inflexions" case or any other. The special points of λ in the parallel inflexions case corresponds to when an A_3 singularity occurs on the special parallel inflexions chord. Note that the A_4 does not occur generically on the parallel inflexions chord.

Conditions for Proposition 5.8. Each chord (apart from the parallel inflexions chords) contains one CSS point. If $f_{20} = 0$, the CSS point coincides with the curve N and if $g_{20} = 0$, the CSS point coincides with the curve M. If $f_{20} = g_{20} = 0$ then we have the parallel inflexions case. Here the whole chord belongs to the CSS and, when the inflexions satisfy $f_{30}g_{30} > 0$, and using the notation of (5), the chord has two special values at $\frac{\lambda}{1\ \lambda} = \pm\frac{b_3}{a_3}$ corresponding to A_3 points (compare Definition 3.3). More degenerate singularities do not occur generically in the parallel inflexions case.

If f_{20} and g_{20} are both nonzero, there exists a CSS point m_0 with $\frac{\lambda}{1\ \lambda} = \frac{g_{20}}{f_{20}}$. The singularity at the point m_0 is of type A_2 unless $g_{20}^2 f_{30} = f_{20}^2 g_{30}$ in which case the singularity will be more degenerate. In particular, it will be of type A_3 if the condition $g_{20}^3 f_{40} \neq f_{20}^3 g_{40}$ holds. If this fails, then the singularity will be of type A_4, assuming that $g_{20}^4 f_{50} \neq f_{20}^4 g_{50}$ which holds generically. No further degeneration can occur generically.

Proposition 5.9. *For a generic one-parameter family of a pair of curves M and N, which do not both have inflexions, at any point $\boldsymbol{x}$ of any parallel tangent chord, away from ($\lambda = 0$ or 1), the germ of the respective generating family Φ is (λ, $\boldsymbol{x}$, ε)-contact equivalent to the germ at the origin of one of the following deformations in parameters $(\lambda, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^3$ of the function germs in the variable t of the type A_k for $k = 2, 3$ and 4.*

$$A_2 : \Phi = t^3 + xt + \lambda, \qquad A_3 : \Phi = t^4 + yt^2 + xt + \lambda,$$
$$A_4 : \Phi = t^5 + \varepsilon t^3 + yt^2 + xt + \lambda.$$

Proposition 5.9 gives the normal forms for the CSS in the $\boldsymbol{x}$-plane as ε varies, but not, of course, in the parallel inflexions case. The normal forms A_2 and A_3 do not contain ε and so no transition occurs. The caustic at an A_2 singularity is diffeomorphic to a smooth curve and at an A_3 singularity it is diffeomorphic to an ordinary cusp. The bifurcation of type A_4 determines the standard swallowtail transition.

Conditions for Proposition 5.9. We assume that f_{20} are g_{20} are both nonzero. Again, there is a single A_2 point $\frac{\lambda}{1\ \lambda} = \frac{g_{20}}{f_{20}}$. The conditions for A_3 and A_4 type singularities are the same as above.

Proposition 5.10. *For a fixed value of $\lambda = \lambda_0$, away from special values as in Definition 3.3, the generating family near a parallel inflexions chord is time-space contact equivalent to the germ at the origin of one of the following deformation in parameters $(\lambda, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^2$ of the function variable t:*

$$A_2^{*\pm} : F = t^3 + (\varepsilon \pm x^2)t + y.$$

This tells us how the equidistants for a fixed λ change with ε away from the special values of λ on the parallel inflexions chord. Compare §4.1 where a direct calculation can handle the case of the special values. The case A_2^{*+} occurs if and only if $f_{30}g_{30} < 0$ and corresponds to the standard "lips" transition and the case A_2^{*} occurs if and only if $f_{30}g_{30} > 0$ and corresponds to the standard "beaks" (bec-à-bec) transition. See Figure 6(c) for the beaks case.

Remark 5.11. The list A_2, A_3, A_4 in Proposition 5.9 is a subset of the list of possible Lagrangian types $A_2, A_3, A_3^1, A_4, D_4^{\pm}$ in [16, p.2727, $n = 2$], occurring away from parallel inflexions. In the parallel inflexions case, the new types of singular points described here are not expected to be in Zakalyukin's list since the surfaces being projected there are smooth.

