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THE MONOMIAL DIFFERENCE MAJORIZED BY SINGLE VARIABLE FUNCTION

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Abstract. We deal with the monomial difference $n!F(y) - \triangle_y^n F(x)$ assuming that its norm is majorized by some function depending upon the variable y.

1. Introduction

In this paper, $\triangle$ denotes the difference operator, which is defined for a function F mapping an Abelian semigroup $(S, +)$ into an Abelian group $(G, +)$ and for a positive integer n by

$$\triangle_y F(x) = F(x+y) - F(x), \quad x, y \in S,$$

and

$$\triangle_y^{n+1} F(x) = \triangle_y \triangle_y^n F(x), \quad x, y \in S.$$

A map $F\colon S \to G$ is called a monomial function of degree n if and only if

$$\triangle_y^n F(x) = n!F(y),$$

for all $x, y \in S$.

We say that the functional equation is stable in the sense of Hyers–Ulam if for any function satisfying the equation with ε-precision, there exists a solution of the equation uniformly close to the function. Stability of functional equations, in this sense, has been extensively studied by many authors. Later, the research was extended to the case when the initial equation is satisfied with an accuracy to a function that is not necessarily constant. Such research on the monomial functional equation $\triangle_y^n F(x) = n!F(y)$ was carried by A. Gilányi. In [3], A. Gilányi considered the functional inequality

$$\|n!F(y) - \triangle_y^n F(x)\| \le \varepsilon(\|x\|^\alpha + \|y\|^\alpha).$$

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His result, for $\alpha = 0$, gives the Hyers–Ulam stability of monomial functional equation.

In this paper, we consider the functional inequality

$$\|n!F(y) - \triangle_y^n F(x)\| \leq \varphi(y)$$

and we look for the conditions under which there exists a monomial function of degree n uniformly approximating the function F.

2. The main result

In the proof of the main theorem of this paper, we will use the scalar version of the result of R. Badora, Z. Páles, and L. Székelyhidi about monomial selections of set-valued maps (see Corollary 2 in [1]).

Theorem 2.1. *Let $(S, +)$ be a semigroup. Let $\Phi\colon S \to 2^{\mathbb{R}}$ be a map with values being compact intervals. Assume that there exists a function $F\colon S \to \mathbb{R}$ such that*

$$\frac{1}{n!} \triangle_y^n F(x) \in \Phi(y), \quad x, y \in S.$$

Then there exists a monomial function $M\colon S \to \mathbb{R}$ of degree n such that $M(x) \in \Phi(x)$ for all $x \in S$.

The main result is the following.

Theorem 2.2. *Let $(S, +)$ be an Abelian semigroup and $(Y, \|\cdot\|)$ be a k-dimensional real normed linear space. Furthermore, let $F\colon S \to Y$ and $\varphi\colon S \to \mathbb{R}$ be mappings such that the inequality*

$$\|n!F(y) - \triangle_y^n F(x)\| \leq \varphi(y) \tag{1}$$

holds for all $x, y \in S$.

Then there exists a monomial mapping $M\colon S \to Y$ of degree n such that

$$\|F(x) - M(x)\| \leq \frac{k}{n!}\varphi(x), \quad x \in S.$$

The idea of the proof of above theorem is due to R. Ger [2].

Proof. Let Y^* be the dual space to the space Y. Then, by (1), for each $y^* \in Y^*$ such that $\|y^*\| = 1$, we have

$$-\varphi(y) \leq n!y^* \circ F(y) - \triangle_y^n y^* \circ F(x) \leq \varphi(y), \quad x, y \in S.$$

Hence

$$-\frac{1}{n!}\varphi(y) - y^* \circ F(y) \leq \frac{1}{n!} \triangle_y^n (-y^* \circ F)(x) \leq \frac{1}{n!}\varphi(y) - y^* \circ F(y), \quad x, y \in S.$$

Let us fix an arbitrary $y^* \in Y^*$ such that $\|y^*\| = 1$ and define the function $\Phi_{y^*}\colon X \to 2^{\mathbb{R}}$ by the formula

$$\Phi_{y^*}(x) := \left[-\frac{1}{n!}\varphi(x) - y^* \circ F(x), \frac{1}{n!}\varphi(x) - y^* \circ F(x)\right], \quad x \in X. \tag{2}$$

Clearly, the values of the function Φ_{y^*} are compact intervals and

$$\frac{1}{n!} \triangle_y^n (-y^* \circ F)(x) \in \Phi_{y^*}(y), \quad x, y \in X.$$

