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HARMONIC MULTIVARIATE OSTROWSKI AND GRÜSS TYPE INEQUALITIES FOR SEVERAL FUNCTIONS

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Abstract. Here we derive very general multivariate Ostrowski and Grüss type inequalities for several functions by involving harmonic polynomials. Estimates are with respect to all basic norms. We give applications.

1. Introduction

The problem of estimating the difference of a value of a function from its average is a paramount one. The answer to it are the Ostrowski type inequalities. Ostrowski type inequalities are very useful among others in Numerical Analysis for approximating integrals. The problem of estimating the difference between the average of a product of functions from the product of their averages is also a very important one. The answer to it are the Grüss type inequalities. Grüss type inequalities are very useful among others in Probability for estimating expected values, etc. There exists a vast literature on Ostrowski and Grüss type inequalities to all possible directions. Mathematical community is very much interested to these inequalities due to their applications. So here we derive very general Ostrowski and Grüss type inequalities for several multivariate functions, acting to all possible directions.

We are motivated by the following results.

THEOREM 1. (1938, Ostrowski [7]) *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e., $\|f'\|_{\infty}^{\sup} := \sup_{t \in (a, b)} |f'(t)| < +\infty$. Then*

$$(1) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \cdot (b-a) \|f'\|_{\infty}^{\sup},$$

for any $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

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THEOREM 2. (1935, Grüss [6]) *Let f, g be integrable functions from $[a, b]$ into $\mathbb{R}$, that satisfy the conditions*

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N, \quad x \in [a, b],$$

where $m, M, n, N \in \mathbb{R}$. Then

$$(2) \quad \left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right| \leq \frac{1}{4} (M-m)(N-n).$$

THEOREM 3. (1998, Dragomir and Wang [4]) *Let $f : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous function with $f' \in L_p([a, b])$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $x \in [a, b]$. Then*

$$(3) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{x-a}{b-a} \right)^{q+1} + \left(\frac{b-x}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} (b-a)^{\frac{1}{q}} \|f'\|_p.$$

THEOREM 4. (1882, Čebyšev [1]) *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous functions with $f', g' \in L_\infty([a, b])$. Then*

$$(4) \quad \left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right| \leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty.$$

Above is also assumed that the involved integrals exist.

2. Background

Let $(P_n)_{n \in \mathbb{N}}$ be a harmonic sequence of polynomials, that is $P'_n = P_{n-1}$, $n \geq 1$, $P_0 = 1$. Furthermore, let $[a, b] \subset \mathbb{R}$, $a \neq b$, and $h : [a, b] \rightarrow \mathbb{R}$ be such that $h^{(n-1)}$ is absolutely continuous function for some fixed $n \geq 1$. We use the notation

$$(5) \quad L_n[h(x)] = \frac{1}{n} \left[h(x) + \sum_{k=1}^{n-1} (-1)^k P_k(x) h^{(k)}(x) + \sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{b-a} \left[P_k(a) h^{(k-1)}(a) - P_k(b) h^{(k-1)}(b) \right] \right],$$

$x \in [a, b]$, for convinience.

For $n = 1$ the above sums are defined to be zero, that is $L_1[h(x)] = h(x)$.

Dedic, Pečarić and Ujević, see [2], [3], established the following identity,

$$(6) \quad L_n[h(x)] - \frac{1}{b-a} \int_a^b h(t) dt = \frac{(-1)^{n+1}}{n(b-a)} \int_a^b P_{n-1}(t) q(x, t) h^{(n)}(t) dt,$$

where

$$(7) \quad q(x, t) = \begin{cases} t-a, & \text{if } t \in [a, x], \\ t-b, & \text{if } t \in (x, b], \end{cases} \quad x \in [a, b].$$

For the harmonic sequence of polynomials $P_k(t) = \frac{(t-x)^k}{k!}$, $k \geq 0$, the identity (6) reduces to the Fink identity in [5], (see also [3], p. 177).

We rewrite (6) as follows:

$$(8) \quad \begin{aligned} h(x) + \sum_{k=1}^{n-1} (-1)^k P_k(x) h^{(k)}(x) \\ + \sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{b-a} \left[P_k(a) h^{(k-1)}(a) - P_k(b) h^{(k-1)}(b) \right] \\ = \frac{n}{b-a} \int_a^b h(t) dt + \frac{(-1)^{n+1}}{b-a} \int_a^b P_{n-1}(t) q(x, t) h^{(n)}(t) dt, \end{aligned}$$

$x \in [a, b]$.

That is the generalized Fink type representation formula.

$$(9) \quad \begin{aligned} h(x) = \sum_{k=1}^{n-1} (-1)^{k+1} P_k(x) h^{(k)}(x) \\ + \sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{b-a} \left[P_k(b) h^{(k-1)}(b) - P_k(a) h^{(k-1)}(a) \right] \\ + \frac{n}{b-a} \int_a^b h(t) dt + \frac{(-1)^{n+1}}{b-a} \int_a^b P_{n-1}(t) q(x, t) h^{(n)}(t) dt, \end{aligned}$$

$x \in [a, b]$, $n \geq 1$, when $n = 1$ the above sums are zero.

3. Main results

Here $\prod_{i=1}^m [a_i, b_i] \subseteq \mathbb{R}^m$, $m, n \in \mathbb{N}$.

We make

GENERAL ASSUMPTIONS 5. Let $f : \prod_{i=1}^m [a_i, b_i] \rightarrow \mathbb{R}$. We assume

- 1) for $j = 1, \dots, m$, we have that $\frac{\partial^{n-1} f}{\partial x_j^{n-1}}(x_1, x_2, \dots, x_{j-1}, s_j, x_{j+1}, \dots, x_m)$ is absolutely continuous in s_j on $[a_j, b_j]$, for every $(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_m) \in \prod_{\substack{i=1 \\ i \neq j}}^m [a_i, b_i]$,

- 2) for $j = 1, \dots, m$, we have that $\frac{\partial^n f(s_1, \dots, s_j, x_{j+1}, \dots, x_m)}{\partial x_j^n}$ is continuous on $\prod_{i=1}^j [a_i, b_i]$, for every $(x_{j+1}, \dots, x_m) \in \prod_{i=j+1}^m [a_i, b_i]$,
- 3) for each $j = 1, \dots, m$, and for every $l = 1, \dots, n-1$, we have that $\frac{\partial^l f}{\partial x_j^l}(s_1, s_2, \dots, s_{j-1}, x_j, \dots, x_m)$ is continuous on $\prod_{i=1}^{j-1} [a_i, b_i]$, for every $(x_j, \dots, x_m) \in \prod_{i=j}^m [a_i, b_i]$,
- 4) f is continuous on $\prod_{i=1}^m [a_i, b_i]$.

BRIEF ASSUMPTIONS 6. Let $f : \prod_{i=1}^m [a_i, b_i] \rightarrow \mathbb{R}$ with $\frac{\partial^l f}{\partial x_i^l}$ for $l = 0, 1, \dots, n$; $i = 1, \dots, m$, are continuous on $\prod_{i=1}^m [a_i, b_i]$.

DEFINITION 7. We put

$$(10) \quad q(x_i, s_i) = \begin{cases} s_i - a_i, & \text{if } s_i \in [a_i, x_i], \\ s_i - b_i, & \text{if } s_i \in (x_i, b_i], \end{cases} \quad x_i \in [a_i, b_i],$$

$i = 1, \dots, m$.