5.1. Details of the proofs

Proof of Proposition 5.8. A theorem in [11] states that if $\frac{\partial \Phi}{\partial \lambda} \neq 0$, then the stability with respect to space-time contact equivalence of the germ Φ coincides with its stability with respect to standard contact equivalence. The

singularities of A_k type are versally unfolded if and only if the first k rows of the following matrix of derivatives has maximal rank, see for example [11, 12].

$$J_{A_k} = \begin{pmatrix} \frac{\partial\Phi}{\partial x} & \frac{\partial\Phi}{\partial y} & \frac{\partial\Phi}{\partial\lambda} & \frac{\partial\Phi}{\partial\varepsilon} \\ \frac{\partial^2\Phi}{\partial t\partial x} & \frac{\partial^2\Phi}{\partial t\partial y} & \frac{\partial^2\Phi}{\partial t\partial\lambda} & \frac{\partial^2\Phi}{\partial t\partial\varepsilon} \\ \frac{\partial^3\Phi}{\partial t^2\partial x} & \frac{\partial^3\Phi}{\partial t^2\partial y} & \frac{\partial^3\Phi}{\partial t^2\partial\lambda} & \frac{\partial^3\Phi}{\partial t^2\partial\varepsilon} \end{pmatrix} \text{ at } t = \varepsilon = \boldsymbol{x} = 0, \lambda = \lambda_0.$$

Substituting the derivatives into the matrix gives

$$J_{A_k} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 2\frac{\lambda}{1\ \lambda}f_{20} & 0 & 0 & g_{11}\lambda \\ 3\frac{\lambda^2}{(1\ \lambda)^2}f_{30} & 0 & g_{20} & \frac{f_{20}\lambda(\lambda\ \ 2)}{(1\ \lambda)^2} & \lambda g_{21} + \frac{\lambda^2}{1\ \lambda}f_{21} \\ 4\frac{\lambda^3}{(1\ \lambda)^3}f_{40} & 0 & g_{30} + \frac{f_{30}\lambda^2(\lambda\ \ 3)^2}{(1\ \lambda)^3} & \lambda g_{31} & \frac{\lambda^3}{(1\ \lambda)^2}f_{31} \end{pmatrix}.$$

Singularities of type A_2 are versally unfolded in the non-parallel inflexions case, since here we assume $f_{20} \neq 0$ as we are not interested in the case $\lambda = 0$. For the parallel inflexions case A_2 singularities are generically versally unfolded because g_{11} is generically nonzero.

Singularities of type A_3 are versally unfolded in the non-parallel inflexions case, since here we assume $f_{20} \neq 0$ as we are not interested in the case $\lambda = 0$, and $f_{20} \neq g_{20}$ as we are not interested in the case when λ is infinite. For the parallel inflexions case A_3 singularities are generically versally unfolded because g_{11} and f_{30} are generically nonzero.

Singularities of the type A_4 do not occur generically in the parallel inflexions case. Away from parallel inflexions, singularities of type A_4 are generically versally unfolded as J_{A_k} has nonzero determinant generically. No further singularities occur generically. □

Proof of Proposition 5.9. This proposition is concerned with the bifurcations of the CSS, away from parallel inflexions, as ε varies. We have the following notion of (λ, $\boldsymbol{x}$, ε)-versality (see Definition 5.6 of (λ, $\boldsymbol{x}$, ε)-equivalence).

Definition 5.12. The germ of a family of functions Φ is called $(\lambda, \boldsymbol{x}, \varepsilon)$-*versal* if for any germ $\phi(t, \lambda, \boldsymbol{x}, \varepsilon)$ there exists a decomposition of the form

$$\phi(t,\lambda,\boldsymbol{x},\varepsilon) = \widetilde{h}(t,\lambda,\boldsymbol{x},\varepsilon)\Phi + \widetilde{T}(t,\lambda,\boldsymbol{x},\varepsilon)\frac{\partial\Phi}{\partial t} + \Lambda(\lambda,\boldsymbol{x},\varepsilon)\frac{\partial\Phi}{\partial\lambda} + \sum_{i=1}^{2} X_i(\boldsymbol{x},\varepsilon)\frac{\partial\Phi}{\partial x_i} + E(\varepsilon)\frac{\partial\Phi}{\partial\varepsilon}.$$

For the case where there are no inflexions with parallel tangents, we are able to show $(\lambda, \boldsymbol{x}, \varepsilon)$-versality using the following lemma:

Lemma 5.13. *If the deformation Φ is space-time versal with respect to λ and $\boldsymbol{x}$ only, keeping ε constant, then it is automatically $(\lambda, \boldsymbol{x}, \varepsilon)$-versal.*

