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SYMPLECTIC SINGULARITIES AND SOLVABLE HAMILTONIAN MAPPINGS

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Abstract. We study singularities of smooth mappings $\bar{F}$ of $\mathbb{R}^{2n}$ into symplectic space $(\mathbb{R}^{2n}, \omega)$ by their isotropic liftings to the corresponding symplectic tangent bundle $(T\mathbb{R}^{2n}, \dot{\omega})$. Using the notion of local solvability of lifting as a generalized Hamiltonian system, we introduce new symplectic invariants and explain their geometric meaning. We prove that a basic local algebra of singularity is a space of generating functions of solvable isotropic mappings over $\bar{F}$ endowed with a natural Poisson structure. The global properties of this Poisson algebra of the singularity among the space of all generating functions of isotropic liftings are investigated. The solvability criterion of generalized Hamiltonian systems is a strong method for various geometric and algebraic investigations in a symplectic space. We illustrate this by explicit classification of solvable systems in codimension one.

1. Introduction

Let M be a submanifold of $T\mathbb{R}^m$, $dimM = m$, transversal to the fibers of the tangent bundle projection $\pi : T\mathbb{R}^m \to \mathbb{R}^m$, then M as a system of first order ordinary differential equations is locally solvable at each point of M. If $\gamma : I \to \mathbb{R}^m$ is a differentiable curve, where I is an open interval $I = (\ \epsilon, \epsilon)$, $\epsilon > 0$, we denote by $\dot{\gamma}(t)$ the vector tangent to γ at $\gamma(t)$ and introduce the prolongation $\dot{\gamma}$ of γ, $\dot{\gamma} : I \to T\mathbb{R}^m : t \mapsto \dot{\gamma}(t)$. A curve $\gamma : I \to \mathbb{R}^m$ is called an integral curve of $M \subset T\mathbb{R}^m$ if $im(\dot{\gamma}) \subset M$. A submanifold M is said to be solvable if for each $p \in M$ there is an integral curve γ of M such that $\dot{\gamma}(0) = p$. If additionally, the integral curve γ depends smoothly on initial conditions in a neighborhood of every point of M, then we

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say that M is smoothly solvable (cf. [6, 7, 4]). If $\pi|_M$ is a diffeomorphism then M is a smoothly solvable vector field on $\mathbb{R}^m$. If M is not transversal to the fibers of π, i.e. the smooth mapping $\pi|_M \to \mathbb{R}^m$ is no longer a diffeomorphism, then M may not be solvable in the critical points of $\pi|_M$ which is a common property for typical position of M (see [3, 14]). The simplest representative example of such situation is given by $M = \{(x, \dot{x}) \in T\mathbb{R} : x = (\dot{x} \quad a)^2\}$ for $a \neq 0$ with non-solvable point $(0, a) \in M$, which is a singular point of the projection $\pi|_M$.

Solvability is a local property of M, thus we suppose M to be the image of an embedding

$$F = (\bar{F}, \dot{F}) : U \to T\mathbb{R}^m$$

of an open set U of $\mathbb{R}^m$ with coordinates $u = (u_1, u_2, \ldots, u_m)$ into $T\mathbb{R}^m$ with coordinates $(x, \dot{x}) = (x_1, x_2, \ldots, x_m, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_m)$, where $\bar{F} = \pi \circ F$.

Definition 1.1. An implicit differential equation $M = F(U)$ of $T\mathbb{R}^m$, where $F = (\bar{F}, \dot{F}) : U \to T\mathbb{R}^m$ is an embedding, is said to be *smoothly solvable* if there exists a smooth tangent vector field X on U such that

$$(\bar{F}(u), \dot{F}(u)) = d\bar{F}(X(u)), \quad \forall u \in U. \tag{1.1}$$

If an implicit differential equation $M = F(U)$ of $T\mathbb{R}^m$ is smoothly solvable with a smooth vector field X on U, then every point $(x_0, \dot{x}_0) \in M$ is a solvable point of M. Indeed, let u_0 be a point in U such that $(\bar{F}(u_0), \dot{F}(u_0)) = (x_0, \dot{x}_0)$, let $\alpha : I \to U$ be an integral curve of the vector field X with $\alpha(0) = u_0$. Then $\gamma(t) := \bar{F}(\alpha(t))$ is a solution of the implicit differential equation of M such that $(\gamma(0), \dot{\gamma}(0)) = (x_0, \dot{x}_0)$. Thus $(x_0, \dot{x}_0) \in M$ is a solvable point of M. Moreover, in this way, integral curves of the vector field X give a family of general solutions of M smoothly depending on initial conditions.

A smooth vector field X on U has the form $X(u) = \sum_{i=1}^{m} a_i(u) \frac{\partial}{\partial u_i}|_u$, where $a_i(u)$ are smooth. Thus an equality (1.1) is equivalent to

$$\dot{F}(u) = J\bar{F}(u)a(u), \tag{1.2}$$

where $J\bar{F}$ is a Jacobian matrix of $\bar{F}$. Thus, we immediately have (cf. [5]) that an implicit differential equation $M = F(U)$ of $T\mathbb{R}^m$ given by an embedding $F = (\bar{F}, \dot{F}) : U \to T\mathbb{R}^m$ is smoothly solvable if and only if (1.2) has a smooth solution $a(u) = (a_1(u), \ldots, a_m(u))$. The condition (1.2) fulfilled to each $u \in U$ is called *tangential solvability condition.*

Now, solvability of implicit differential equations becomes equivalent to a smooth solvability of linear algebraic equations. Using the classical result by J. Mather [13], we get the basic solvability result.

Let $\mathcal{E}_m$ denote the germs at $0 \in \mathbb{R}^m$ of smooth functions of m variables. Let $M(m)$ denote the set of all $m \times m$ real matrices and $\Sigma_r(m)$ denote the

set of all $m \times m$ real matrices with rank r. If we suppose that (1.2) has a solution $a(u)$ at every point $u \in U$ and that the rank of the jacobian matrix $J\bar{F}(0)$ of $(\bar{F}_1(u), \cdots, \bar{F}_m(u))$ at the origin is r. Then using the classical result by J. Mather [13], we can get the basic solvability result. It is proved in [11] that if $J(\bar{F}_1, \ldots, \bar{F}_m) : U \to M(m)$ is transversal to $\Sigma_r(m)$ at the origin 0, then an implicit differential equation $M = F(U)$ of $T\mathbb{R}^m$, given by an embedding $F = (\bar{F}, \dot{F}) : U \to T\mathbb{R}^m$, is smoothly solvable in a neighborhood of $(\bar{F}(0), \dot{F}(0))$.

The more general algebraic version of this result reads,

Theorem 1.2. (see [5]) *Suppose that* (1.2) *has a solution* $a(u)$ *at every point* $u \in U$. *If the ideal* $\langle \det J\bar{F}(u) \rangle$ *has property of zeros (i.e. if any function* $h(u)$ *vanishes on the variety defined by* $\langle \det J\bar{F}(u) \rangle$, *then* $h(u)$ *belongs to* $\langle \det J\bar{F}(u) \rangle$), *then* (1.2) *has a smooth solution defined in a neighborhood of each* $u \in U$.

In what follows, we consider $\mathbb{R}^{2n}$ $(m = 2n)$ endowed with a symplectic structure ω and generalize the notion of Hamiltonian system (cf. [3, 11]). An implicit Hamiltonian system is a solvable isotropic embedding $F : \mathbb{R}^{2n} \supset U \to T\mathbb{R}^{2n}$ into the tangent bundle $T\mathbb{R}^{2n}$ endowed with a symplectic structure $\dot{\omega}$ defined by the canonical flat morphism between tangent and co-tangent bundles of the symplectic space $(\mathbb{R}^{2n}, \omega)$, (see [15]). The solvability properties of $F(U)$ were partially investigated in [5]. In this paper, we extend the notion of implicit Hamiltonian system allowing $\bar{F}$ to be singular (see [2, 12]). In this case, all the properties of the implicit Hamiltonian system are defined by its parametrization F and we will call F a Hamiltonian mapping if it is isotropic, $F^*\dot{\omega} = 0$ solvable and $F^*\dot{\theta} = \quad dh$ for some smooth function h (called the generating function of F). To each F we associate $\bar{F} = \pi \circ F$, where $\pi : T\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a tangent bundle projection and look at F as a vector field along $\bar{F}$. We investigate the space $\mathcal{R}_{\bar{F}}$ of all generating functions of F for fixed singular $\bar{F}$ (all vector fields along $\bar{F}$). Thus for the corank 1 case of $\bar{F}$ the generating function h for isotropic F along $\bar{F}$, or more precisely its derivative $\partial_e h$ belongs to the ideal generated by the determinant of a Jacobian matrix $\Delta_{\bar{F}} = det(J\bar{F})$, where e spans the kernel of the Jacobian matrix at singular point. The sufficient solvability condition for isotropic mappings we prove reads as follows

Theorem A. *Let* $\bar{F} : \mathbb{R}^{2n} \supset U \to \mathbb{R}^{2n}$ *be a smooth mapping with corank* k *singularity at the origin* $(0, 0) \in \mathbb{R}^{2n}$ *and we assume that the jet extension* $j^1\bar{F} : U \to J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ *is transversal to the corank* k *stratum* Σ^k *of* $J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. *If an isotropic mapping* F *along* $\bar{F}$ *satisfies the tangential solvability condition, then* F *is smoothly solvable on* U.

The smoothly solvable isotropic mappings F are characterized by the image property, $F = d\bar{F}(X_h)$ for some smooth vector field, called Hamiltonian vector field on U generated by smooth function h called Hamiltonian function associated to $\bar{F}$. Flows of solvable Hamiltonian mappings are characterized by the following

Theorem B. *Let $F : U \to T\mathbb{R}^{2n}$ be a solvable isotropic mapping along $\bar{F} : U \to \mathbb{R}^{2n}$ and let h be a Hamiltonian function of F. Suppose that fold singular points of $\bar{F}$ are dense in the singular point set of $\bar{F}$. Then integral curves of the vector field X_h preserve the singular point set of $\bar{F}$.*

In the space $\mathcal{H}_{\bar{F}}$ of all Hamiltonians associated to $\bar{F}$, we introduce the Poisson bracket $\{.,.\}_{\bar{F}*\omega}$ and show its basic meaning.

Theorem C. *Let $F : (U, 0) \to T\mathbb{R}^{2n}$ be a smooth isotropic map-germ along a smooth map-germ $\bar{F} : (U, 0) \to \mathbb{R}^{2n}$ such that the regular point set of $\bar{F}$ is dense in U. Let $h : (U, 0) \to \mathbb{R}$ be a generating function-germ of F. Then F is smoothly solvable if and only if $h \in \mathcal{H}_{\bar{F}}$, i.e. h is a Hamiltonian function. Moreover the space of Hamiltonians associated to $\bar{F}$, $(\mathcal{H}_{\bar{F}}, \{.,.\}_{\bar{F}*\omega})$ is a local Poisson algebra.*

In this way, we found the fundamental object of singularity theory which traditionally is a local algebra of singular point. In our case which is a symplectically invariant singularity of $\bar{F}$ it is the corresponding Poisson algebra $(\mathcal{H}_{\bar{F}}, \{.,.\}_{\bar{F}*\omega})$. This structurally invariant property is discovered by collecting all of solvable Hamiltonian systems over the singularity of $\bar{F}$.