We present the following general representation result of Fink type.

THEOREM 8. Let f be as in General Assumptions 5 or Brief Assumptions 6. Then

$$(11) \quad f(x_1, \dots, x_n) = \frac{n^m}{\prod_{i=1}^m (b_i - a_i)} \int_{\prod_{i=1}^m [a_i, b_i]} f(s_1, \dots, s_m) ds_1 \dots ds_m \\ + \sum_{i=1}^m T_i(x_i, x_{i+1}, \dots, x_m),$$

where

$$(12) \quad T_i(x_i, \dots, x_m) := \frac{n^{i-1}}{\prod_{j=1}^{i-1} (b_j - a_j)} \\ \cdot \left[\sum_{k=1}^{n-1} (-1)^{k+1} P_k(x_i) \int_{a_1}^{b_1} \dots \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^k f(s_1, \dots, s_{i-1}, x_i, \dots, x_m)}{\partial x_i^k} ds_1 \dots ds_{i-1} \right. \\ + \sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{b_i - a_i} \left[P_k(b_i) \int_{a_1}^{b_1} \dots \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^{k-1} f(s_1, \dots, s_{i-1}, b_i, x_{i+1}, \dots, x_m)}{\partial x_i^{k-1}} ds_1 \dots ds_{i-1} \right. \\ \left. - P_k(a_i) \int_{a_1}^{b_1} \dots \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^{k-1} f(s_1, \dots, s_{i-1}, a_i, x_{i+1}, \dots, x_m)}{\partial x_i^{k-1}} ds_1 \dots ds_{i-1} \right] \\ \left. + \frac{(-1)^{n+1}}{(b_i - a_i)} \int_{a_1}^{b_1} \dots \int_{a_i}^{b_i} P_{n-1}(s_i) q(x_i, s_i) \frac{\partial^n f(s_1, \dots, s_i, x_{i+1}, \dots, x_m)}{\partial x_i^n} ds_1 \dots ds_i \right],$$

are continuous functions for all $i = 1, \dots, m$.

Proof. We apply (9) repeatedly.

We have

$$\begin{aligned}
 (13) \quad f(x_1, \dots, x_m) &= \sum_{k=1}^{n-1} (-1)^{k+1} P_k(x_1) \frac{\partial^k f(x_1, \dots, x_m)}{\partial x_1^k} \\
 &+ \sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{(b_1 - a_1)} \left[P_k(b_1) \frac{\partial^{k-1} f(b_1, x_2, \dots, x_m)}{\partial x_1^{k-1}} \right. \\
 &\quad \left. - P_k(a_1) \frac{\partial^{k-1} f(a_1, x_2, \dots, x_m)}{\partial x_1^{k-1}} \right] + \frac{n}{(b_1 - a_1)} \int_{a_1}^{b_1} f(s_1, x_2, \dots, x_m) ds_1 \\
 &+ \frac{(-1)^{n+1}}{(b_1 - a_1)} \int_{a_1}^{b_1} P_{n-1}(s_1) q(x_1, s_1) \frac{\partial^n f(s_1, x_2, \dots, x_m)}{\partial x_1^n} ds_1,
 \end{aligned}$$

for any $x_1 \in [a_1, b_1]$, under the assumption that $\frac{\partial^{n-1} f(\cdot, x_2, \dots, x_m)}{\partial x_1^{n-1}} \in AC([a_1, b_1])$.

Call

$$\begin{aligned}
 (14) \quad T_1(x_1, \dots, x_m) &:= \sum_{k=1}^{n-1} (-1)^{k+1} P_k(x_1) \frac{\partial^k f(x_1, \dots, x_m)}{\partial x_1^k} \\
 &+ \sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{(b_1 - a_1)} \left[P_k(b_1) \frac{\partial^{k-1} f(b_1, x_2, \dots, x_m)}{\partial x_1^{k-1}} \right. \\
 &\quad \left. - P_k(a_1) \frac{\partial^{k-1} f(a_1, x_2, \dots, x_m)}{\partial x_1^{k-1}} \right] \\
 &+ \frac{(-1)^{n+1}}{(b_1 - a_1)} \int_{a_1}^{b_1} P_{n-1}(s_1) q(x_1, s_1) \frac{\partial^n f(s_1, x_2, \dots, x_m)}{\partial x_1^n} ds.
 \end{aligned}$$

Hence, it holds

$$(15) \quad f(x_1, \dots, x_m) = \frac{n}{(b_1 - a_1)} \int_{a_1}^{b_1} f(s_1, x_2, \dots, x_m) ds_1 + T_1(x_1, \dots, x_m).$$

Next similarly, we get

$$\begin{aligned}
 (16) \quad f(s_1, x_2, \dots, x_m) &= \sum_{k=1}^{n-1} (-1)^{k+1} P_k(x_2) \frac{\partial^k f(s_1, x_2, \dots, x_m)}{\partial x_2^k} \\
 &+ \sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{(b_2 - a_2)} \\
 &\cdot \left[P_k(b_2) \frac{\partial^{k-1} f(s_1, b_2, x_3, \dots, x_m)}{\partial x_2^{k-1}} - P_k(a_2) \frac{\partial^{k-1} f(s_1, a_2, x_3, \dots, x_m)}{\partial x_2^{k-1}} \right]
 \end{aligned}$$

$$\begin{aligned}
& + \frac{n}{(b_2 - a_2)} \int_{a_2}^{b_2} f(s_1, s_2, x_3, \dots, x_m) ds_2 \\
& + \frac{(-1)^{n+1}}{(b_2 - a_2)} \int_{a_2}^{b_2} P_{n-1}(s_2) q(x_2, s_2) \frac{\partial^n f(s_1, s_2, x_3, \dots, x_m)}{\partial x_2^n} ds_2,
\end{aligned}$$

for any $x_2 \in [a_2, b_2]$.

Hence, it holds

$$\begin{aligned}
(17) \quad f(x_1, \dots, x_m) &= \frac{n^2}{(b_1 - a_1)(b_2 - a_2)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(s_1, s_2, x_3, \dots, x_m) ds_1 ds_2 \\
&+ T_1(x_1, \dots, x_m) + T_2(x_2, x_3, \dots, x_m),
\end{aligned}$$

where

$$\begin{aligned}
(18) \quad T_2(x_2, x_3, \dots, x_m) &:= \frac{n}{(b_1 - a_1)} \left\{ \sum_{k=1}^{n-1} (-1)^{k+1} P_k(x_2) \int_{a_1}^{b_1} \frac{\partial^k f(s_1, x_2, \dots, x_m)}{\partial x_2^k} ds_1 \right. \\
&+ \sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{(b_2 - a_2)} \left[P_k(b_2) \int_{a_1}^{b_1} \frac{\partial^{k-1} f(s_1, b_2, x_3, \dots, x_m)}{\partial x_2^{k-1}} ds_1 \right. \\
&\quad \left. \left. - P_k(a_2) \int_{a_1}^{b_1} \frac{\partial^{k-1} f(s_1, a_2, x_3, \dots, x_m)}{\partial x_2^{k-1}} ds_1 \right] \right. \\
&\left. + \frac{(-1)^{n+1}}{(b_2 - a_2)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} P_{n-1}(s_2) q(x_2, s_2) \frac{\partial^n f(s_1, s_2, x_3, \dots, x_m)}{\partial x_2^n} ds_1 ds_2 \right\}.
\end{aligned}$$