In view of Theorem 2.1, there exists a monomial function $M_{y^*}: X \to \mathbb{R}$ of degree n such that

$$M_{y^*}(x) \in \Phi_{y^*}(x), \quad x \in X. \tag{3}$$

By (2) and (3), we obtain

$$|y^*(F(x)) + M_{y^*}(x)| \leq \frac{1}{n!}\varphi(x), \quad x \in X. \tag{4}$$

Let $\{e_1, \ldots, e_k\}$ be a basis of Y such that $\|e_i\| = 1$ for all $i \in \{1, \ldots, k\}$. Further, let $y_i^*: Y \to \mathbb{R}$ be a function such that $y_i^*(y_1 e_1 + \ldots + y_k e_k) = y_i$ for $(y_1, \ldots, y_k) \in \mathbb{R}^k$, $i \in \{1, \ldots, k\}$.

Clearly, $y_i^* \in Y^*$ and $\|y_i^*\| = 1$ for all $i \in \{1, \ldots, k\}$.

Let us define a function $M: X \to Y$ by the formula

$$M(x) = -M_{y_1^*}(x)e_1 - \ldots - M_{y_k^*}(x)e_k, \quad x \in X.$$

The map M is a monomial function of degree n. From (4), we deduce that, for any $x \in X$,

$$\begin{aligned}
\|F(x) - M(x)\| &= \Big\|\sum_{i=1}^{k}(y_i^*(F(x)) + M_{y_i^*}(x))e_i\Big\| \leq \sum_{i=1}^{k} \|(y_i^*(F(x)) + M_{y_i^*}(x))e_i\| \\
&= \sum_{i=1}^{k} |y_i^*(F(x)) + M_{y_i^*}(x)|\, \|e_i\| \\
&= \sum_{i=1}^{k} |y_i^*(F(x)) + M_{y_i^*}(x)| \leq \frac{k}{n!} \cdot \varphi(x),
\end{aligned}$$

which completes the proof. ■

If we take $\varphi(x) = \varepsilon > 0$, $x \in S$, in Theorem 2.2, we obtain the following stability result of Hyers–Ulam type for monomial functions:

Corollary 2.3. *Let $(S, +)$ be an Abelian semigroup and let $(Y, \|\cdot\|)$ be a k-dimensional real normed linear space. If a function $F: S \to Y$ satisfies the inequality*

$$\|n!F(y) - \triangle_y^n F(x)\| \leq \varepsilon,$$

for all $x, y \in S$, then there exists a monomial mapping $M: S \to Y$ of degree n such that

$$\|F(x) - M(x)\| \leq \frac{k}{n!}\varepsilon, \quad x \in S.$$

Remark 2.4. *A monomial function F of degree 1 is an additive function, i.e., $F(x)+F(y)=F(x+y)$, $x,y\in S$.*

If we take $n=1$ and $\varphi(x)=\varepsilon>0$, $x\in S$, in Theorem 2.2, then we obtain the following stability result of Hyers–Ulam type for additive functions:

Corollary 2.5. *Let $(S,+)$ be an Abelian semigroup and let $(Y,\|\cdot\|)$ be a k-dimensional real normed linear space. If a function $F\colon S\to Y$ satisfies the inequality*

$$\|F(x+y)-F(x)-F(y)\|\leq\varepsilon,$$

for all $x,y\in S$, then there exists an additive mapping $A\colon S\to Y$ of degree n such that

$$\|F(x)-A(x)\|\leq k\varepsilon,\quad x\in S.$$

On the basis of Theorem 2.2, we get also the following corollary:

Corollary 2.6. *Let $(X,\|\cdot\|)$ be a normed linear space, $(Y,\|\cdot\|)$ a k-dimensional real normed linear space, n a positive integer, and α a real number. If a function $F\colon X\to Y$ satisfies the inequality*

$$\|n!F(y)-\triangle_y^n F(x)\|\leq\varepsilon\|y\|^\alpha,\quad x,y\in X,$$

then there exists a monomial function $M\colon X\to Y$ of degree n such that

$$\|F(x)-M(x)\|\leq\frac{k\varepsilon}{n!}\|x\|^\alpha,\quad x\in X.$$

Proof. It is sufficient to take $\varphi(y)=\varepsilon\|y\|^\alpha$, $y\in X$, in Theorem 2.2. ∎

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