Isotropic mappings into tangent symplectic space are investigated in Section 2. Smoothly solvable isotropic mappings with the solvability conditions and flows of solvable generalized Hamiltonian systems are studied in Section 3. In Section 4, a Lie algebra of generating functions based on the space of solvable isotropic mappings is constructed and relation to its Poisson structure is described. Solvability condition in the case of corank 1 singularity is also formulated. The canonical ideals of Poisson algebra of the singularity are characterized in Section 5, and existence of periodic solutions in the singular case is investigated in Section 6.

2. Isotropic mappings

Let $(\mathbb{R}^{2n}, \omega)$ be a symplectic space with $\omega = \sum_{i=1}^{n} dy_i \wedge dx_i$ in canonical Darboux coordinates $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Let θ be the Liouville 1-form on the cotangent bundle $T^*\mathbb{R}^{2n}$. Then $d\theta$ is a standard symplectic structure on $T^*\mathbb{R}^{2n}$. Let $\beta : T\mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}$ be the canonical bundle map defined by ω,

$$\beta : T\mathbb{R}^{2n} \ni v \mapsto \omega(v, \cdot) \in T^*\mathbb{R}^{2n}.$$

Then we can define the canonical symplectic structure $\dot{\omega}$ on $T\mathbb{R}^{2n}$,

$$\dot{\omega} = \beta^* d\theta = d(\beta^*\theta) = \sum_{i=1}^{n} (d\dot{y}_i \wedge dx_i \quad d\dot{x}_i \wedge dy_i),$$

where $(x, y, \dot{x}, \dot{y})$ are local coordinates on $T\mathbb{R}^{2n}$ and $\beta^*\theta = \sum_{i=1}^{n} (\dot{y}_i dx_i$ $\dot{x}_i dy_i)$.

Throughout the paper unless otherwise stated all objects are germs at 0 of smooth functions, mappings, forms etc. or their representatives on an open neighborhood of 0 in $\mathbb{R}^{2n}$.

Definition 2.1. Let $F : (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n}$ be a smooth map-germ. We say that F is *isotropic* if $F^*\dot{\omega} = 0$.

If we assume that $F : (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n}$ is an isotropic map-germ, then the germ of a differential of a 1-form $(\beta \circ F)^*\theta$ vanishes, $d(\beta \circ F)^*\theta = F^*\beta^* d\theta = F^*\dot{\omega} = 0$. Thus $(\beta \circ F)^*\theta$ is a germ of a closed 1-form. And there exists a smooth function-germ $h : (\mathbb{R}^{2n}, 0) \to \mathbb{R}$ such that

$$(\beta \circ F)^*\theta = \quad dh. \tag{2.1}$$

For each smooth isotropic map-germ F, the function-germ h is uniquely defined up to an additive constant.

Let $(u, v) = (u_1, \ldots, u_n, v_1, \ldots, v_n)$ denote coordinates of the source space $U \cong \mathbb{R}^{2n}$. In local coordinates we define

$$F = (f, g, \dot{f}, \dot{g}) : (U, 0) \to T\mathbb{R}^{2n},$$

and

$$\bar{F} = \pi \circ F = (f, g) : (U, 0) \to \mathbb{R}^{2n},$$

where π denotes the canonical projection, $\pi : T\mathbb{R}^{2n} \to \mathbb{R}^{2n}$.

In general, F can be regarded as a vector field along $\bar{F}$, i.e. a section of an induced fiber bundle $\bar{F}^* T\mathbb{R}^{2n}$. By $\mathcal{E}_U$ ($\mathcal{E}_{\mathbb{R}^{2n}}$-respectively) we denote the $\mathbb{R}$-algebra of smooth function germs at 0 on U (and on "the target space" $\mathbb{R}^{2n}$, respectively). To each isotropic map-germ F along $\bar{F}$, there exists a unique h belonging to the maximal ideal $\mathbf{m}_U$ of $\mathcal{E}_U$, $h \in \mathbf{m}_U$, which is a generating function-germ for F.

Let $F : (U, 0) \to T\mathbb{R}^{2n}$ and $G : (U, 0) \to T\mathbb{R}^{2n}$ be two isotropic map-germs along $\bar{F} : (U, 0) \to \mathbb{R}^{2n}$ and $\bar{G} : (U, 0) \to \mathbb{R}^{2n}$, respectively. Now we introduce the natural equivalence group acting on isotropic mappings through a natural lifting of diffeomorphic or symplectic equivalences of $\bar{F}$ and $\bar{G}$. The C^∞ map-germs $\bar{F} : (U, 0) \to \mathbb{R}^{2n}$ and $\bar{G} : (U, 0) \to \mathbb{R}^{2n}$ are said to be symplectomorphic or symplectically equivalent if there exist a diffeomorphism-germ $\varphi : (U, 0) \to (U, 0)$ and a symplectomorphism-germ $\Phi : (T\mathbb{R}^{2n}, 0 \to (T\mathbb{R}^{2n}, 0)$ such that $\bar{G} = \Phi \circ \bar{F} \circ \varphi$.

First we recall the standard equivalence of Lagrange projections (cf. [10]).

Let $F : (U, 0) \to T\mathbb{R}^{2n}$ and $G : (U, 0) \to T\mathbb{R}^{2n}$ be two isotropic map-germs. We say that F and G are *Lagrangian equivalent* (*L-equivalent* [1]) if there exist a diffeomorphism-germ $\varphi : (U, 0) \to (U, 0)$, and a symplectomorphism-germ $\Psi : (T\mathbb{R}^{2n}, 0) \to (T\mathbb{R}^{2n}, 0)$, $\Psi^*\dot{\omega} = \dot{\omega}$, preserving the fibering π such that $G = \Psi \circ F \circ \varphi$.

Definition 2.2. Let $F : (U, 0) \to T\mathbb{R}^{2n}$ and $G : (U, 0) \to T\mathbb{R}^{2n}$ be two isotropic map-germs along $\bar{F} : (U, 0) \to \mathbb{R}^{2n}$ and $\bar{G} : (U, 0) \to \mathbb{R}^{2n}$, respectively. We say that F and G are *L-symplectic equivalent* if there exist a diffeomorphism-germ $\varphi : (U, 0) \to (U, 0)$, and a symplectomorphism-germ $\Psi : (T\mathbb{R}^{2n}, 0) \to (T\mathbb{R}^{2n}, 0)$, $\Psi^*\dot{\omega} = \dot{\omega}$, preserving the fibering π and a symplectomorphism-germ $\Phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$, $\Phi^*\omega = \omega$, $\pi \circ \Psi = \Phi \circ \pi$, such that $G = \Psi \circ F \circ \varphi$ and $\bar{G} = \Phi \circ \bar{F} \circ \varphi$. In this case $\bar{F}$ and $\bar{G}$ are naturally *symplectomorphic*.

To $\bar{F}$ we associate a symplectically invariant algebra $\mathcal{R}_{\bar{F}}$ of all generating function-germs,

$$\mathcal{R}_{\bar{F}} = \{h \in \mathcal{E}_U : h \text{ generates an isotropic map-germ along } \bar{F}\}.$$

It is easy to check that if $\bar{F}$ and $\bar{G}$ are symplectomorphic, $\bar{G} = \Phi \circ \bar{F} \circ \varphi$, then we have an isomorphism $\varphi^* : \mathcal{R}_{\bar{F}} \to \mathcal{R}_{\bar{G}}$. And if $\bar{F}$ has a maximal rank, then $\mathcal{R}_{\bar{F}} = \mathcal{E}_U$. It seems that if $\bar{F}$ and $\bar{G}$ are symplectomorphic, then for $h \in \mathcal{R}_{\bar{F}}$, the isotropic map-germ F generated by h and the isotropic map-germ G generated by $\varphi^*(h)$ are L-symplectic equivalent, $G = \Psi \circ F \circ \varphi$. In this case $\Psi : T\mathbb{R}^{2n} \to T\mathbb{R}^{2n}$ is a symplectic lifting of the symplectomorphism $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. The aim of this section is to study the case when $\bar{F}$ does not have maximal rank and establish the structure of $\mathcal{R}_{\bar{F}}$. In the rest of this section, we study isotropic mappings with $\bar{F}$ of corank 1.

Let $e \in T_0U$ span the kernel of the Jacobian matrix $J\bar{F}$ of a corank one map-germ $\bar{F}$ at zero. By $\Delta_{\bar{F}}$, we denote the determinant of $J\bar{F}$ and by ∂_e the derivation into e-direction.

Theorem 2.3. (cf. [7]) *Let F be a smooth map-germ such that $\bar{F}$ has a corank one singularity at 0. If F is isotropic then there exists uniquely defined function-germ $h : (U, 0) \to (\mathbb{R}, 0)$ such that $\partial_e h \in \langle \Delta_{\bar{F}} \rangle$ and $(\beta \circ F)^*\theta = dh$, where $\langle \Delta_{\bar{F}} \rangle$ is the ideal generated by $\Delta_{\bar{F}}$ in $\mathcal{E}_U$. Conversely, for every smooth function-germ $h : (U, 0) \to \mathbb{R}$ such that $\partial_e h \in \langle \Delta_{\bar{F}} \rangle$ there is a uniquely defined isotropic map-germ $F : (U, 0) \to T\mathbb{R}^{2n}$ such that $\bar{F} = \pi \circ F$ and $(\beta \circ F)^*\theta = dh$.*

Proof. In coordinates of a source space U we write

$$J\bar{F} = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}$$

and by I_n we denote the unit matrix of dimension n.

In matrix form we get that a smooth map-germ F is isotropic if and only if there exists a smooth function-germ $h : (U, 0) \to \mathbb{R}$ such that

$$\begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix} = {}^t \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} \begin{pmatrix} \dot{f} \\ \dot{g} \end{pmatrix}. \tag{2.2}$$

Since we have assumed that the corank of $\bar{F} = (f, g) : (U, 0) \to \mathbb{R}^{2n}$ is one at the origin, then we can choose coordinates in U and $\mathbb{R}^{2n}$ such that

$$\begin{aligned} &f_i(u, v) = u_i, \quad i = 1, \ldots, n, \\ &g_i(u, v) = v_i, \quad i = 1, \ldots, n \quad 1, \\ &\frac{\partial g_n}{\partial v_n}(0, 0) = 0 \end{aligned} \tag{2.3}$$

and $e = \frac{\partial}{\partial v_n}$. Then

$$J\bar{F} = \begin{pmatrix} I_n & O & 0 \\ O & I_{n\ 1} & 0 \\ \frac{\partial g_n}{\partial u} & \frac{\partial g_n}{\partial \bar{v}} & \frac{\partial g_n}{\partial v_n} \end{pmatrix},$$

where $\bar{v} = (v_1, \ldots, v_{n\ 1})$.