Next, we see similarly that

$$\begin{aligned}
(19) \quad f(s_1, s_2, x_3, \dots, x_m) &= \sum_{k=1}^{n-1} (-1)^{k+1} P_k(x_3) \frac{\partial^k f(s_1, s_2, x_3, \dots, x_m)}{\partial x_3^k} \\
&+ \sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{(b_3 - a_3)} \left[P_k(b_3) \frac{\partial^{k-1} f(s_1, s_2, b_3, x_4, \dots, x_m)}{\partial x_3^{k-1}} \right. \\
&\quad \left. - P_k(a_3) \frac{\partial^{k-1} f(s_1, s_2, a_3, x_4, \dots, x_m)}{\partial x_3^{k-1}} \right] \\
&+ \frac{n}{(b_3 - a_3)} \int_{a_3}^{b_3} f(s_1, s_2, s_3, x_4, \dots, x_m) ds_3 \\
&+ \frac{(-1)^{n+1}}{(b_3 - a_3)} \int_{a_3}^{b_3} P_{n-1}(s_3) q(x_3, s_3) \frac{\partial^n f(s_1, s_2, s_3, x_4, \dots, x_m)}{\partial x_3^n} ds_3,
\end{aligned}$$

for any $x_3 \in [a_3, b_3]$.

So that, we get

$$\begin{aligned}
 (20) \quad & f(x_1, \dots, x_m) \\
 &= \frac{n^3}{\prod_{j=1}^3 (b_j - a_j)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(s_1, s_2, s_3, x_4, \dots, x_m) ds_1 ds_2 ds_3 \\
 &\quad + T_1(x_1, \dots, x_m) + T_2(x_2, \dots, x_m) + T_3(x_3, \dots, x_m),
 \end{aligned}$$

where

$$\begin{aligned}
 (21) \quad & T_3(x_3, \dots, x_m) := \frac{n^2}{(b_1 - a_1)(b_2 - a_2)} \\
 & \cdot \left[\sum_{k=1}^{n-1} (-1)^{k+1} P_k(x_3) \int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\partial^k f(s_1, s_2, x_3, \dots, x_m)}{\partial x_3^k} ds_1 ds_2 \right. \\
 & + \sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{(b_3 - a_3)} \left[P_k(b_3) \int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\partial^{k-1} f(s_1, s_2, b_3, x_4, \dots, x_m)}{\partial x_3^{k-1}} ds_1 ds_2 \right. \\
 & \quad \left. - P_k(a_3) \int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\partial^{k-1} f(s_1, s_2, a_3, x_4, \dots, x_m)}{\partial x_3^{k-1}} ds_1 ds_2 \right] \\
 & \left. + \frac{(-1)^{n+1}}{(b_3 - a_3)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} P_{n-1}(s_3) q(x_3, s_3) \frac{\partial^n f(s_1, s_2, s_3, x_4, \dots, x_m)}{\partial x_3^n} ds_1 ds_2 ds_3 \right].
 \end{aligned}$$

Furthermore, we can write

$$\begin{aligned}
 (22) \quad & f(s_1, s_2, s_3, x_4, \dots, x_m) \\
 &= \sum_{k=1}^{n-1} (-1)^{k+1} P_k(x_4) \frac{\partial^k f(s_1, s_2, s_3, x_4, \dots, x_m)}{\partial x_4^k} \\
 & + \sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{(b_4 - a_4)} \left[P_k(b_4) \frac{\partial^{k-1} f(s_1, s_2, s_3, b_4, x_5, \dots, x_m)}{\partial x_4^{k-1}} \right. \\
 & \quad \left. - P_k(a_4) \frac{\partial^{k-1} f(s_1, s_2, s_3, a_4, x_5, \dots, x_m)}{\partial x_4^{k-1}} \right] \\
 & + \frac{n}{(b_4 - a_4)} \int_{a_4}^{b_4} f(s_1, s_2, s_3, s_4, x_5, \dots, x_m) ds_4 \\
 & + \frac{(-1)^{n+1}}{(b_4 - a_4)} \int_{a_4}^{b_4} P_{n-1}(s_4) q(x_4, s_4) \frac{\partial^n f(s_1, s_2, s_3, s_4, x_5, \dots, x_m)}{\partial x_4^n} ds_4,
 \end{aligned}$$

for any $x_4 \in [a_4, b_4]$.

Therefore, it holds

$$\begin{aligned}
(23) \quad f(x_1, \dots, x_m) &= \frac{n^4}{\prod_{j=1}^4 (b_j - a_j)} \\
&\cdot \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \int_{a_4}^{b_4} f(s_1, s_2, s_3, s_4, x_5, \dots, x_m) ds_1 ds_2 ds_3 ds_4 \\
&+ T_1(x_1, \dots, x_m) + T_2(x_2, \dots, x_m) + T_3(x_3, \dots, x_m) + T_4(x_4, \dots, x_m),
\end{aligned}$$

where

$$\begin{aligned}
(24) \quad T_4(x_4, \dots, x_m) &:= \frac{n^3}{\prod_{j=1}^4 (b_j - a_j)} \left[\sum_{k=1}^{n-1} (-1)^{k+1} P_k(x_4) \right. \\
&\cdot \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \frac{\partial^k f(s_1, s_2, s_3, x_4, \dots, x_m)}{\partial x_4^k} ds_1 ds_2 ds_3 + \sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{(b_4 - a_4)} \\
&\cdot \left[P_k(b_4) \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \frac{\partial^{k-1} f(s_1, s_2, s_3, b_4, x_5, \dots, x_m)}{\partial x_4^{k-1}} ds_1 ds_2 ds_3 \right. \\
&\quad \left. - P_k(a_4) \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \frac{\partial^{k-1} f(s_1, s_2, s_3, a_4, x_5, \dots, x_m)}{\partial x_4^{k-1}} ds_1 ds_2 ds_3 \right] \\
&+ \frac{(-1)^{n+1}}{(b_4 - a_4)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \int_{a_4}^{b_4} P_{n-1}(s_4) q(x_4, s_4) \\
&\quad \cdot \frac{\partial^n f(s_1, s_2, s_3, s_4, x_5, \dots, x_m)}{\partial x_4^n} ds_1 ds_2 ds_3 ds_4 \Big],
\end{aligned}$$

etc.