Proof. Assume that Φ is space-time versal with respect to λ and $\boldsymbol{x}$ only. This means that for any germ $\phi(t, \lambda, \boldsymbol{x}, \varepsilon)$, we can write

$$\phi(t, \lambda, \boldsymbol{x}, \varepsilon) = \widetilde{h}(t, \lambda, \boldsymbol{x}, \varepsilon)\Phi + \widetilde{T}(t, \lambda, \boldsymbol{x}, \varepsilon)\frac{\partial \Phi}{\partial t} + \widetilde{\Lambda}(\lambda, \boldsymbol{x})\frac{\partial \Phi}{\partial t} + \sum_{i=1}^{2} \widetilde{X_i}(\boldsymbol{x}),$$

for some smooth function germs $\widetilde{h}, \widetilde{T}, \widetilde{\Lambda}$ and X_i in the respective variables. Setting $E(\varepsilon) = 0$, $X_i(\boldsymbol{x}, \varepsilon) = \widetilde{X_i}(\boldsymbol{x})$, $\Lambda(\lambda, \boldsymbol{x}, \varepsilon) = \widetilde{\Lambda}(\lambda, \boldsymbol{x})$ in 5.12 gives the required decomposition. □

We now use the following lemma from [11].

Lemma 5.14. *Let $F(t, \lambda, \boldsymbol{x})$ be a (right) infinitesimally versal deformation with parameters λ and $\boldsymbol{x}$ of a quasi-homogeneous germ of a function $f(t)$ at the origin. If $\frac{\partial F}{\partial \lambda} \neq 0$ at the origin, then F is space-time contact infinitesimally versal.*

Proposition 5.15. *The singularities of type A_1, A_2 and A_3 are versally unfolded. The singularity A_4 is generically versally unfolded.*

Proof. Consider the following matrix of derivatives which does not include the column of derivatives with respect to ε:

$$\widetilde{J}_{A_k} = \begin{pmatrix} 0 & 1 & & 1 \\ 2\frac{\lambda}{(1\ \lambda)}f_{20} & 0 & & 0 \\ 3\frac{\lambda^2}{(1\ \lambda)^2}f_{30} & 0 & g_{20} & \frac{f_{20}\lambda(\lambda\ 2)}{(1\ \lambda)^2} \end{pmatrix}.$$

Singularities of type A_k for $k = 1, 2, 3$ are versally unfolded if the first k rows of this matrix have maximal rank. The singularity of type A_2 occurs if $\frac{\lambda}{1\ \lambda} = \frac{g_{20}}{f_{20}}$ and $g_{30}f_{20}^2 \neq f_{30}g_{20}^2$.

Here we are not concerned with the parallel inflexions case, nor the case near $\lambda = 0$, so we assume $f_{20} \neq 0$. Therefore, A_2 singularities are $(\lambda, \boldsymbol{x}, \varepsilon)$-versally unfolded.

Singularities of type A_3 occur if $\frac{\lambda}{1\ \lambda} = \frac{g_{20}}{f_{20}}$, $g_{30}f_{20}^2 = f_{30}g_{20}^2$ and $g_{40}f_{20}^3 \neq f_{40}g_{20}^3$. Substituting the conditions for an A_3 singularity into the matrix reveals that it has non-vanishing determinant if f_{20} and g_{20} are both nonzero and $f_{20} \neq g_{20}$. Therefore, A_3 singularities are versally unfolded. Note that this means that no beaks or lips bifurcations occur.

The singularity A_4 occurs if $\frac{\lambda}{1\ \lambda} = \frac{g_{20}}{f_{20}}$ and $g_{30}f_{20}^2 = f_{30}g_{20}^2$, $g_{40}f_{20}^3 = f_{40}g_{20}^3$ and $g_{50}f_{20}^4 \neq f_{50}g_{20}^4$. The singularity is versally unfolded if J_{A_k} has nonzero determinant and also the determinant of the 4×4 matrix J_{A_k} used in the proof of Proposition 5.8 is nonzero. The first condition has already shown to be true and the matrix J_{A_k} has nonzero determinant generically. Therefore, singularities of type A_4 are generically versally unfolded. □

Proof of Proposition 5.10. Consider the family of wavefronts for a fixed value $\lambda = \lambda_0$. We consider the family Φ up to time-space equivalence: here ε plays the role of time and (x, y) as space.