Since $\dot{f}, \dot{g}$ in the equation (2.2) are smooth, we can write equivalently

$$\begin{pmatrix} \dot{f} \\ \dot{g} \end{pmatrix} = \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} {}^t \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}^{1} \begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix}. \tag{2.4}$$

From the form of

$${}^tJ\bar{F}^{\ 1} = {}^t \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}^{1} = \begin{pmatrix} I_n & O & \frac{\partial g_n}{\partial u}/\Delta_{\bar{F}} \\ O & I_{n\ 1} & \frac{\partial g_n}{\partial \bar{v}}/\Delta_{\bar{F}} \\ 0 & 0 & 1/\Delta_{\bar{F}} \end{pmatrix}. \tag{2.5}$$

We get

$$\frac{\partial h}{\partial v_n} \in \langle \Delta_{\bar{F}} \rangle. \tag{2.6}$$

For the other implication, if we have $h \in \mathbf{m}_U$ which fulfills the condition (2.6) then by the formula (2.4), we construct F in the unique way. ∎

Remark 2.4. Instead of isotropic F associated to $\bar{F}$, we consider pairs $(\bar{F}, h)$ with a smooth function-germ h belonging to $\mathcal{R}_{\bar{F}}$. An algebra $\mathcal{R}_{\bar{F}}$ of all generating function-germs associated to $\bar{F}$ is represented by $\bar{F}$ in the following form (cf. [9]),

$$\mathcal{R}_{\bar{F}} = \{h \in \mathcal{E}_U : dh \in \mathcal{E}_U d(\bar{F}^* \mathcal{E}_{\mathbb{R}^{2n}})\}.$$

Thus by Theorem 2.3, we get an algebra $\mathcal{R}_{\bar{F}}$ of all generating function-germs (which is also an $\mathcal{E}_{\mathbb{R}^{2n}}$-module) for a smooth map-germ $\bar{F}$ of corank one,

$$\mathcal{R}_{\bar{F}} = \{h \in \mathcal{E}_U : \partial_e h \in \langle \Delta_{\bar{F}} \rangle\}.$$

Remark 2.5. Let $F : (U, 0) \to T\mathbb{R}^{2n}$ be a smooth isotropic map-germ such that $\bar{F} = \pi \circ F : (U, 0) \to \mathbb{R}^{2n}$ has corank one singular point at $(0, 0)$. Then F has corank at most one at $(0, 0)$. The corank of F is exactly one if and only if

$$\partial_e(\partial_e h/\Delta_{\bar{F}})(0, 0) = 0.$$

2.1. Symplectic classification of corank 1 mappings. Let $F = (\bar{F}, \dot{F})$, $G = (\bar{G}, \dot{G}) : U \subset \mathbf{R}^{2n} \to T\mathbb{R}^{2n}$ be two smooth mappings. Suppose that $\bar{F}$ and $\bar{G}$ are symplectomorphic with a diffeomorphism-germ $\phi : (U, 0) \to (U, 0)$ and a symplectomorphism $\Phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ such that $\bar{G} = \Phi \circ \bar{F} \circ \phi$. Moreover, suppose that $\dot{G}$ is given by

$$\dot{G}(u, v) = J\Phi(\bar{F} \circ \phi(u, v))\dot{F}(\phi(u, v)), \tag{2.7}$$

where $J\Phi(\bar{F}(\phi(u, v)))$ is the Jacobian matrix of Φ at $\bar{F} \circ \phi(u, v)$ and is regarded as a linear transformation of the fiber over $\bar{F} \circ \phi(u, v)$ of the tangent bundle $T\mathbb{R}^{2n}$. Then *F is isotropic if and only if G is isotropic and F is smoothly solvable if and only if G is smoothly solvable.* Moreover, if $\gamma : (a, b) \to \mathbb{R}^{2n}$ is a solution of implicit differential equation $F(U) \subset T\mathbb{R}^{2n}$, then $\Phi \circ \gamma : (a, b) \to \mathbb{R}^{2n}$ is a solution of implicit differential equation $G(\phi^{-1}(U)) \subset T\mathbb{R}^{2n}$.

To describe $\mathcal{R}_{\bar{F}}$ in more clear way, we will classify corank 1 stable map-germs $\bar{F}$ up to symplectic equivalence. If $\bar{F} : (U, 0) \to (\mathbb{R}^{2n}, 0)$ is a corank 1 stable map-germ, then $\bar{F}$ is diffeomorphically equivalent (or diffeomorphic, [12]) to one of the A_k-type normal forms $(0 < k < 2n)$,

$$(w_1, \ldots, w_{2n}) \mapsto (w_1, \ldots, w_{2n-1}, w_{2n}^{k+1} + \sum_{i=1}^{k-1} w_i w_{2n}^{k-i}), \tag{2.8}$$

where we use the notation $(w_1, \ldots, w_{2n}) = (u_1, \ldots, u_n, v_1, \ldots, v_n)$.

Theorem 2.6. *Let $\bar{F} : (U, 0) \to (\mathbb{R}^{2n}, 0)$ be an A_k-type singular map-germ. Then $\bar{F}$ is symplectically equivalent to the following map-germ*

$$w = (w_1, \ldots, w_{2n}) \mapsto (w_1, \ldots, w_{2n-1}, w_{2n}^{k+1} + \sum_{i=1}^{k-1} a_i(w) w_{2n}^{k-i}), \tag{2.9}$$

where $a_1(w), \ldots, a_{k-1}(w)$ are smooth function-germs such that $da_1, \ldots, da_{k-1}$ and dw_{2n} are linearly independent at the origin.

Proof. Let $\bar{F} : (U, 0) \to (\mathbb{R}^{2n}, 0)$ be an A_k type singularity. Let $(\bar{w}_1, \ldots, \bar{w}_{2n})$ be coordinates in U. Then there exist diffeomorphism-germs

$\phi = (\phi_1, \phi_2, \ldots, \phi_{2n}) : (U, 0) \to (U, 0)$ and $\psi = (\psi_1, \psi_2, \ldots, \psi_{2n}) : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ such that

$$\psi_i \circ \bar{F} \circ \phi(\bar{w}_1, \ldots, \bar{w}_{2n}) = \bar{w}_i, \qquad i = 1, \ldots, 2n \quad 1,$$

$$\psi_{2n} \circ \bar{F} \circ \phi(\bar{w}_1, \ldots, \bar{w}_{2n}) = \bar{w}_{2n}^{k+1} + \sum_{i=1}^{k \quad 1} \bar{w}_i \bar{w}_{2n}^{k \quad i}. \tag{2.10}$$

We replace coordinates ψ by the symplectic ones. In fact, since $d\psi_{2n}$ does not vanish at the origin, there exists a symplectic coordinate system $(\varphi_1, \ldots, \varphi_{2n})$ on $(\mathbb{R}^{2n}, 0)$ with $\varphi_{2n} = \psi_{2n}$. Set

$$w_i = \varphi_i \circ \bar{F} \circ \phi(\bar{w}_1, \ldots, \bar{w}_{2n}), \qquad i = 1, \ldots, 2n \quad 1,$$

$$w_{2n} = \bar{w}_{2n}. \tag{2.11}$$

We see that $(w_1, \ldots, w_{2n})$ is a new coordinate system in $(U, 0)$. Indeed, for functions $\alpha_1 \ldots, \alpha_k$ and variables $\bar{w}_1, \ldots, \bar{w}_m$, let us denote the Jacobian matrix at the origin of $\alpha_1, \ldots, \alpha_k$ with respect to $\bar{w}_1, \ldots, \bar{w}_m$ by

$$J\left(\frac{\alpha_1, \alpha_2, \ldots, \alpha_k}{\bar{w}_1, \ldots, \bar{w}_m}\right)(0).$$

We have

$$\operatorname{rank} J\left(\frac{w_1, \ldots, w_{2n \quad 1}}{\bar{w}_1, \ldots, \bar{w}_{2n \quad 1}}\right)(0) =$$

$$= \operatorname{rank} J\left(\frac{\varphi_1 \circ \bar{F} \circ \phi, \ldots, \varphi_{2n \quad 1} \circ \bar{F} \circ \phi}{\bar{w}_1, \ldots, \bar{w}_{2n \quad 1}}\right)(0) = 2n \quad 1.$$

Thus $(w_1, \ldots, w_{2n \quad 1}, w_{2n} = \bar{w}_{2n})$ is a coordinate system. Now, from (2.10) and (2.11), we have

$$\varphi_i \circ \bar{F} \circ \phi = w_i, \qquad i = 1, \ldots, 2n \quad 1,$$

$$\varphi_{2n} \circ \bar{F} \circ \phi = w_{2n}^{k+1} + \sum_{i=1}^{k \quad 1} \bar{w}_i w_{2n}^{k \quad i}.$$

Taking inverse of (2.11), we write $a_i(w) = \bar{w}_i$, and obtain (2.9). ∎

Corollary 2.7. (symplectic fold) *Let $\bar{F} : (U, 0) \to (\mathbb{R}^{2n}, 0)$ be an A_1-type singularity, i.e. fold singularity. Then $\bar{F}$ is symplectically equivalent to*

$$(u_1, \ldots, u_n, v_1, \ldots, v_n) \mapsto (u_1, \ldots, u_n, v_1 \ldots, v_{n \quad 1}, v_n^2), \tag{2.12}$$

which is a simple symplectic normal form.

3. Smoothly solvable isotropic mappings

The natural property of smooth dynamical systems defined by smooth vector fields is their local solvability. This notion was generalized in [11, 5]

to smooth submanifolds of tangent bundle with possible singular projection into the base space.

Let $(M,0) \subset T\mathbb{R}^{2n}$ be a submanifold-germ defined as an image of smooth $F:(U,0)\to T\mathbb{R}^{2n}$ which has a maximal rank at 0. Then a point $(x,y)\in M$ is called *solvable point* of M if there exists a smooth curve $\gamma_{(x,y)}:(\ \epsilon,\epsilon)\to\mathbb{R}^{2n}$ such that $\gamma_{(x,y)}(0)=(x,y)$, $\gamma'_{(x,y)}(0)=(\dot{x},\dot{y})$, and

$$\kappa_{(x,y)}(t):=(\gamma_{(x,y)}(t),\gamma'_{(x,y)}(t))\in M,$$

for all $t\in(\ \epsilon,\epsilon)$, $\epsilon>0$, and the map $(x,y,t)\mapsto\kappa_{(x,y)}(t)$ is at least continuous. $(M,0)$ is called solvable if M (a representative of the germ $(M,0)$) consists of only solvable points.

A necessary condition for a smooth submanifold $M\subset T\mathbb{R}^{2n}$ to be solvable was found in [11] (cf. [5]). If π is a tangent bundle projection then the necessary solvability condition

$$(\dot{x},\dot{y})\in d(\pi|_M)_{(x,y,\dot{x},\dot{y})}(T_{(x,y,\dot{x},\dot{y})}M)$$

at $(x,y,\dot{x},\dot{y})\in M$ is called a *tangential solvability condition* and extended to the general smooth mapping $F=(f,g,\dot{f},\dot{g}):(U,0)\to T\mathbb{R}^{2n}$ is written in the form

$$(\dot{f},\dot{g})(u,v)\in J\bar{F}(u,v)(\mathbb{R}^{2n}), \tag{3.1}$$

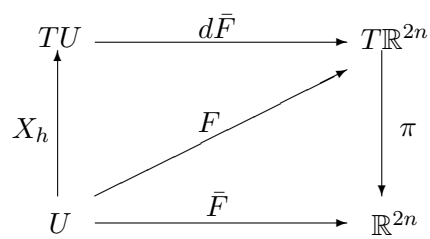
where $F(u,v)=(x,y,\dot{x},\dot{y})$.

Conditions for smooth solvability of implicit differential systems were investigated in [5] (cf. [14]). Now, we extend the solvability property introduced for a smooth submanifold of a tangent bundle defined by an immersion mapping F to the general smooth isotropic mappings into tangent bundle.