The theorem is proved. ■

We make

REMARK 9. Let f_λ , $\lambda = 1, \dots, r \in \mathbb{N} - \{1\}$, be as in Assumptions 5 or Brief Assumptions 6; $n_\lambda \in \mathbb{N}$ associated with f_λ . Here $x = (x_1, \dots, x_m)$, $s = (s_1, \dots, s_m) \in \prod_{i=1}^m [a_i, b_i]$. Then

$$(25) \quad f_\lambda(x) = \frac{n_\lambda^m}{\prod_{i=1}^m (b_i - a_i)} \int_{\prod_{i=1}^m [a_i, b_i]} f_\lambda(s) ds + \sum_{i=1}^m T_{i\lambda}(x_i, x_{i+1}, \dots, x_m).$$

Here, we have

$$(26) \quad T_{i\lambda}(x_i, \dots, x_m) := \frac{n_\lambda^{i-1}}{\prod_{j=1}^{i-1} (b_j - a_j)} \left[\sum_{k=1}^{n_\lambda-1} (-1)^{k+1} P_k(x_i) \right.$$

$$\begin{aligned}
 & \cdot \int_{a_1}^{b_1} \dots \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^k f_\lambda(s_1, \dots, s_{i-1}, x_i, \dots, x_m)}{\partial x_i^k} ds_1 \dots ds_{i-1} \\
 & + \sum_{k=1}^{n_\lambda-1} \frac{(-1)^k (n_\lambda - k)}{b_i - a_i} \\
 & \cdot \left[P_k(b_i) \int_{a_1}^{b_1} \dots \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^{k-1} f_\lambda(s_1, \dots, s_{i-1}, b_i, x_{i+1}, \dots, x_m)}{\partial x_i^{k-1}} ds_1 \dots ds_{i-1} \right. \\
 & \left. - P_k(a_i) \int_{a_1}^{b_1} \dots \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^{k-1} f_\lambda(s_1, \dots, s_{i-1}, a_i, x_{i+1}, \dots, x_m)}{\partial x_i^{k-1}} ds_1 \dots ds_{i-1} \right] \\
 & + \frac{(-1)^{n_\lambda+1}}{(b_i - a_i)} \int_{a_1}^{b_1} \dots \int_{a_i}^{b_i} P_{n_\lambda-1}(s_i) q(x_i, s_i) \\
 & \cdot \frac{\partial^{n_\lambda} f_\lambda(s_1, \dots, s_i, x_{i+1}, \dots, x_m)}{\partial x_i^{n_\lambda}} ds_1 \dots ds_i \Big],
 \end{aligned}$$

are continuous functions, $i = 1, \dots, m$; $\lambda = 1, \dots, r$.

Hence, it holds

$$\begin{aligned}
 (27) \quad \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) f_\lambda(x) &= \left(\frac{n_\lambda^m}{\prod_{i=1}^m (b_i - a_i)} \right) \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) \left(\int_{\prod_{i=1}^m [a_i, b_i]} f_\lambda(s) ds \right) \\
 &+ \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) \left(\sum_{i=1}^m T_{i\lambda}(x_1, \dots, x_m) \right).
 \end{aligned}$$

Therefore, we derive

$$\begin{aligned}
 (28) \quad \sum_{\lambda=1}^r & \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) f_\lambda(x) \right) \\
 & - \frac{1}{\prod_{i=1}^m (b_i - a_i)} \left\{ \sum_{\lambda=1}^r \left(n_\lambda^m \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) \left(\int_{\prod_{i=1}^m [a_i, b_i]} f_\lambda(s) ds \right) \right) \right\} \\
 & = \sum_{\lambda=1}^r \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) \left(\sum_{i=1}^m T_{i\lambda}(x_1, \dots, x_m) \right) \right).
 \end{aligned}$$

We notice that

$$(29) \quad T_{i\lambda}(x_1, \dots, x_m) = A_{i\lambda}(x_1, \dots, x_m) + B_{i\lambda}(x_1, \dots, x_m),$$

$i = 1, \dots, m$; where

$$\begin{aligned}
(30) \quad A_{i\lambda}(x_i, \dots, x_m) &:= \frac{n_\lambda^{i-1}}{\prod_{j=1}^{i-1} (b_j - a_j)} \left[\sum_{k=1}^{n_\lambda-1} (-1)^{k+1} P_k(x_i) \right. \\
&\cdot \int_{a_1}^{b_1} \dots \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^k f_\lambda(s_1, \dots, s_{i-1}, x_i, \dots, x_m)}{\partial x_i^k} ds_1 \dots ds_{i-1} \\
&+ \sum_{k=1}^{n_\lambda-1} \frac{(-1)^k (n_\lambda - k)}{b_i - a_i} \\
&\cdot \left[P_k(b_i) \int_{a_1}^{b_1} \dots \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^{k-1} f_\lambda(s_1, \dots, s_{i-1}, b_i, x_{i+1}, \dots, x_m)}{\partial x_i^{k-1}} ds_1 \dots ds_{i-1} \right. \\
&\left. \left. - P_k(a_i) \int_{a_1}^{b_1} \dots \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^{k-1} f_\lambda(s_1, \dots, s_{i-1}, a_i, x_{i+1}, \dots, x_m)}{\partial x_i^{k-1}} ds_1 \dots ds_{i-1} \right] \right],
\end{aligned}$$

and

$$\begin{aligned}
(31) \quad B_{i\lambda}(x_i, \dots, x_m) &:= \frac{n_\lambda^{i-1} (-1)^{n_\lambda+1}}{\prod_{j=1}^i (b_j - a_j)} \\
&\cdot \left[\int_{a_1}^{b_1} \dots \int_{a_i}^{b_i} P_{n_\lambda-1}(s_i) q(x_i, s_i) \frac{\partial^{n_\lambda} f_\lambda(s_1, \dots, s_i, x_{i+1}, \dots, x_m)}{\partial x_i^{n_\lambda}} ds_1 \dots ds_i \right],
\end{aligned}$$

for all $i = 1, \dots, m$; $\lambda = 1, \dots, r$.

We call and have the identity

$$\begin{aligned}
(32) \quad S(f_1, \dots, f_r)(x) &:= \sum_{\lambda=1}^r \left\{ \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) \left[f_\lambda(x) - \frac{n_\lambda^m}{\prod_{i=1}^m (b_i - a_i)} \left(\int_{\prod_{i=1}^m [a_i, b_i]} f_\lambda(s) ds \right) \right. \right. \\
&\left. \left. - \sum_{i=1}^m A_{i\lambda}(x_i, \dots, x_m) \right] \right\} = \sum_{\lambda=1}^r \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) \left(\sum_{i=1}^m B_{i\lambda}(x_i, \dots, x_m) \right) \right),
\end{aligned}$$

true for any fixed $x \in \prod_{i=1}^m [a_i, b_i]$.

Then, we have

$$(33) \quad |S(f_1, \dots, f_r)(x)| \leq \sum_{\lambda=1}^r \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r |f_\rho(x)| \right) \left(\sum_{i=1}^m |B_{i\lambda}(x_i, \dots, x_m)| \right) \right).$$

We estimate the right hand side of (33).

We also make

REMARK 10. We observe that

$$(34) \quad |B_{i\lambda}(x_i, \dots, x_m)| \leq \frac{n_\lambda^{i-1}}{\prod_{j=1}^i (b_j - a_j)} \left[\int_{a_1}^{b_1} \dots \int_{a_i}^{b_i} |P_{n_\lambda-1}(s_i)| |q(x_i, s_i)| \right. \\ \left. \cdot \left| \frac{\partial^{n_\lambda} f_\lambda(s_1, \dots, s_i, x_{i+1}, \dots, x_m)}{\partial x_i^{n_\lambda}} \right| ds_1 \dots ds_i \right] =: (\xi),$$

for all $i = 1, \dots, m$; $\lambda = 1, \dots, r$.