Lemma 5.16. *If* $\frac{\lambda_0^2}{(1\ \lambda_0)^2} \neq \frac{g_{30}}{f_{30}}$, *the generating family is time-space contact equivalent to the germ at the origin of the following deformation in parameters* $(\lambda, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^2$ *of the function variable* t*:*

$$A_2^* \pm : F = t^3 + (\varepsilon \pm x^2)t + y.$$

Proof. Here, as in [12], we show time-space versality by showing versality with respect to contact equivalence without involving ε. The singularity is versal if and only if the matrix of derivatives, that does not include the column that contains the derivatives with respect to ε, has maximal rank. Since this matrix of derivatives

$$\widehat{J}_{A_2} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \\ \frac{\partial^2 \Phi}{\partial t \partial x} & \frac{\partial^2 \Phi}{\partial t \partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ at } t = \varepsilon = \boldsymbol{x} = 0, \lambda = \lambda_0$$

does not have maximum rank, the wave fronts are not of type A_2, i.e. an ordinary cusp.

In order to be of type A_2^*, the following conditions must hold (see [11]):

1) The first row of $\widehat{J}_{A_2}$ has maximal rank. This is true as it has a nonzero entry.
2) The matrix of derivatives that includes the column with derivatives with respect to epsilon has maximal rank:

$$J_{A_2} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & g_{11} \end{pmatrix}.$$

This condition is satisfied so long as $g_{11} \neq 0$, which we assume throughout.

The family can therefore be reduced to the form:

$$F = t^3 + (\varepsilon + \alpha(x))t + y,$$

for some $\alpha(x)$ with linear part in x vanishing.

Proposition 5.17. *If f_{30} and g_{30} have the same sign then the equidistants for a fixed λ ($\neq 0, 1$ and away from special values) at $\varepsilon = 0$ are of beaks type. If f_{30} and g_{30} are of opposite sign then the equidistants for a fixed λ ($\neq 0, 1$) at $\varepsilon = 0$ are of lips type. In particular, the generating family is time-space equivalent to*

$$F = t^3 + (\varepsilon \pm x^2)t + y.$$

At an A_2^* singularity, the generating family can be reduced to the form $F = at^3 + (\varepsilon + bx^2) + y$, where a and b are nonzero coefficients. The necessary and sufficient condition for beaks to occur is $ab < 0$ and for lips it is $ab > 0$, see for example [2]. We have

$$\Phi(t, x, \varepsilon) = \phi_0(x, \varepsilon) + \phi_1(x, \varepsilon)t + \phi_2(x, \varepsilon)t^2 + \phi_3(x, \varepsilon)t^3 + \phi_4(x, \varepsilon)t^4 + \cdots$$

for some functions ϕ_i.

Substituting $t = \xi(T, x, \varepsilon)$ for some function ξ, the generating family Φ can be reduced to the form

$$\widetilde{\Phi}(T, x, \varepsilon) = \widetilde{\phi}_0(x, \varepsilon) + \widetilde{\phi}_1(x, \varepsilon)T + \widetilde{\phi}_2(x, \varepsilon)T^2 + T^3.$$

Solving as a power series reveals that the necessary function is

$$\xi(T, x, \varepsilon) = \left(\frac{1}{\phi_3}\right)^{\frac{1}{3}} T \quad \frac{1}{3}\left(\frac{1}{\phi_3}\right)^{\frac{5}{3}} \phi_4 T^2 + \cdots$$

Now make a further substitution $T = t_1 \quad \widetilde{\phi_2}/3$ to give $\widehat{\Phi} = t_1^3 + \widehat{\phi}_1 t_1 + \widehat{\phi}_0$ for some functions $\widehat{\phi}_1$ and $\widehat{\phi}_0$.

Keeping track of terms yields that $\widehat{\phi}_1 = c\varepsilon + dx^2 + h.o.t.$ for some c and d where $d^3 = \frac{27\lambda^6(1 \quad \lambda)^6 f_{30}^3 g_{30}^3}{(f_{30}\lambda^4 \quad (1 \quad \lambda)^4 g_{30})^4}$ which can be written as some positive factor multiplied by $\quad f_{30}g_{30}$. So we have that if f_{30} and g_{30} are of the same sign, this expression is negative and hence the beaks transition occurs, whereas if f_{30} and g_{30} are of opposite sign, the expression is positive and then the lips transition occurs. □

Remark 5.18. Note that further changes of variables ε and x respecting the time-space equivalence can reduce the generating family to one of the normal forms in 5.10. These further changes do not affect the sign of the coefficients in $\widehat{\Phi}$ corresponding to the monomials t_1^3 and $x^2 t_1$.

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