Definition 3.1. Let $F:(U,0)\to T\mathbb{R}^{2n}$ be a smooth isotropic map-germ with a generating function $h:(U,0)\to\mathbb{R}$. We say that F is *smoothly solvable* if there exists a smooth vector field X_h on U such that

$$F=d\bar{F}(X_h).$$

In other words the following diagram commutes



Example 3.2. It was shown in [5] (Example 5.1) that the tangential solvability condition is not sufficient for M to be solvable. An example of

isotropic map-germ $F : (U, 0) \to T\mathbb{R}^{2n}$, which fulfills the tangential solvability condition but is not solvable, is given by

$$F(u, v) = (v(1 \quad u^2), v^2u + u^3, u + 1, v),$$

with a generating function $h \in \mathcal{R}_{\bar{F}}$,

$$h(u, v) = \frac{3}{2}v^2u^2 \quad v^2u \quad \frac{3}{4}u^4 \quad u^3 + \frac{1}{2}v^2.$$

In this case does not exist a smooth vector field (germ) $X = a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}$ such that

$$\begin{pmatrix} \dot{f} \\ \dot{g} \end{pmatrix} = J\bar{F}(X) = \begin{pmatrix} 2vu & 1 & u^2 \\ v^2 + 3u^2 & & 2vu \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} u + 1 \\ v \end{pmatrix}. \tag{3.2}$$

Indeed, if X exists then there is a local smooth solution $t \mapsto (u(t), v(t))$ of X (i.e. $u' = a, v' = b$) such that

$$\begin{aligned} u + 1 = \quad & 2vuu' + (1 \quad u^2)v', \\ v = \quad & (v^2 + 3u^2)u' + 2vuv'. \end{aligned}$$

From the first equation, we have $v(t) = t + t^2\phi(t)$ and because $u(t) = \alpha t + t^2\psi(t)$ from the second equation, we get a contradiction.

The geometric meaning of the solvability property is explained in the following sufficient condition.

Theorem 3.3. *Let $\bar{F} = (f, g) : U \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a smooth mapping such that $\bar{F}$ has a corank k singularity at the origin $(0, 0) \in \mathbb{R}^{2n}$ and that the jet extension $j^1\bar{F} : U \to J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is transversal to the corank k stratum Σ^k of $J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. If an isotropic mapping F along $\bar{F}$ satisfies the tangential solvability condition, then F is smoothly solvable.*

Proof. Let $\bar{F} = (f, g) : U \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a smooth mapping such that $\bar{F}$ has a corank k singularity at the origin $(0, 0) \in \mathbb{R}^{2n}$ and that the jet extension $j^1\bar{F} : U \to J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is transversal to the corank k stratum Σ^k of $J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$.

Let $F = (f, g, \dot{f}, \dot{g})$ be an isotropic mapping along $\bar{F}$ which satisfies the tangential solvability condition:

$$\begin{pmatrix} \dot{f}(u, v) \\ \dot{g}(u, v) \end{pmatrix} \in Image\ J\bar{F}(u, v). \tag{3.3}$$

Since F is a smooth isotropic mapping, F is generated by a smooth function h:

$$\begin{pmatrix} \dot{f}(u, v) \\ \dot{g}(u, v) \end{pmatrix} = \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} {}^tJ\bar{F}^{\ 1} \begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix}. \tag{3.4}$$

We know that F is smoothly solvable if and only if

$$J\bar{F}^{\ 1}\begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix}{}^{t}J\bar{F}^{\ 1}\begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix} \quad \text{is smooth,} \tag{3.5}$$

which, on the basis of (3.4) is the case if and only if

$$J\bar{F}^{\ 1}\begin{pmatrix} \dot{f}(u,v) \\ \dot{g}(u,v) \end{pmatrix} \quad \text{is smooth,} \tag{3.6}$$

which is true if and only if the linear equation

$$J\bar{F}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \dot{f}(u,v) \\ \dot{g}(u,v) \end{pmatrix} \tag{3.7}$$

has a smooth solution $(a(u,v), b(u,v))$.

Since, from (3.3),

$$\begin{pmatrix} \dot{f}(u,v) \\ \dot{g}(u,v) \end{pmatrix} \in Image\ J\bar{F}(u,v), \qquad \text{for every point } (u,v) \in U$$

and $j^1\bar{F} : U \to J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is transversal to the corank k stratum Σ^k of $J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$, then from J. Mather's theorem [13], Equation (3.7) has a smooth solution and F is smoothly solvable. This completes the proof. ∎

3.1. Flows of solvable generalized Hamiltonian systems. A generalized Hamiltonian system is the image of $F : U \subset \mathbb{R}^{2n} \to T\mathbb{R}^{2n}$, which is an isotropic map generated by a smooth function h. If it is solvable then solutions of a generalized Hamiltonian system $F(U) \subset T\mathbb{R}^{2n}$ are the images under $\bar{F} = (f, g) : U \to \mathbb{R}^{2n}$ of integral curves of the vector field

$$X(u,v) = \sum_{i=1}^{n} \xi_i(u,v)\frac{\partial}{\partial u_i} + \eta_i(u,v)\frac{\partial}{\partial v_i} \tag{3.8}$$

on U.

Proposition 3.4. *Let $F : U \to T\mathbb{R}^{2n}$ be a solvable isotropic mapping along $\bar{F} : U \to \mathbb{R}^{2n}$ and let h be a generating function of F. Then the vector field X_h is tangent to the fold singular point set* $\mathrm{Fold}(\bar{F})$ *of $\bar{F}$ and integral curves of the vector filed X_h preserve the fold singular point set* $\mathrm{Fold}(\bar{F})$ *of $\bar{F}$.*

Proof. Suppose that $\bar{F} : (U, 0) \to (\mathbb{R}^{2n}, 0)$ has a fold singular point at 0. Then from the normal form of fold we may assume that in U, $\bar{F}$ has the form

$$(u_1, \cdots, u_n, v_1, \ldots, v_n) \mapsto (u_1, \ldots, u_n, v_1, \cdots, v_{n\ 1}, v_n^2). \tag{3.9}$$

Therefore

$$J\bar{F}(u,v) = \begin{pmatrix} I_n & \mathrm{O} & 0 \\ \mathrm{O} & I_{n\ 1} & 0 \\ 0 & 0 & 2v_n \end{pmatrix} \quad \text{and} \quad J\bar{F}^{\ 1}(u,v) = \begin{pmatrix} I_n & \mathrm{O} & 0 \\ \mathrm{O} & I_{n\ 1} & 0 \\ 0 & 0 & 1/2v_n \end{pmatrix}$$

and the singular point set $\Sigma(\bar{F})$ of $\bar{F}$ is

$$\Sigma(\bar{F}) = \{(u,v) \mid v_n = 0\}. \tag{3.10}$$

The vector field X_h has a form

$$X_h(u,v) = \sum_{i=1}^{n} \xi_i(u,v)\frac{\partial}{\partial u_i} + \eta_i(u,v)\frac{\partial}{\partial v_i}, \quad \dot{F} = J\bar{F}X_h, \tag{3.11}$$

$$\begin{aligned} \begin{pmatrix} \xi \\ \eta \end{pmatrix} &= J\bar{F}^{\ 1} \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} {}^{t}J\bar{F}^{\ 1} \begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{O} & 0 & I_{n\ 1} & 0 \\ 0 & 0 & 0 & 1/2v_n \\ I_{n\ 1} & 0 & \mathrm{O} & 0 \\ 0 & 1/2v_n & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix}. \end{aligned}$$

Since X_h is a smooth vector field,

$$\frac{\partial h}{\partial u_n}/v_n \quad \text{and} \quad \frac{\partial h}{\partial v_n}/v_n$$

must be smooth and $h(u,v)$ has the form

$$h(u,v) = {v_n}^2\alpha(u,v) + \beta(u',v'), \qquad (u',v') = (u_1,\dots,u_{n\ 1}, v_1,\dots,v_{n\ 1}),$$

for some smooth functions $\alpha(u,v)$ and $\beta(u',v')$. Therefore

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} v_n^2\frac{\partial\alpha}{\partial v'}(u,v) + \frac{\partial\beta}{\partial v'}(u',v') \\ 2\alpha(u,v) + v_n\frac{\partial\alpha}{\partial v_n}(u,v) \\ v_n^2\frac{\partial\alpha}{\partial u'}(u,v) \quad \frac{\partial\beta}{\partial u'}(u',v') \\ v_n\frac{\partial\alpha}{\partial u_n}(u,v) \end{pmatrix}.$$

Thus, the restriction of X_h to the singular point set $\Sigma(\bar{F}) = \{(u,v) \mid v_n = 0\}$ of $\bar{F}$ has the form

$$\begin{aligned} X_h = &\sum_{i=1}^{n\ 1} \frac{\partial\beta}{\partial v_i}(u',v') \left(\frac{\partial}{\partial u_i}\right) \quad \sum_{i=1}^{n\ 1} \frac{\partial\beta}{\partial u_i}(u',v') \left(\frac{\partial}{\partial v_i}\right) \\ &+ 2\alpha(u,v) \left(\frac{\partial}{\partial u_n}\right) \quad 0 \cdot \left(\frac{\partial}{\partial v_n}\right). \end{aligned}$$

Thus, the vector field X_h is tangent to the singular point set $\Sigma(\bar{F})$. This completes the proof of Proposition 3.4. ∎

Theorem 3.5. *Let $F : U \to T\mathbb{R}^{2n}$ be a solvable isotropic mapping along $\bar{F} : U \to \mathbb{R}^{2n}$ and let h be a generating function of F. Suppose that fold singular points of $\bar{F}$ are dense in the singular point set of $\bar{F}$. Then integral curves of the vector filed X_h preserve the singular point set of $\bar{F}$.*

Proof. From Proposition 3.4, integral curves of the vector field X_h preserve the fold singular point set $\mathrm{Fold}(\bar{F})$. Since integral curves of the vector field X_h depend smoothly on initial conditions and fold singular points of $\bar{F}$ are dense in the singular point set $\Sigma(\bar{F})$ thus the integral curves of the vector field X_h preserve the whole singular point set $\Sigma(\bar{F})$. ∎

Now we consider a global situation. Let M^{2n} be a compact smooth manifold of dimension $2n$. The isotropicity and the solvability are local notions, we may define isotropicity and solvability for global smooth mappings $F = (\bar{F}.\dot{F}) : M \to T\mathbb{R}^{2n}$. A *Hamiltonian mapping* is a smoothly solvable isotropic mapping $F = (\bar{F}.\dot{F}) : M \to T\mathbb{R}^{2n}$. Then F is locally generated by a function h and there exists a global vector field X, which is locally of the form (3.8) such that $\dot{F} = d\bar{F}X$.

Theorem 3.6. *Let M^{2n} be a compact smooth manifold. Let $F = (\bar{F}.\dot{F}) : M \to T\mathbb{R}^{2n}$ be a Hamiltonian mapping such that fold singular points are dense in the singular point set $\Sigma(\bar{F})$ of $\bar{F}$. Then integral curves of the vector filed X preserve the singular point set of $\bar{F}$. Consequently, solutions of the generalized Hamiltonian system $F(M) \subset T\mathbb{R}^{2n}$ preserve the singular value set of $\bar{F}$.*

3.2. Poincaré's recurrence theorem. In the present situation, Poincaré's recurrence theorem (see [8]) can be summarized as follows,

Theorem 3.7. (Poincaré's recurrence theorem) *Let M be a smooth manifold having a countable basis. Suppose that M has a measure m with $m(M) < \infty$. Let $\varphi : M \to M$ be a volume preserving homeomorphism. Then,*

1) *almost every point (with respect to m) on M is a recurrent point; for almost every $x \in M$, there is a sequence $n_j \uparrow \infty$ satisfying*

$$\lim_{j\to\infty} \varphi^{n_j}(x) = x,$$

or, equivalently

2) *for any point $x \in M$ and for any neighborhood U of x, there exist a point $y \in U$ and a number $n \in \mathbf{N}$ such that $\varphi^n(y) \in U$.*

We can apply this theorem to our global situation. Let M^{2n} be a compact smooth manifold of dimension $2n$. $F : M \to T\mathbb{R}^{2n}$ be a solvable isotropic

mapping along $\bar{F} : M \to \mathbb{R}^{2n}$ such that fold singular points of $\bar{F}$ are dense in the singular point set $\Sigma(\bar{F})$ of $\bar{F}$. Let X be the unique smooth vector field on M such that $F = d\bar{F}X$.