We know that

$$(35) \quad |q(x_i, s_i)| \leq \max(x_i - a_i, b_i - x_i) = \frac{(b_i - a_i) + |a_i + b_i - 2x_i|}{2}.$$

We have

$$(36) \quad (\xi) \stackrel{(35)}{\leq} \frac{n_\lambda^{i-1}}{\prod_{j=1}^i (b_j - a_j)} \left[\|P_{n_\lambda-1}\|_{\infty, [a_i, b_i]} \left(\frac{(b_i - a_i) + |a_i + b_i - 2x_i|}{2} \right) \right. \\ \left. \cdot \left\| \frac{\partial^{n_\lambda} f_\lambda(\dots, x_{i+1}, \dots, x_m)}{\partial x_i^{n_\lambda}} \right\|_{L_1(\prod_{j=1}^i [a_j, b_j])} \right].$$

Thus

$$(37) \quad |S(f_1, \dots, f_r)(x)| \leq \sum_{\lambda=1}^r \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r |f_\rho(x)| \right) \left(\sum_{i=1}^m \frac{n_\lambda^{i-1}}{\prod_{j=1}^i (b_j - a_j)} \right. \right. \\ \left. \cdot \left[\|P_{n_\lambda-1}\|_{\infty, [a_i, b_i]} \left(\frac{(b_i - a_i) + |a_i + b_i - 2x_i|}{2} \right) \right. \right. \\ \left. \left. \cdot \left\| \frac{\partial^{n_\lambda} f_\lambda(\dots, x_{i+1}, \dots, x_m)}{\partial x_i^{n_\lambda}} \right\|_{L_1(\prod_{j=1}^i [a_j, b_j])} \right] \right) \right) =: \theta_1(x).$$

Next let $p_{li} > 1 : \sum_{l=1}^3 \frac{1}{p_{li}} = 1$. Then

$$(38) \quad |B_{i\lambda}(x_i, \dots, x_m)| \stackrel{(34)}{\leq} (\xi) \leq \frac{n_\lambda^{i-1}}{\prod_{j=1}^i (b_j - a_j)} \\ \cdot \left[\left(\int_{a_1}^{b_1} \dots \int_{a_i}^{b_i} |P_{n_\lambda-1}(s_i)|^{p_{1i}} ds_1 \dots ds_i \right)^{\frac{1}{p_{1i}}}$$

$$\begin{aligned}
& \cdot \left(\int_{a_1}^{b_1} \dots \int_{a_i}^{b_i} |q(x_i, s_i)|^{p_{2i}} ds_1 \dots ds_i \right)^{\frac{1}{p_{2i}}} \\
& \cdot \left\| \frac{\partial^{n_\lambda} f_\lambda(\dots, x_{i+1}, \dots, x_m)}{\partial x_i^{n_\lambda}} \right\|_{L_{p_{3i}}(\prod_{j=1}^i [a_j, b_j])} \Bigg] \\
& = \frac{n_\lambda^{i-1}}{\prod_{j=1}^i (b_j - a_j)} \left[\|P_{n_\lambda-1}\|_{L_{p_{1i}}([a_i, b_i])} \left(\prod_{j=1}^{i-1} (b_j - a_j) \right)^{\frac{1}{p_{1i}} + \frac{1}{p_{2i}}} \right. \\
& \cdot \left(\frac{(b_i - x_i)^{p_{2i}+1} + (x_i - a_i)^{p_{2i}+1}}{p_{2i} + 1} \right)^{\frac{1}{p_{2i}}} \\
(39) \quad & \cdot \left. \left\| \frac{\partial^{n_\lambda} f_\lambda(\dots, x_{i+1}, \dots, x_m)}{\partial x_i^{n_\lambda}} \right\|_{L_{p_{3i}}(\prod_{j=1}^i [a_j, b_j])} \right].
\end{aligned}$$

Therefore we get

$$\begin{aligned}
(40) \quad |S(f_1, \dots, f_r)(x)| & \stackrel{(39)}{\leq} \sum_{\lambda=1}^r \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r |f_\rho(x)| \right) \right. \\
& \cdot \left(\sum_{i=1}^m \frac{n_\lambda^{i-1}}{\prod_{j=1}^i (b_j - a_j)} \left[\|P_{n_\lambda-1}\|_{L_{p_{1i}}([a_i, b_i])} \left(\prod_{j=1}^{i-1} (b_j - a_j) \right)^{\frac{1}{p_{1i}} + \frac{1}{p_{2i}}} \right. \right. \\
& \cdot \left(\frac{(b_i - x_i)^{p_{2i}+1} + (x_i - a_i)^{p_{2i}+1}}{p_{2i} + 1} \right)^{\frac{1}{p_{2i}}} \\
& \cdot \left. \left. \left\| \frac{\partial^{n_\lambda} f_\lambda(\dots, x_{i+1}, \dots, x_m)}{\partial x_i^{n_\lambda}} \right\|_{L_{p_{3i}}(\prod_{j=1}^i [a_j, b_j])} \right] \right) \Bigg) \\
& =: \theta_2(x).
\end{aligned}$$

We also have

$$\begin{aligned}
(41) \quad |B_{i\lambda}(x_i, \dots, x_m)| & \leq (\xi) \leq n_\lambda^{i-1} \left[\|P_{n_\lambda-1}\|_{\infty, [a_i, b_i]} \right. \\
& \cdot \left. \left\| \frac{\partial^{n_\lambda} f_\lambda(\dots, x_{i+1}, \dots, x_m)}{\partial x_i^{n_\lambda}} \right\|_{\infty, \prod_{j=1}^i [a_j, b_j]} \left(\frac{(b_i - x_i)^2 + (x_i - a_i)^2}{2(b_i - a_i)} \right) \right],
\end{aligned}$$

$i = 1, \dots, m; \lambda = 1, \dots, r.$

Consequently, we find

$$\begin{aligned}
 (42) \quad |S(f_1, \dots, f_r)(x)| &\leq \sum_{\lambda=1}^r \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r |f_\rho(x)| \right) \right. \\
 &\cdot \left(\sum_{i=1}^m \left[n_\lambda^{i-1} \left(\frac{(b_i - x_i)^2 + (x_i - a_i)^2}{2(b_i - a_i)} \right) \right. \right. \\
 &\cdot \left. \left. \|P_{n_\lambda-1}\|_{\infty, [a_i, b_i]} \left\| \frac{\partial^{n_\lambda} f_\lambda(\dots, x_{i+1}, \dots, x_m)}{\partial x_i^{n_\lambda}} \right\|_{\infty, \prod_{j=1}^i [a_j, b_j]} \right] \right) \Bigg) =: \theta_3(x).
 \end{aligned}$$

Finally, we derive that

$$(43) \quad |S(f_1, \dots, f_r)(x)| \leq \min \{ \theta_1(x), \theta_2(x), \theta_3(x) \}.$$

We have proved the following general multivariate Ostrowski type inequality for several functions.

THEOREM 11. *Let f_λ , $\lambda = 1, \dots, r \in \mathbb{N} - \{1\}$, be as in Assumptions 5 or Brief Assumptions 6; $n_\lambda \in \mathbb{N}$ associated with f_λ , $x = (x_1, \dots, x_m)$, $s = (s_1, \dots, s_m) \in \prod_{i=1}^m [a_i, b_i]$. Here $A_{i\lambda}(x_i, \dots, x_m)$ is as in (30), $i = 1, \dots, m$. We put*

$$\begin{aligned}
 (44) \quad S(f_1, \dots, f_r)(x) &:= \sum_{\lambda=1}^r \left\{ \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) \left[f_\lambda(x) - \frac{n_\lambda^m}{\prod_{i=1}^m (b_i - a_i)} \right. \right. \\
 &\cdot \left. \left(\int_{\prod_{i=1}^m [a_i, b_i]} f_\lambda(s) ds \right) - \sum_{i=1}^m A_{i\lambda}(x_i, \dots, x_m) \right] \Bigg\}.
 \end{aligned}$$

Here $\theta_1(x)$ is as in (37). Let $p_{li} > 1 : \sum_{l=1}^3 \frac{1}{p_{li}} = 1$, $i = 1, \dots, m$, and $\theta_2(x)$ be as in (40), and $\theta_3(x)$ as in (42). Then

$$(45) \quad |S(f_1, \dots, f_r)(x)| \leq \min \{ \theta_1(x), \theta_2(x), \theta_3(x) \}.$$

We continue with

REMARK 12. Additionally assume that $\frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}}$ are continuous on $\prod_{j=1}^m [a_j, b_j]$ for all $i = 1, \dots, m$; $\lambda = 1, \dots, r$.

We define and observe

$$\begin{aligned}
 (46) \quad W &:= \int_{\prod_{j=1}^m [a_j, b_j]} S(f_1, \dots, f_r)(x) dx = r \int_{\prod_{j=1}^m [a_j, b_j]} \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) dx \\
 &\quad - \frac{1}{\prod_{j=1}^m (b_j - a_j)} \sum_{\lambda=1}^r n_\lambda^m \left(\int_{\prod_{j=1}^m [a_j, b_j]} \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) dx \right) \left(\int_{\prod_{i=1}^m [a_i, b_i]} f_\lambda(s) ds \right) \\
 &\quad - \sum_{\lambda=1}^r \int_{\prod_{j=1}^m [a_j, b_j]} \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) \left(\sum_{i=1}^m A_{i\lambda}(x_i, \dots, x_m) \right) \right) dx \\
 (47) \quad &= \sum_{\lambda=1}^r \left\{ \int_{\prod_{j=1}^m [a_j, b_j]} \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) \left(\sum_{i=1}^m B_{i\lambda}(x_i, \dots, x_m) \right) \right) dx \right\}.
 \end{aligned}$$

Clearly, here $B_{i\lambda}(x_i, \dots, x_m)$ is a continuous function for all $i = 1, \dots, m$; $\lambda = 1, \dots, r$.

Hence

$$\begin{aligned}
 (48) \quad |W| &\stackrel{(3)}{\leq} \sum_{\lambda=1}^r \left\{ \int_{\prod_{j=1}^m [a_j, b_j]} \left\{ \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r |f_\rho(x)| \right) \left(\sum_{i=1}^m |B_{i\lambda}(x_i, \dots, x_m)| \right) \right\} dx \right\} \\
 &=: (\omega_1).
 \end{aligned}$$

That is

$$\begin{aligned}
 (49) \quad |W| &\leq \sum_{\lambda=1}^r \left\{ \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r \|f_\rho\|_{\infty, \prod_{j=1}^m [a_j, b_j]} \right) \left(\sum_{i=1}^m \left(\int_{\prod_{j=1}^m [a_j, b_j]} |B_{i\lambda}(x_i, \dots, x_m)| dx \right) \right) \right\}.
 \end{aligned}$$

From (41), we obtain

$$\begin{aligned}
 (50) \quad |B_{i\lambda}(x_i, \dots, x_m)| &\leq n_\lambda^{i-1} \|P_{n_\lambda-1}\|_{\infty, [a_i, b_i]} \left\| \frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}} \right\|_{\infty, \prod_{j=1}^m [a_j, b_j]} \left(\frac{(b_i - x_i)^2 + (x_i - a_i)^2}{2(b_i - a_i)} \right),
 \end{aligned}$$

for all $i = 1, \dots, m$; $\lambda = 1, \dots, r$.

Then

$$\begin{aligned}
 (51) \quad & \int_{\prod_{j=1}^m [a_j, b_j]} |B_{i\lambda}(x_i, \dots, x_m)| dx \\
 & \leq \frac{n_\lambda^{i-1} \|P_{n_\lambda-1}\|_{\infty, [a_i, b_i]} \left\| \frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}} \right\|_{\infty, \prod_{j=1}^m [a_j, b_j]}}{2(b_i - a_i)} \\
 & \quad \cdot \left(\prod_{\substack{j=1 \\ j \neq i}}^m (b_j - a_j) \right) \left(\int_{a_i}^{b_i} \left((b_i - x_i)^2 + (x_i - a_i)^2 \right) dx_i \right),
 \end{aligned}$$

and consequently, it holds

$$\begin{aligned}
 (52) \quad & \int_{\prod_{j=1}^m [a_j, b_j]} |B_{i\lambda}(x_i, \dots, x_m)| dx \\
 & \leq \frac{(b_i - a_i) \left(\prod_{j=1}^m (b_j - a_j) \right)}{3} n_\lambda^{i-1} \|P_{n_\lambda-1}\|_{\infty, [a_i, b_i]} \left\| \frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}} \right\|_{\infty, \prod_{j=1}^m [a_j, b_j]},
 \end{aligned}$$

for all $i = 1, \dots, m$; $\lambda = 1, \dots, r$.

Hence

$$\begin{aligned}
 (53) \quad & \sum_{i=1}^m \left(\int_{\prod_{j=1}^m [a_j, b_j]} |B_{i\lambda}(x_i, \dots, x_m)| dx \right) \leq \left(\frac{\left(\prod_{j=1}^m (b_j - a_j) \right)}{3} \right) \\
 & \quad \cdot \left(\sum_{i=1}^m \left[(b_i - a_i) \|P_{n_\lambda-1}\|_{\infty, [a_i, b_i]} \left\| \frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}} \right\|_{\infty, \prod_{j=1}^m [a_j, b_j]} n_\lambda^{i-1} \right] \right),
 \end{aligned}$$

for $\lambda = 1, \dots, r$.

Using (49) and (53), we obtain

$$\begin{aligned}
 (54) \quad |W| & \leq \left(\frac{\left(\prod_{j=1}^m (b_j - a_j) \right)}{3} \right) \left[\sum_{\lambda=1}^r \left\{ \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r \|f_\rho\|_{\infty, \prod_{j=1}^m [a_j, b_j]} \right) \right. \right. \\
 & \quad \cdot \left. \left(\sum_{i=1}^m \left[(b_i - a_i) n_\lambda^{i-1} \|P_{n_\lambda-1}\|_{\infty, [a_i, b_i]} \left\| \frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}} \right\|_{\infty, \prod_{j=1}^m [a_j, b_j]} \right] \right) \right\} \right] =: A_1.
 \end{aligned}$$

Notice next that

$$(55) \quad (\omega_1) = \sum_{\lambda=1}^r \sum_{i=1}^m \left(\int_{\prod_{j=1}^m [a_j, b_j]} \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r |f_\rho(x)| \right) |B_{i\lambda}(x_i, \dots, x_m)| dx \right) =: (\omega_2).$$

Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$(56) \quad (\omega_2) \leq \sum_{\lambda=1}^r \sum_{i=1}^m \left\| \prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho \right\|_{L_p(\prod_{j=1}^m [a_j, b_j])} \|B_{i\lambda}\|_{L_q(\prod_{j=1}^m [a_j, b_j])} \left(\prod_{j=1}^{i-1} (b_j - a_j) \right)^{\frac{1}{q}}.$$

We have also proved that

$$(57) \quad |W| \leq \sum_{\lambda=1}^r \sum_{i=1}^m \left\| \prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho \right\|_{L_p(\prod_{j=1}^m [a_j, b_j])} \|B_{i\lambda}\|_{L_q(\prod_{j=i}^m [a_j, b_j])} \left(\prod_{j=1}^{i-1} (b_j - a_j) \right)^{\frac{1}{q}} =: A_2.$$

From (50), we get

$$(58) \quad |B_{i\lambda}(x_i, \dots, x_m)| \leq n_\lambda^{i-1} \|P_{n_\lambda-1}\|_{\infty, [a_i, b_i]} \left\| \frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}} \right\|_{\infty, \prod_{j=1}^m [a_j, b_j]} \left(\frac{b_i - a_i}{2} \right),$$

for all $i = 1, \dots, m; \lambda = 1, \dots, r$.