The map $\bar{F} : M \to \mathbb{R}^{2n}$ induces a symplectic structure $\bar{F}^*\omega$ on the regular point set $M \quad \Sigma(\bar{F})$, where ω is the canonical symplectic structure on $\mathbb{R}^{2n}$. We see that the vector field X is a complete hamiltonian vector field on the symplectic manifold $(M \quad \Sigma(\bar{F}), \bar{F}^*\omega)$ and the flow of X is volume preserving. Thus Poincaré's recurrence theorem holds as a straightforward consequence of Theorem 3.6.

Theorem 3.8. *Let M^{2n} be a compact smooth manifold of dimension $2n$. $F : M \to T\mathbb{R}^{2n}$ be a solvable isotropic mapping along $\bar{F} : M \to \mathbb{R}^{2n}$ such that fold singular points of $\bar{F}$ are dense in the singular point set $\Sigma(\bar{F})$ of $\bar{F}$. Let X be the unique smooth vector field on M such that $F = d\bar{F}X$. Then almost every regular point p of $\bar{F}$ is a recurrent point of the integral curve $\varphi^t(p)$ of X; there is a sequence $t_j \uparrow \infty$ satisfying*

$$\lim_{j\to\infty} \varphi^{t_j}(p) = p.$$

4. Poisson algebra of solvable isotropic mappings

Let $\bar{F} : \mathbb{R}^{2n} \supset U \to (\mathbb{R}^{2n}, \omega)$ be a smooth map-germ, then $\bar{F}$ induces a possibly degenerate two-form $\bar{F}^*\omega$ on U. For a smooth function h defined on U, we formally define the Hamiltonian vector field X_h (which may not be smooth) on U by the equality

$$\bar{F}^*\omega(X_h, \xi) = \quad \xi(h) \quad \text{for each vector field} \quad \xi \quad \text{on} \quad U. \tag{4.1}$$

For smooth functions k, h defined on $U \subset \mathbb{R}^{2n}$, we can define also the formal brackets $\{k, h\}_{\bar{F}^*\omega}$, by

$$\{k, h\}_{\bar{F}^*\omega} := \bar{F}^*\omega(X_k, X_h). \tag{4.2}$$

It may happen that X_h, X_k and $\{k, h\}_{\bar{F}^*\omega}$ diverge on the singular point set of $\bar{F}$. However, they are ordinary Poisson brackets outside of this set. Now, we search for conditions on h such that X_h is smooth.

Definition 4.1. Let $h : \mathbb{R}^{2n} \supset U \to \mathbb{R}$ be a smooth function. If X_h defined by (4.1) is smooth then X_h is called a *Hamiltonian vector field* and h is called the *Hamiltonian function*. By

$$\mathcal{H}_{\bar{F}} = \{h \in C^\infty(U) : X_h \text{ is smooth}\} \tag{4.3}$$

we denote the space of all Hamiltonians associated to $\bar{F}$ ($\bar{F}$ Poisson algebra).

We notice that if $h, k \in \mathcal{H}_{\bar{F}}$, then $hk \in \mathcal{H}_{\bar{F}}$.

Theorem 4.2. *Let $\bar{F} : \mathbb{R}^{2n} \supset U \to (\mathbb{R}^{2n}, \omega)$ be a smooth map whose regular point set is dense in U. Then $\mathcal{H}_{\bar{F}}$ is closed under the brackets $\{\cdot,\cdot\}_{\bar{F}^*\omega}$ and the space $(\mathcal{H}_{\bar{F}}, \{\cdot,\cdot\}_{\bar{F}^*\omega})$ is a Poisson algebra.*

Proof. Let U be an open ball neighborhood of the origin of $\mathbb{R}^m$. Let $\Delta(x_1, \ldots, x_m)$ be a smooth function defined on U and let Ω be the set $\{x \in U \mid \Delta(x) \neq 0\}$. Suppose that Ω is dense in U. Let $a(x)$ be a fractional function whose numerator is a smooth function defined on U and whose denominator is $\Delta(x)$:

$$a(x) = \frac{\alpha(x)}{\Delta(x)}.$$

If the restriction $a|_\Omega$ to Ω is extendable to a smooth function on U, then $a(x)$ itself is smooth on U, i.e. α is divisible by Δ.

Let U be an open ball neighborhood of the origin $(0,0)$ in $\mathbb{R}^{2n}$. Let $\bar{F} : \mathbb{R}^{2n} \supset U \to (\mathbb{R}^{2n}, \omega)$ be a map whose regular point set is dense in U. Let $\Delta_{\bar{F}}(u,v)$ be the Jacobian determinant of $\bar{F}$.

Let $\Omega = \{(u,v) \in U \mid \Delta_{\bar{F}}(u,v) \neq 0\}$ be the set of regular points of $\bar{F}$ which we assume is dense in U. Then the restriction $\bar{F}^*\omega|_\Omega$ to Ω of the 2-form $\bar{F}^*\omega$ is non-degenerate. Let h be a smooth function defined on U. Then the Hamiltonian vector field X_h is defined by the equality

$$\bar{F}^*\omega(X_h, \xi) = \quad \xi(h), \qquad \text{for each vector field } \xi \text{ on } U.$$

Let us express X_h in the form

$$X_h = \sum_{i=1}^{n} \left(a_i(u,v) \frac{\partial}{\partial u_i} + b_i(u,v) \frac{\partial}{\partial v_i} \right). \tag{4.4}$$

Then, after some calculations we have that each coefficient $a_i(u,v)$ or $b_i(u,v)$ of X_h is a sum of a smooth function, a fractional function whose numerator is a smooth function and denominator is $\Delta_{\bar{F}}$ and a fractional function whose numerator is a smooth function and denominator is $\Delta_{\bar{F}}^2$, in which numerators may vanish as well.

For any smooth function h, the restriction $X_h|_\Omega$ to Ω of the vector field X_h is always smooth. Therefore, the restrictions $a_i|_\Omega, b_i|_\Omega$'s to Ω of the coefficients a_i, b_i's are also always smooth. Thus from the form of (4.4), we see that X_h is smooth if and only if $a_i|_\Omega, b_i|_\Omega$'s are extendable to smooth functions defined on U.

Now let $h, k \in \mathcal{H}_{\bar{F}}$. Then h, k, X_h, X_k are all smooth on U. Hence $\{h,k\}_{\bar{F}^*\omega} = X_h(k)$ is smooth on U. And we have

$$X_{\{h,k\}_{(f,g)^*\omega}}|_\Omega = [X_h|_\Omega, X_k|_\Omega] = X_h|_\Omega X_k|_\Omega \quad X_k|_\Omega X_k|_\Omega. \tag{4.5}$$

Since X_h and X_k are smooth on U, the right-hand side of (4.5) is extendable to the bracket vector field $[X_h, X_k]$ which is smooth on U. Since the coefficients

of $X_{\{h,k\}_{F^*\omega}}|_\Omega$ are extendable to the coefficients of $[X_h, X_k]$ which are smooth on U, then the coefficients of $X_{\{h,k\}_{F^*\omega}}$ themselves are smooth on U. Thus $X_{\{h,k\}_{F^*\omega}}$ is also smooth on U. Thus $\{h,k\}_{\bar{F}^*\omega} \in \mathcal{H}_{\bar{F}}$. ∎

Definition 4.3. The space $(\mathcal{H}_{\bar{F}}, \{\cdot,\cdot\}_{\bar{F}^*\omega})$ endowed with

$$\{k,h\}_{\bar{F}^*\omega} := \bar{F}^*\omega(X_k, X_h), \quad h,k \in \mathcal{H}_{\bar{F}},$$

is called the *Poisson algebra associated* to $\bar{F}$ (or $\bar{F}$*-Poisson algebra*) endowed with the Poisson brackets $\{k,h\}_{\bar{F}^*\omega}$.

Theorem 4.4. *Let $F : (U,0) \to T\mathbb{R}^{2n}$ be a smooth isotropic map-germ along a smooth map-germ $\bar{F} : (U,0) \to \mathbb{R}^{2n}$ such that the regular point set of $\bar{F}$ is dense in U. Let $h : (U,0) \to \mathbb{R}$ be a generating function-germ of F. Then F is smoothly solvable if and only if $h \in \mathcal{H}_{\bar{F}}$, i.e. h is a Hamiltonian function.*

Proof. Following the proof of Theorem 4.2, we need to show that the equation (4.1) defining the Hamiltonian vector field X_h is equivalent to the equation (2.1) expressed in the form

$$(\beta \circ d\bar{F}(X_h))^*\theta = \quad dh. \tag{4.6}$$

Then we get solvability of an isotropic map F immediately.

Let $X_h = \sum_{i=1}^n (a_i(u,v)\frac{\partial}{\partial u_i} + b_i(u,v)\frac{\partial}{\partial v_i})$. Putting $\frac{\partial}{\partial u_i}$, $\frac{\partial}{\partial v_i}$ into (4.1) instead of ξ, we obtain

$$\begin{aligned}\frac{\partial h}{\partial w_i} = \quad \bar{F}^*\omega\left(X_h, \frac{\partial}{\partial w_i}\right) &= \sum_{j=1}^n \sum_{k=1}^n a_j(u,v)\left(\frac{\partial f_k}{\partial w_i}\frac{\partial g_k}{\partial u_j} + \frac{\partial g_k}{\partial w_i}\frac{\partial f_k}{\partial u_j}\right) \\ &+ \sum_{j=1}^n \sum_{k=1}^n b_j(u,v)\left(\frac{\partial f_k}{\partial w_i}\frac{\partial g_k}{\partial v_j} + \frac{\partial g_k}{\partial w_i}\frac{\partial f_k}{\partial v_j}\right),\end{aligned} \tag{4.7}$$

where $(w_1,\dots,w_{2n}) = (u_1,\dots,u_n,v_1,\dots,v_n)$. It is easy to see that (4.7) is equivalent in the matrix form to the equation

$$\begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix} = {}^t \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Thus (4.6) is smoothly invertible for X_h. ∎

Remark 4.5. Since smooth solvability of an isotropic map F generated by a smooth function $h : U \to \mathbb{R}$ is defined by smoothness of X_h, then an equivalent condition for smooth solvability of F can be given in terms of the Poisson bracket, namely:

F is smoothly solvable or equivalently h is a Hamiltonian function on U if $\{h,\alpha\}_{\bar{F}^*\omega}$ is smooth on U for all smooth functions α defined on U.