Thus, it holds

$$(59) \quad \sum_{i=1}^m |B_{i\lambda}(x_i, \dots, x_m)| \leq \frac{1}{2} \left\{ \sum_{i=1}^m \left[(b_i - a_i) n_\lambda^{i-1} \|P_{n_\lambda-1}\|_{\infty, [a_i, b_i]} \left\| \frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}} \right\|_{\infty, \prod_{j=1}^m [a_j, b_j]} \right] \right\},$$

$\lambda = 1, \dots, r$.

Using (48) and (59), we finally derive

$$(60) \quad |W| \leq \frac{1}{2} \left\{ \sum_{\lambda=1}^r \left\{ \left\| \prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho \right\|_{L_1(\prod_{j=1}^m [a_j, b_j])} \cdot \left[\sum_{i=1}^m \left[(b_i - a_i) n_\lambda^{i-1} \|P_{n_\lambda-1}\|_{\infty, [a_i, b_i]} \left\| \frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}} \right\|_{\infty, \prod_{j=1}^m [a_j, b_j]} \right] \right] \right\} \right\} =: A_3.$$

We have proved the following general multivariate Grüss inequality.

THEOREM 13. Let f_λ , $\lambda = 1, \dots, r \in \mathbb{N} - \{1\}$, be as in Assumptions 5 and let $\frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}}$ be continuous on $\prod_{j=1}^m [a_j, b_j]$ for all $i = 1, \dots, m$; $\lambda = 1, \dots, r$, or Brief Assumptions 6; $n_\lambda \in \mathbb{N}$ associated with f_λ . Here $A_{i\lambda}(x_i, \dots, x_m)$ are as in (30), and $B_{i\lambda}(x_i, \dots, x_m)$ are as in (31), $i = 1, \dots, m$. We set

$$(61) \quad W := r \int_{\prod_{j=1}^m [a_j, b_j]} \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) dx \\ - \frac{1}{\prod_{j=1}^m (b_j - a_j)} \sum_{\lambda=1}^r n_\lambda^m \left(\int_{\prod_{j=1}^m [a_j, b_j]} \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) dx \right) \left(\int_{\prod_{i=1}^m [a_i, b_i]} f_\lambda(s) ds \right) \\ - \sum_{\lambda=1}^r \int_{\prod_{j=1}^m [a_j, b_j]} \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) \left(\sum_{i=1}^m A_{i\lambda}(x_i, \dots, x_m) \right) \right) dx.$$

Let A_1 be as in (54); $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, A_2 as in (57), and A_3 as in (60).

Then

$$(62) \quad |W| \leq \min \{A_1, A_2, A_3\}.$$

4. Applications

We apply Theorems 8, 11 and 13 for the case of $n_1 = n_2 = \dots = n_r = 1$.

We simplify General Assumptions 5 and Brief Assumptions 6, respectively, as follows:

GENERAL ASSUMPTIONS 14. Let $f : \prod_{j=1}^m [a_j, b_j] \rightarrow \mathbb{R}$ satisfy:

- 1) for $j = 1, \dots, m$, we have that $f(x_1, x_2, \dots, x_{j-1}, s_j, x_{j+1}, \dots, x_m)$ is absolutely continuous in $s_j \in [a_j, b_j]$, for every $(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_m) \in \prod_{\substack{i=1 \\ i \neq j}}^m [a_i, b_i]$,
- 2) for $j = 1, \dots, m$, we have that $\frac{\partial f(s_1, \dots, s_j, x_{j+1}, \dots, x_m)}{\partial x_j}$ is continuous on $\prod_{i=1}^j [a_i, b_i]$, for every $(x_{j+1}, \dots, x_m) \in \prod_{i=j+1}^m [a_i, b_i]$,
- 3) f is continuous on $\prod_{i=1}^m [a_i, b_i]$.

BRIEF ASSUMPTIONS 15. Let $f : \prod_{j=1}^m [a_j, b_j] \rightarrow \mathbb{R}$ with $\frac{\partial^l f}{\partial x_i^l}$ for $l = 0, 1$; $j = 1, \dots, m$, are continuous on $\prod_{j=1}^m [a_j, b_j]$.

We give the following multivariate representation result, which is an application of Theorem 8

THEOREM 16. *Let f be as in General Assumptions 14 or Brief Assumptions 15. Then*

$$(63) \quad f(x_1, \dots, x_n) = \frac{1}{\prod_{j=1}^m (b_j - a_j)} \int_{\prod_{j=1}^m [a_j, b_j]} f(s_1, \dots, s_m) ds_1 \dots ds_m + \sum_{i=1}^m T_i^*(x_i, \dots, x_m),$$

where

$$(64) \quad T_i^*(x_i, \dots, x_m) = \frac{1}{\prod_{j=1}^i (b_j - a_j)} \cdot \int_{a_1}^{b_1} \dots \int_{a_i}^{b_i} q(x_i, s_i) \frac{\partial f(s_1, \dots, s_i, x_{i+1}, \dots, x_m)}{\partial x_i} ds_1 \dots ds_i,$$

are continuous functions for all $i = 1, \dots, m$.

Next, we make

REMARK 17. Let f_λ be as in Assumptions 14 or 15, $\lambda = 1, \dots, r$. Then

$$(65) \quad f_\lambda(x) = \frac{1}{\prod_{j=1}^m (b_j - a_j)} \int_{\prod_{j=1}^m [a_j, b_j]} f_\lambda(s) ds + \sum_{i=1}^m T_{i\lambda}^*(x_i, \dots, x_m).$$

Here the corresponding

$$A_{i\lambda}^*(x_i, \dots, x_m) = 0,$$

and

$$(66) \quad B_{i\lambda}^*(x_i, \dots, x_m) = \frac{1}{\prod_{j=1}^i (b_j - a_j)} \cdot \int_{a_1}^{b_1} \dots \int_{a_i}^{b_i} q(x_i, s_i) \frac{\partial f_\lambda(s_1, \dots, s_i, x_{i+1}, \dots, x_m)}{\partial x_i} ds_1 \dots ds_i,$$

for all $i = 1, \dots, m$; $\lambda = 1, \dots, r$.

That is

$$(67) \quad T_{i\lambda}^*(x_i, \dots, x_m) = B_{i\lambda}^*(x_i, \dots, x_m),$$

$i = 1, \dots, m$; $\lambda = 1, \dots, r$.