4.1. Smooth solvability related to Poisson structure. Smooth solvability is a structural property preserved by Poisson bracket defined on the space of Hamiltonians $\mathcal{H}_{\bar{F}}$. However, the space of generating functions $\mathcal{R}_{\bar{F}}$ is not preserved by the Poisson bracket $\{\cdot,\cdot\}_{\bar{F}*_\omega}$. As an example, we consider the fold map

$$\bar{F}:\mathbb{R}^2\to\mathbb{R}^2,\quad \bar{F}(u,v)=(u,v^2/2).$$

In this case $\mathcal{R}_{\bar{F}}=\{h:\frac{\partial h}{\partial v}\in\langle v\rangle\}$. Taking $h=u\in\mathcal{R}_{\bar{F}},\quad k=v^3\in\mathcal{R}_{\bar{F}}$ we find $\{h,k\}_{\bar{F}*_\omega}=\quad 3v$ thus

$$\frac{\partial\{h,k\}_{\bar{F}*_\omega}}{\partial v}\notin\langle v\rangle \text{ and } \{h,k\}_{\bar{F}*_\omega}\notin\mathcal{R}_{\bar{F}}.$$

Let us consider the natural subspace $\mathcal{R}^T_{\bar{F}}$ of the space of generating functions for isotropic mappings along $\bar{F}$ satisfying the tangential solvability condition (3.1).

$$\mathcal{R}^T_{\bar{F}}=\{h\in C^\infty(U):h\in\mathcal{R}_{\bar{F}} \text{ and } F \text{ generated by } h \text{ satisfies (3.1)}\},$$

which will be called *the space of tangential generating functions.*

In the case if $\bar{F}$ has a corank k singularity at 0 and the transversality assumption of Theorem 3.3 is satisfied then $\mathcal{R}^T_{\bar{F}}=\mathcal{H}_{\bar{F}}$. In general, $\mathcal{H}_{\bar{F}}$ is a proper subset of $\mathcal{R}^T_{\bar{F}}$ and there is a natural question if the Poisson structure $\{.,.\}_{\bar{F}*_\omega}$can be extended to $\mathcal{R}^T_{\bar{F}}$? By the following example, we know that this is impossible.

Example 4.6. Let $\bar{F}:\mathbb{R}^2\to(\mathbb{R}^2,\omega)$ be defined by

$$\bar{F}(u,v)=\left(u,u^2v+\frac{1}{3}v^3\right).$$

We show that $\mathcal{R}^T_{\bar{F}}$ is not closed under the Poisson bracket. First we calculate the jacobian matrix of $\bar{F}$

$$J\bar{F}(u,v)=\begin{pmatrix}1 & 0\\ 2uv & u^2+v^2\end{pmatrix},\qquad J\bar{F}^{\ 1}(u,v)=\begin{pmatrix}1 & 0\\ \frac{2uv}{u^2+v^2} & \frac{1}{u^2+v^2}\end{pmatrix}.$$

From the condition of isotropicity (cf. Theorem 2.3), we have

$$\frac{\partial h}{\partial v}\in\langle\Delta_{\bar{F}}\rangle=\langle u^2+v^2\rangle$$

thus

$$h=(u^2+v^2)^2\alpha(u,v)+\beta(u).$$

Now, we check the tangential solvability condition at $(0,0)$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^tJ\bar{F}^{-1}(u,v) \begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix}_{(u,v)=(0,0)} \in Image J\bar{F}(0,0).$$

And obtain

$$h(u,v) = (u^2+v^2)^2\alpha(u,v) + u^4\beta(u) + const. \tag{4.8}$$

Thus,

$$\mathcal{R}^T_{\bar{F}} = \{h \in C^\infty(U) \mid h(u,v) = (u^2+v^2)^2\alpha(u,v) + u^4\beta(u) + const \text{ for some smooth } \alpha(u,v) \text{ and } \beta(u)\}. \tag{4.9}$$

Consider the following two elements of $\mathcal{R}^T_{\bar{F}}$

$$h(u,v) = (u^2+v^2)^2 + u^4,$$
$$k(u,v) = (u^2+v^2)^2 v + u^4.$$

The Poisson bracket of h and k is given by

$$\{h,k\}_{\bar{F}*_\omega} = \quad 4u(u^2+v^2)^2 \quad 4u^3(u^2+v^2) \quad 16u^3v^2 + 16u^3v.$$

And consequently

$$\{h,k\}_{\bar{F}*_\omega} \notin \mathcal{R}^T_{\bar{F}}.$$

Thus, $\mathcal{R}^T_{\bar{F}}$ is not closed under the Poisson bracket.

We can easily see that the transversality condition of Theorem 3.3 is only a sufficient condition. We can find examples of $\bar{F}$ such that the jet extension $j^1\bar{F} : U \to J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is not transversal to the corank k stratum Σ^k of $J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ but $\mathcal{R}^T_{\bar{F}}$ is closed under the Poisson bracket. In fact, we can take

$$\bar{F} : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0), \qquad \bar{F}(u,v) = \left(u, \frac{1}{k+1}v^{k+1}\right).$$

We see that $\bar{F}$ has corank 1 at $(u,0)$ but $j^1\bar{F}$ is not transversal to the corank 1 stratum in the jet space for $k \geq 2$. Then by straightforward calculations, we show also that $\mathcal{R}^T_{\bar{F}}$ is closed under the Poisson bracket. Moreover, in this example we have $\mathcal{R}^T_{\bar{F}} = \mathcal{H}_{\bar{F}}$. Then the natural question arises: If there is any smooth mapping $\bar{F}$ such that $\mathcal{R}^T_{\bar{F}}$ is closed under the Poisson bracket but $\mathcal{R}^T_{\bar{F}} \neq \mathcal{H}_{\bar{F}}$, or we conjecture that

$\mathcal{R}^T_{\bar{F}} = \mathcal{H}_{\bar{F}}$ holds always if $\mathcal{R}^T_{\bar{F}}$ is closed under the Poisson bracket.

4.2. Solvability condition for corank 1 case. Now we find conditions describing the Poisson space associated to $\bar{F}$ which has a corank 1 singularity at the origin $(0,0) \in U \subset \mathbb{R}^{2n}$.

Theorem 4.7. *Let $F : (U, 0) \to T\mathbb{R}^{2n}$ be a smooth isotropic map-germ such that $\bar{F} = \pi \circ F$ has a corank 1 singularity at $(0, 0) \in U \subset \mathbb{R}^{2n}$ expressed in local coordinates (u, v) defined in (2.3). Let $h : (U, 0) \to \mathbb{R}$ be a smooth generating function-germ for F defined on U. Then F is smoothly solvable if and only if*

$$\frac{\partial h}{\partial v_n} \in \langle \Delta_{\bar{F}} \rangle, \tag{4.10}$$

and

$$\sum_{i=1}^{n\ \ 1} \left(\frac{\partial g_n}{\partial v_i} \frac{\partial h}{\partial u_i} \quad \frac{\partial g_n}{\partial u_i} \frac{\partial h}{\partial v_i} \right) \quad \frac{\partial h}{\partial u_n} \in \langle \Delta_{\bar{F}} \rangle. \tag{4.11}$$

Proof. From (4.1) taking $X_h = \sum_{i=1}^{n} (a_i(u, v) \frac{\partial}{\partial u_i} + b_i(u, v) \frac{\partial}{\partial v_i})$ for the local form of $\bar{F}$ given by (2.3), we calculate the coefficients of X_h

$$\begin{aligned}
a_i &= \frac{\partial h}{\partial v_i} \quad \frac{\partial g_n}{\partial v_i} \frac{\partial h}{\partial v_n} / \Delta_{\bar{F}}, && i = 1, \ldots, n \quad 1, \\
a_n &= \frac{\partial h}{\partial v_n} / \Delta_{\bar{F}}, \\
b_i &= \quad \frac{\partial h}{\partial u_i} + \frac{\partial g_n}{\partial u_i} \frac{\partial h}{\partial v_n} / \Delta_{\bar{F}}, && i = 1, \ldots, n \quad 1, \\
b_n &= \frac{1}{\Delta_{\bar{F}}} \left(\quad \frac{\partial h}{\partial u_n} + \sum_{i=1}^{n\ \ 1} \frac{\partial g_n}{\partial v_i} \frac{\partial h}{\partial u_i} \quad \sum_{i=1}^{n\ \ 1} \frac{\partial g_n}{\partial u_i} \frac{\partial h}{\partial v_i} \right),
\end{aligned} \tag{4.12}$$

which are smooth if and only if (4.10) and (4.11) are fulfilled. ∎

Remark 4.8. By straightforward calculations, we get

$$\begin{aligned}
\{h, v_n\}_{\bar{F}*\omega} &:= \bar{F}^*(\omega)(X_h, X_{v_n}) = X_h(v_n) = \sum_{i=1}^{n} \left(a_i \frac{\partial v_n}{\partial u_i} + b_i \frac{\partial v_n}{\partial v_i} \right) \\
&= \frac{1}{\Delta_{\bar{F}}} \left(\quad \frac{\partial h}{\partial u_n} + \sum_{i=1}^{n\ \ 1} \frac{\partial g_n}{\partial v_i} \frac{\partial h}{\partial u_i} \quad \sum_{i=1}^{n\ \ 1} \frac{\partial g_n}{\partial u_i} \frac{\partial h}{\partial v_i} \right) \frac{\partial v_n}{\partial v_n}.
\end{aligned}$$

The condition (4.11) may be rewritten in the form

$$\{h, v_n\}_{\bar{F}*\omega} \quad \text{is smooth on } U.$$

Remark 4.9. The space of Hamiltonian functions $\mathcal{H}_{\bar{F}}$ and its corresponding space of smoothly solvable isotropic mappings along $\bar{F}$ are symplectically invariant Poisson algebras. $\mathcal{H}_{\bar{F}}$ is an $\mathbb{R}$-subalgebra of the $\mathbb{R}$-algebra $\mathcal{R}_{\bar{F}}$ which is an $\mathcal{E}_{\mathbb{R}^{2n}}$-submodule of $\mathcal{E}_U$,

$$\mathcal{H}_{\bar{F}} \subset \mathcal{R}_{\bar{F}} \subset \mathcal{E}_U.$$

For the corank 1 mapping $\bar{F} = (f, g) : (U, 0) \to \mathbb{R}^{2n}$, we can write the Poisson bracket

$$\{h,k\}_{\bar{F}*\bar{\omega}} = \left(\frac{\partial k}{\partial u}, \frac{\partial k}{\partial v}\right) \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}^{-1} \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} {}^t \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix},$$

and

$$\begin{aligned}
&\begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}^{-1} \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} {}^t \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} I_n & O & 0 \\ O & I_{n-1} & 0 \\ \frac{\partial g_n}{\partial u_j} & \frac{\partial g_n}{\partial v_j} & \frac{\partial g_n}{\partial v_n} \end{pmatrix}^{-1} \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} {}^t \begin{pmatrix} I_n & O & 0 \\ O & I_{n-1} & 0 \\ \frac{\partial g_n}{\partial u_j} & \frac{\partial g_n}{\partial v_j} & \frac{\partial g_n}{\partial v_n} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} I_n & O & 0 \\ O & I_{n-1} & 0 \\ \frac{\partial g_n}{\partial u_j}/\Delta_{\bar{F}} & \frac{\partial g_n}{\partial v_j}/\Delta_{\bar{F}} & 1/\Delta_{\bar{F}} \end{pmatrix} \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} \begin{pmatrix} I_n & O & \frac{\partial g_n}{\partial u_j}/\Delta_{\bar{F}} \\ O & I_{n-1} & \frac{\partial g_n}{\partial v_j}/\Delta_{\bar{F}} \\ 0 & 0 & 1/\Delta_{\bar{F}} \end{pmatrix} \\
&= \begin{pmatrix} O & 0 & I_{n-1} & \frac{\partial g_n}{\partial v_j}/\Delta_{\bar{F}} \\ 0 & 0 & 0 & 1/\Delta_{\bar{F}} \\ I_{n-1} & 0 & O & \frac{\partial g_n}{\partial u_j}/\Delta_{\bar{F}} \\ \frac{\partial g_n}{\partial v_j}/\Delta_{\bar{F}} & 1/\Delta_{\bar{F}} & \frac{\partial g_n}{\partial u_j}/\Delta_{\bar{F}} & 0 \end{pmatrix}.
\end{aligned}$$

Thus, for the fold singularity (2.12)

$$\{h,k\}_{\bar{F}*\bar{\omega}} = \sum_{i=1}^{n-1} \left(\frac{\partial h}{\partial v_i}\frac{\partial k}{\partial u_i} \quad \frac{\partial k}{\partial v_i}\frac{\partial h}{\partial u_i}\right) + \frac{1}{2v_n}\left(\frac{\partial h}{\partial v_n}\frac{\partial k}{\partial u_n} \quad \frac{\partial k}{\partial v_n}\frac{\partial h}{\partial u_n}\right),$$

where $h, k \in \mathcal{H}_{\bar{F}}$, and

$$\mathcal{H}_{\bar{F}} = \{h : \frac{\partial h}{\partial v_n}, \frac{\partial h}{\partial u_n} \in \langle \Delta_{\bar{F}} \rangle\}.$$

5. Structure of the Poisson algebra $\mathcal{H}_{\bar{F}}$

The natural ideals of $\mathcal{H}_{\bar{F}}$ are those generated by powers of the Jacobian determinant. We recall that a function h belongs to $\mathcal{H}_{\bar{F}}$ if and only if

$$J\bar{F}^{-1} \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} {}^t J\bar{F}^{-1} \begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix}$$

is smooth. Let $\Delta_{\bar{F}}$ denote the jacobian determinant $\det J\bar{F}$ and let $\widetilde{J\bar{F}}$

denote the cofactor matrix of $J\bar{F}$. Then we have

$$J\bar{F}^{-1} = \frac{1}{\Delta_{\bar{F}}}\,\widetilde{J\bar{F}}.$$

Therefore, h belongs to $\mathcal{H}_{\bar{F}}$ if and only if

$$\frac{1}{{\Delta_{\bar{F}}}^2}\,\widetilde{J\bar{F}}\begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix}\,{}^t\widetilde{J\bar{F}}\begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix}$$

is smooth. Thus, if h belongs to the ideal $\langle {\Delta_{\bar{F}}}^3 \rangle$, then $h \in \mathcal{H}_{\bar{F}}$. Now, we prove the following stronger result.

Theorem 5.1. *Let $\bar{F} = (f, g) : (U, 0) \to \mathbb{R}^{2n}$ be a smooth map-germ, then following holds:*

1) $\langle {\Delta_{\bar{F}}}^2 \rangle \subset \mathcal{H}_{\bar{F}}$.
2) *For $\ell \geq 3$, $\langle {\Delta_{\bar{F}}}^{\ell} \rangle$ is a Poisson subalgebra of $\mathcal{H}_{\bar{F}}$.*
3) *For $\ell \geq 3$, $\langle {\Delta_{\bar{F}}}^{\ell} \rangle$ is an ideal of $\langle {\Delta_{\bar{F}}}^3 \rangle$.*

Before we prove this theorem, we need the following

Lemma 5.2. *Let J_n denote the matrix*

$$J_n = \begin{pmatrix} \mathrm{O} & I_n \\ I_n & \end{pmatrix}.$$

Let $A = (a_{ij})$ be a square matrix of size $2n$ and let $\widetilde{A}$ denote its cofactor matrix. Let

$$B = (b_{k\ell}) = \widetilde{A} J_n\ {}^t\widetilde{A}, \qquad \text{where } {}^t\widetilde{A} \text{ denotes the transpose of } \widetilde{A}.$$

Then we have

$$b_{k\ell} \in \langle \det A \rangle_{\mathbb{R}[a_{11}, a_{12}, \ldots, a_{nn}]}. \tag{5.1}$$

In other words, $\det A$ *divides every entry $b_{k\ell}$ of the matrix $\widetilde{A} J_n\ {}^t\widetilde{A}$ as polynomials of the variables $a_{11}, a_{12}, \ldots, a_{nn}$.*

Proof. Let us denote by $A_{(k,\ell;i,n+i)}$, the square matrix of size $2n \quad 2$ obtained from A deleting kth and ℓth rows and ith and $n + i$th columns. Then we can state our Lemma in more precise form:

$$b_{k\ell} = \Big(\sum_{i=1}^{n} \det A_{(k,\ell;i,n+i)}\Big) \det A. \tag{5.2}$$

Let $J_{i,n+i}$ denote the matrix $J_{i,n+i} = (\epsilon_{kl})$ given by

$$\epsilon_{kl} = \begin{cases} 1, & \text{for } (k, l) = (i, n + i), \\ 1, & \text{for } (k, l) = (n + i, i), \\ 0, & \text{otherwise.} \end{cases}$$

Namely

$$J_{i,n+i} = \begin{pmatrix} 0 & & & & & & 0 \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ 0 & & & & & & 0 \end{pmatrix}$$

Consider the matrix

$$C_i = (c_{k\ell}) = \widetilde{A} J_{i,n+i}\ {}^t\widetilde{A}.$$

Since

$$J_n = J_{1,n+1} + \cdots + J_{n,2n} \qquad \text{and} \qquad \widetilde{A} J_n\ {}^t\widetilde{A} = \widetilde{A} J_{1,n+1}\ {}^t\widetilde{A} + \cdots + \widetilde{A} J_{n,2n}\ {}^t\widetilde{A},$$

to prove (5.2), it suffices to prove

$$c_{k\ell} = \det A_{(k,\ell;i,n+i)} \cdot \det A, \tag{5.3}$$

which can be proved as follows:

We calculate C_i.

$$\begin{aligned} C_i = (c_{k\ell}) &= \widetilde{A} J_{i,n+i}\ {}^t\widetilde{A} \\ &= \widetilde{A} \begin{pmatrix} 0 & & & & & & 0 \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ 0 & & & & & & 0 \end{pmatrix} \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{21} & \cdots & & \tilde{a}_{2n,1} \\ \tilde{a}_{12} & \tilde{a}_{22} & \cdots & & \tilde{a}_{2n,1} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \tilde{a}_{1,2n} & \tilde{a}_{2,2n} & \cdots & & \tilde{a}_{2n,2n} \end{pmatrix} \\ &= \widetilde{A} \begin{pmatrix} 0 & \cdots & 0 \\ \tilde{a}_{1,n+i} & \cdots & \tilde{a}_{2n,n+i} \\ 0 & \cdots & 0 \\ \tilde{a}_{1,i} & \cdots & \tilde{a}_{2n,i} \\ 0 & \cdots & 0 \end{pmatrix}. \end{aligned}$$

Thus we have

$$c_{k,l} = \tilde{a}_{k,i}\tilde{a}_{\ell,n+i} \quad \tilde{a}_{k,n+i}\tilde{a}_{\ell,i}. \tag{5.4}$$

Now consider the matrix obtained from A permuting rows and columns so that the n 1th row is replaced by the ith row, the nth row by n ith row, the n 1th column is replaced by the kth column, the nth column by ℓth column;

$$\begin{pmatrix} & & & a_{1k} & a_{1\ell} \\ & (a_{pq}) & & \vdots & \vdots \\ & & & a_{2n\ 1,k} & a_{2n\ 1,\ell} \\ & & & a_{2n,k} & a_{2n,\ell} \\ a_{i1} & \cdots & a_{i,2n} & a_{ik} & a_{i\ell} \\ a_{n+i,1} & \cdots & a_{n+i,2n} & a_{n+i,k} & a_{n+i,\ell} \end{pmatrix},$$

and consider the multiplication of it by a matrix obtained from the transpose ${}^t\widetilde{A}$ of $\widetilde{A}$:

$$\begin{pmatrix} & & & a_{k1} & a_{\ell 1} \\ & (a_{qp}) & & \vdots & \vdots \\ & & & a_{k,2n\ 1} & a_{\ell,2n\ 1} \\ & & & a_{k,2n} & a_{\ell,2n} \\ a_{1i} & \cdots & a_{2n,i} & a_{ki} & a_{\ell i} \\ a_{1,n+i} & \cdots & a_{2n,n+i} & a_{k,n+i} & a_{\ell,n+i} \end{pmatrix} \begin{pmatrix} & \tilde{a}_{1,i} & \tilde{a}_{1,n+i} \\ I_{2n\ 2} & \vdots & \vdots \\ & \tilde{a}_{2n\ 1,i} & \tilde{a}_{2n\ 1,n+i} \\ & \tilde{a}_{2n,i} & \tilde{a}_{2n,n+i} \\ 0 & \tilde{a}_{k,i} & \tilde{a}_{k,n+i} \\ 0 & \tilde{a}_{\ell,i} & \tilde{a}_{\ell,n+i} \end{pmatrix}. \tag{5.5}$$

Then the $(2n \quad 2) \times (2n \quad 2)$ minor (a_{qp}) at the upper left corner of the left-hand matrix of (2) is the transpose ${}^tA_{(k,\ell;i,n+i)}$ of $A_{(k,\ell;i,n+i)}$.

Note that

(5.6) the determinant of the matrix on the left is equal to $\det A$

and the determinant of the matrix on the right is equal to

$$\tilde{a}_{k,i}\tilde{a}_{\ell,n+i} \quad \tilde{a}_{k,n+i}\tilde{a}_{\ell,i} = c_{k\ell}. \tag{5.7}$$

Since ${}^tA\ {}^t\widetilde{A} = I_{2n}$ and

$$\sum_{q=1}^{2n} a_{qp}\tilde{a}_{q,r} = \delta_{pr} \det A,$$

we see that (5.5) is equal to

$$\begin{pmatrix} & & & 0 & 0 \\ & {}^tA_{(k,\ell;i,n+i)} & & \vdots & \vdots \\ & & & 0 & 0 \\ a_{1i} & \cdots & a_{2n,i} & \det A & 0 \\ a_{1,n+i} & \cdots & a_{2n,n+i} & 0 & \det A \end{pmatrix}. \tag{5.8}$$

From (5.5)–(5.8), we have

$$\det A \cdot (\tilde{a}_{k,i}\tilde{a}_{\ell,n+i} \quad \tilde{a}_{k,n+i}\tilde{a}_{\ell,i}) = \det A_{(k,\ell;i,n+i)} \cdot \det A^2.$$

Thus we have

$$c_{k\ell} = (\tilde{a}_{k,i}\tilde{a}_{\ell,n+i} \quad \tilde{a}_{k,n+i}\tilde{a}_{\ell,i}) = \det A_{(k,\ell;i,n+i)} \cdot \det A. \tag{5.9}$$

This proves Lemma 5.2. ∎

Proof of Theorem 5.1. Let $\bar{F} : U(\subset \mathbb{R}^{2n}) \to \mathbb{R}^{2n}$ be a smooth mapping. A function h generates a solvable isotropic mapping if and only if

$$J\bar{F}^{\ -1} \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} {}^t J\bar{F}^{\ -1} \begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix}$$

is smooth. Let $\Delta_{\bar{F}}$ denote the jacobian determinant $\det J\bar{F}$ and let $\widetilde{J\bar{F}}$ denote the cofactor matrix of $J\bar{F}$. Then we have

$$J\bar{F}^{\ -1} = \frac{1}{\Delta_{\bar{F}}} \widetilde{J\bar{F}}.$$

Therefore h belongs to $\mathcal{H}_{\bar{F}}$ if and only if

$$\frac{1}{{\Delta_{\bar{F}}}^2} \widetilde{J\bar{F}} J_n {}^t\widetilde{J\bar{F}} \begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix}$$

is smooth. Now applying Lemma 5.2 to $A = J\bar{F}$, we see that every entry of $\widetilde{J\bar{F}} J_n {}^t\widetilde{J\bar{F}}$ is an element of $\langle \Delta_{\bar{F}} \rangle$. Therefore if $h \in \langle \Delta_{\bar{F}}^2 \rangle$, then

$$\frac{1}{{\Delta_{\bar{F}}}^2} \widetilde{J\bar{F}} J_n {}^t\widetilde{J\bar{F}} \begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix}$$

is smooth and $h \in \mathcal{H}_{\bar{F}}$. Thus $\langle \Delta_{\bar{F}}^2 \rangle \subset \mathcal{H}_{\bar{F}}$. This proves 1).