We call and have the identity

$$\begin{aligned}
(68) \quad S^*(f_1, \dots, f_r)(x) &:= \sum_{\lambda=1}^r \left\{ \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) \left[f_\lambda(x) - \frac{1}{\prod_{j=1}^m (b_j - a_j)} \left(\int_{\prod_{j=1}^m [a_j, b_j]} f_\lambda(s) ds \right) \right] \right\} \\
&= \sum_{\lambda=1}^r \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) \left(\sum_{i=1}^m B_{i\lambda}^*(x_i, \dots, x_m) \right) \right),
\end{aligned}$$

true for any fixed $x \in \prod_{j=1}^m [a_j, b_j]$.

Hence, it holds

$$(69) \quad |S^*(f_1, \dots, f_r)(x)| \leq \sum_{\lambda=1}^r \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r |f_\rho(x)| \right) \left(\sum_{i=1}^m |B_{i\lambda}^*(x_i, \dots, x_m)| \right) \right).$$

We obtain:

1) From (37), we derive

$$\begin{aligned}
(70) \quad |S^*(f_1, \dots, f_r)(x)| &\leq \sum_{\lambda=1}^r \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r |f_\rho(x)| \right) \left(\sum_{i=1}^m \frac{1}{\prod_{j=1}^i (b_j - a_j)} \right. \right. \\
&\quad \cdot \left. \left[\left(\frac{(b_i - a_i) + |a_i + b_i - 2x_i|}{2} \right) \left\| \frac{\partial f_\lambda(\dots, x_{i+1}, \dots, x_m)}{\partial x_i} \right\|_{L_1(\prod_{j=1}^i [a_j, b_j])} \right] \right) \Bigg) \\
&=: \theta_1^*(x).
\end{aligned}$$

2) Let $p_{li} > 1 : \sum_{l=1}^3 \frac{1}{p_{li}} = 1$. Then by (40), we derive

$$\begin{aligned}
(71) \quad |S^*(f_1, \dots, f_r)(x)| &\leq \sum_{\lambda=1}^r \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r |f_\rho(x)| \right) \left(\sum_{i=1}^m \frac{1}{\prod_{j=1}^i (b_j - a_j)} \left[(b_i - a_i)^{\frac{1}{p_{1i}}} \left(\prod_{j=1}^{i-1} (b_j - a_j) \right)^{\frac{1}{p_{1i}} + \frac{1}{p_{2i}}} \right. \right. \right. \\
&\quad \cdot \left(\frac{(b_i - x_i)^{p_{2i}+1} + (x_i - a_i)^{p_{2i}+1}}{p_{2i} + 1} \right)^{\frac{1}{p_{2i}}} \\
&\quad \cdot \left. \left. \left\| \frac{\partial f_\lambda(\dots, x_{i+1}, \dots, x_m)}{\partial x_i} \right\|_{L_{p_{3i}}(\prod_{j=1}^i [a_j, b_j])} \right] \right) \Bigg) =: \theta_2^*(x).
\end{aligned}$$

3) We get by (42) that

$$(72) \quad |S^*(f_1, \dots, f_r)(x)| \leq \sum_{\lambda=1}^r \left\{ \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r |f_\rho(x)| \right) \left[\sum_{i=1}^m \left(\frac{(b_i - x_i)^2 + (x_i - a_i)^2}{2(b_i - a_i)} \right) \right. \right. \\ \left. \left. \cdot \left\| \frac{\partial f_\lambda(\dots, x_{i+1}, \dots, x_m)}{\partial x_i} \right\|_{\infty, \prod_{j=1}^i [a_j, b_j]} \right] \right\} =: \theta_3^*(x).$$

So, as an applications of Theorem 11, we give the following multivariate Ostrowski type inequality.

THEOREM 18. *All as in Remark 17. Then*

$$(73) \quad |S^*(f_1, \dots, f_r)(x)| \leq \min \{ \theta_1^*(x), \theta_2^*(x), \theta_3^*(x) \}.$$

We also make

REMARK 19. Here Assumptions 14 hold and $\frac{\partial f_\lambda}{\partial x_i}$ are continuous on $\prod_{j=1}^m [a_j, b_j]$ for all $i = 1, \dots, m$; $\lambda = 1, \dots, r$, or Assumptions 15 are valid.

We set

$$(74) \quad W^* = \int_{\prod_{j=1}^m [a_j, b_j]} S^*(f_1, \dots, f_r)(x) dx \\ = r \int_{\prod_{j=1}^m [a_j, b_j]} \left(\prod_{\rho=1}^r f_\rho(x) \right) dx \\ - \frac{1}{\prod_{j=1}^m (b_j - a_j)} \sum_{\lambda=1}^r \left(\int_{\prod_{j=1}^m [a_j, b_j]} \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) dx \right) \left(\int_{\prod_{i=1}^m [a_i, b_i]} f_\lambda(s) ds \right) \\ (75) \quad = \sum_{\lambda=1}^r \left\{ \int_{\prod_{j=1}^m [a_j, b_j]} \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho(x) \right) \left(\sum_{i=1}^m B_{i\lambda}^*(x_i, \dots, x_m) \right) \right) dx \right\}.$$

Here $B_{i\lambda}^*$ is continuous for any $i = 1, \dots, m$; $\lambda = 1, \dots, r$.

Then

1) following (49), we find

$$(76) \quad |W^*| \leq \sum_{\lambda=1}^r \left\{ \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r \|f_\rho\|_{\infty, \prod_{j=1}^m [a_j, b_j]} \right) \left(\sum_{i=1}^m \left(\int_{\prod_{j=1}^m [a_j, b_j]} |B_{i\lambda}^*(x_i, \dots, x_m)| dx \right) \right) \right\}.$$

We also get that

$$(77) \quad |W^*| \stackrel{(54)}{\leq} \left(\frac{\prod_{j=1}^m (b_j - a_j)}{3} \right) \left[\sum_{\lambda=1}^r \left\{ \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r \|f_\rho\|_{\infty, \prod_{j=1}^m [a_j, b_j]} \right) \cdot \left(\sum_{i=1}^m (b_i - a_i) \left\| \frac{\partial f_\lambda}{\partial x_i} \right\|_{\infty, \prod_{j=1}^m [a_j, b_j]} \right) \right\} \right] =: A_1^*.$$

2) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$(78) \quad |W^*| \stackrel{(57)}{\leq} \sum_{\lambda=1}^r \sum_{i=1}^m \left\| \prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^r f_\rho \right\|_{L_p(\prod_{j=1}^m [a_j, b_j])} \|B_{i\lambda}^*\|_{L_q(\prod_{j=1}^m [a_j, b_j])} \cdot \left(\prod_{j=1}^{i-1} (b_j - a_j) \right)^{\frac{1}{q}} =: A_2^*.$$

3) From (60), we obtain

$$(79) \quad |W^*| \leq \frac{1}{2} \left\{ \sum_{\lambda=1}^r \left\{ \left\| \prod_{\rho=1}^r f_\rho \right\|_{L_1(\prod_{j=1}^m [a_j, b_j])} \left[\sum_{i=1}^m \left[(b_i - a_i) \left\| \frac{\partial f_\lambda}{\partial x_i} \right\|_{\infty, \prod_{j=1}^m [a_j, b_j]} \right] \right] \right\} \right\} =: A_3^*.$$

We have proved the following multivariate Grüss type inequality as an application of Theorem 13.

THEOREM 20. *Here all as in Remark 19. We derive*

$$(80) \quad |W^*| \leq \min \{A_1^*, A_2^*, A_3^*\}.$$

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