Let $\ell \geq 3$ and let $h, k \in \langle {\Delta_{\bar{F}}}^{\ell} \rangle$. From the Definition 4.3

$$\{h,k\}_{\bar{F}*\bar{\omega}} = \left(\frac{\partial k}{\partial u}, \frac{\partial k}{\partial v}\right) \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}^{-1} \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} {}^t \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix}.$$

Since $h, k \in \langle {\Delta_{\bar{F}}}^{\ell} \rangle$, then

$$\frac{\partial h}{\partial u}, \frac{\partial h}{\partial v}, \frac{\partial k}{\partial u}, \frac{\partial k}{\partial v} \in \langle {\Delta_{\bar{F}}}^{\ell \ -1} \rangle$$

and on the basis of Lemma 5.2

$$\{h,k\}_{\bar{F}*\bar{\omega}} \in \langle {\Delta_{\bar{F}}}^{2\ell \ -2 \ -1} \rangle.$$

Since $\ell \geq 3$, $2\ell \quad -2 \quad -1 \geq \ell$. This proves 2).

3) can be proved in the same way. ∎

6. Existence of periodic solutions

Let (M,ω) be a symplectic manifold and let H be a smooth function on M. For a value $\lambda \in \mathbb{R}$, the level set $S_\lambda := H^{-1}(\lambda)$ is called an energy surface of the Hamiltonian vector field X_H. An energy surface S_λ is said to be regular if $dH \neq 0$ on S_λ.

Theorem 6.1. [8] *Let $S = S_1$ be a compact regular energy surface for the Hamiltonian vector field X_H on (M,ω). Assume that there is an open neighborhood U of S such that the symplectic capacity $c_0(U,\omega) < \infty$. Then*

1) *there exists a sequence $\lambda_j \to 1$ of energy values, such that X_H possesses a periodic solution on every energy surface S_{λ_j}.*
2) *Moreover, there is a small open interval I with $1 \in I$ such that*

$$\bigcup_{\lambda \in I} S_{\lambda_j} \subset U$$

and in this case, there is a dense set $\Lambda \subset I$ such that for $\lambda \in \Lambda$, the energy surface S_λ has a periodic solution of X_H.

Let M^{2n} be a compact manifold of dimension $2n$ and let $\bar{F} = (f_1, \ldots, f_n, g_1, \ldots, g_n) : M \to (\mathbb{R}^{2n}, \omega)$ be a smooth mapping.

Let $\mathcal{H}_{\bar{F}}$ denotes the set of all functions h on M such that X_h is smooth:

$$\mathcal{H}_{\bar{F}} = \{h \in C^\infty(M) \mid \ h \text{ generates a solvable isotropic mapping}\}.$$

For a point $p \in M$, let $C^\infty(M,p)$ denote the ring of the germs at p of smooth functions on M.

Let

$$\langle {\Delta_{\bar{F}}}^{\ell} \rangle_{C^\infty(M)} = \{h \in C^\infty(M) \mid \quad \text{at any singular point } p \in \Sigma(\bar{F}) \text{ the germ of } h \text{ at } p \text{ belongs to } \langle {\Delta_{\bar{F}}}^{\ell} \rangle_{C^\infty(M,p)}\}. \tag{6.1}$$

Then $\langle {\Delta_{\bar{F}}}^{\ell} \rangle_{C^\infty(M)}$ is an ideal in $C^\infty(M)$.

From Theorem 5.1, we have

1) $\langle {\Delta_{\bar{F}}}^{3} \rangle_{C^\infty(M)} \subset \mathcal{H}_{\bar{F}}$,

2) if the corank of the Jacobian matrix $J\bar{F}(p)$ is at most 1 everywhere, then $\langle {\Delta_{\bar{F}}}^{2} \rangle_{C^\infty(M)} \subset \mathcal{H}_{\bar{F}}$.

Note that as in the proof of Theorem 5.1, $\langle {\Delta_{\bar{F}}}^{3} \rangle_{C^\infty(M)}$ (or $\langle {\Delta_{\bar{F}}}^{2} \rangle_{C^\infty(M)}$ in the case that the corank of $J\bar{F}(p)$ is at most 1) dominates an essential part of $\mathcal{H}_{\bar{F}}$. Actually, it is not easy to find an element $h \in \mathcal{H}_{\bar{F}} \quad \langle {\Delta_{\bar{F}}}^{3} \rangle_{C^\infty(M)}$ (or $h \in \mathcal{H}_{\bar{F}} \quad \langle {\Delta_{\bar{F}}}^{2} \rangle_{C^\infty(M)}$ in the case that the corank of $J\bar{F}(p)$ is at most 1).

Now, we consider existence of periodic solutions for the global solvable isotropic mappings.

Let M^{2n} be a compact manifold of dimension $2n$ and let $\bar{F} = (f_1, \ldots, f_n, g_1, \ldots, g_n) : M \to (\mathbb{R}^{2n}, \omega)$ be a smooth mapping with fold singular points being dense in $\Sigma(\bar{F})$. Let $h \in \langle {\Delta_{\bar{F}}}^3 \rangle_{C^\infty(M)} \subset \mathcal{H}_{\bar{F}}$ (respectively $h \in \langle {\Delta_{\bar{F}}}^2 \rangle_{C^\infty(M)}$ in the case $\bar{F}$ has only corank 1 singularities). Then h generates a Hamiltonian vector field X_h and flows of X_h preserve the singular point set $\Sigma(\bar{F})$.

Let $\lambda_0 \neq 0$ be a regular value of h such that $h^{-1}(\lambda_0) \neq \emptyset$. Let S_{λ_0} be a connected component of $h^{-1}(\lambda_0)$ and let Ω be the connected component of the regular point set $M \setminus \Sigma(\bar{F})$ of $\bar{F}$ such that $S_{\lambda_0} \subset \Omega$. Then $(\Omega, \bar{F}^*\omega)$ is a symplectic manifold which contains S_{λ_0}.

Let $U \subset \Omega$ be a small open neighborhood of S_{λ_0}. Take an interval $I = (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ so small that it does not contain 0 and that for every $\lambda \in I, S_\lambda := h^{-1}(\lambda) \cap U \neq \emptyset$, S_λ is a connected component of $h^{-1}(\lambda)$ and is a regular hypersurface. Consider the set

$$\bigcup_{\lambda \in I} S_\lambda.$$

Then, from the Hofer–Zender theorem (See Theorem 1, p. 106 of [8])), we obtain

Theorem 6.2. *If $c_0(U, \bar{F}^*\omega) < \infty$, then there is a dense set $\Lambda \subset I$ such that for $\lambda \in \Lambda$, the energy surface S_λ has a periodic solution of X_h.*

So far, we did not mention any thing about the Hofer–Zender capacity c_0. However, it is known that for any bounded open subset O of $(\mathbb{R}^{2n}, \omega)$, we have $c_0(O, \omega) < \infty$. And, since the Hofer–Zender capacity c_0 is a symplectic invariant, if $\bar{F} : U \to \mathbb{R}^{2n}$ is an embedding, then $\bar{F} : (U, \bar{F}^*\omega) \to (\mathbb{R}^{2n}, \omega)$ is a symplectic embedding, $\bar{F}(U)$ is open subset of $\mathbb{R}^{2n}$ and $c_0(U, \bar{F}^*\omega) = c_0(\bar{F}(U), \omega)$. Note that since $\bar{F}$ is a smooth mapping from a compact manifold, its image is a bounded subset of $\mathbb{R}^{2n}$ and so is $\bar{F}(U)$.

Suppose that the restricted mapping $\bar{F} : S_{\lambda_0} \to \mathbb{R}^{2n}$ is an embedding. Then, since $\bar{F} : \Omega \to \mathbb{R}^{2n}$ is an immersion and since S_{λ_0} is compact, there is an open neighborhood U of S_{λ_0} such that $\bar{F} : U \to \mathbb{R}^{2n}$ is an embedding, so that $c_0(U, \bar{F}^*\omega) < \infty$. Thus we have

Corollary 6.3. *Suppose that the restricted mapping $\bar{F} : S_{\lambda_0} \to \mathbb{R}^{2n}$ is an embedding. Then there is a dense set $\Lambda \subset I$ such that for $\lambda \in \Lambda$, the energy surface S_λ has a periodic solution of X_h.*

There is a trivial example. Let p_0 be an isolated local minimal or maximal point of our Hamiltonian function $h = \alpha\Delta^\ell$. Then there exists a small neighborhood of p_0 such that U contains no critical points except for p and that $\bar{F} : U \to \mathbb{R}^{2n}$ is an embedding. Then $\bar{F} : (U, \bar{F}^*\omega) \to (\mathbb{R}^{2n}, \omega)$ is a

symplectic embedding and $c_0(U, \bar{F}^*\omega) < \infty$. Let p be a local maximal point of h and let $h(p_0) = c_0$. Then there exists a small positive number $\epsilon_0 > 0$ such that for any point q in U with $c_0 \quad \epsilon_0 < h(q) < c_0$, the connected component containing q of the energy surface $h \ ^1(h(q))$ is a subset of U. In this situation, we have

Corollary 6.4. *In the above situation, there is a dense set* $\Lambda \subset (c_0 \quad \epsilon_0, c_0)$ *such that for* $\lambda \in \Lambda$*, the energy surface* $S_\lambda \cap U$ *has a periodic solution of* X_h*.*

In the end of this section we came to the following,

Problem. Let M be a smooth manifold of dimension $2n$ and let $\bar{F} : M \to (\mathbb{R}^{2n}, \omega)$ be a submersion such that $\bar{F}(M)$ is a bounded open subset of $\mathbb{R}^{2n}$ and that the numbers of elements of inverse images $\bar{F} \ ^1(q), \ q \in \mathbb{R}^{2n}$ are bounded:

$$\sup\{\sharp\bar{F} \ ^1(q) \mid q \in \mathbb{R}^{2n}\} < \infty.$$

Then

$$c_0(M, \bar{F}^*\omega) < \infty \quad ?$$

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