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THE LIFTING-EXTENSION PROBLEM FOR DUCHAIN COMPLEXES OF DWYER AND KAN

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Abstract. A duchain complex of W. Dwyer and D. Kan is a common extension of the notions of a chain complex and a cochain complex.

Given a square commutative diagram of duchain complexes, the lifting-extension problem asks whether there exists a diagonal map making the two resulting triangles commute. Duchain complexes have a model category structure, and hence a lift exists if the left vertical map is a cofibration, the right vertical map is a fibration, and one of them is a weak equivalence.

We show that it is possible to replace the two conditions above, by a countably infinite, bigraded, family of conditions which guarantee the existence of a lift.

1. Introduction

The notion of duchain complex (Definition 1 below) of W. Dwyer and D. Kan [2] is a common extension of the notions of a chain complex and a cochain complex. As observed in [2] and [3], duchain complexes are closely related to algebraic structures laying at the foundations of cyclic homology.

In this paper, we study the lifting-extension problem for the category of duchain complexes. Given a commutative diagram of duchain complexes

$$\begin{array}{ccc} K & \longrightarrow & U \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ L & \longrightarrow & V \end{array}$$

the problem is whether there exists a lifting (or lift), i.e. a map $L \to U$ making the two resulting triangles commute. As observed already by S. T. Hu [6], a vast number of problems in topology can be phrased as special cases of this problem.

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Duchain complexes have a model category structure (see the next section), and hence a lift exists if i is a cofibration, p is a fibration, and either the left or the right vertical map is a weak equivalence.

By a careful analysis of spheres and balls (see 2.1) in the category of duchain complexes, we show that it is possible to replace the two conditions above, by a countably infinite family of conditions which guarantee the existence of a lift (see Theorem 2). In different settings, the existence of such a lifting has been studied by a number of authors, J. Grossman [5], D. Isaksen [7], and the author [9]. However, the category of duchain complexes is not a stratified model category in the sense of [9], since the presence of two families of boundary maps makes the splitting of conditions for a map to be a weak equivalences more complex.

Similarly, the model category axioms guarantee the existence of two factorizations of a map of duchain complexes into a cofibration followed by a fibration, where either the first or the second map can be chosen to be a weak equivalence. We show that there is in fact a countably infinite family of such factorizations, where the conditions for being a weak equivalence are split between the two maps (see Theorem 1).

2. The category of duchain complexes

In this section, we recall the main notions of [2]. Let $\mathbf{R}$ be a ring with $1 \neq 0$.

Definition 1. A duchain complex is a diagram of modules:

$$U_0 \underset{\partial}{\overset{\delta}{\rightleftarrows}} U_1 \underset{\partial}{\overset{\delta}{\rightleftarrows}} U_2 \underset{\partial}{\overset{\delta}{\rightleftarrows}} \dots$$

such that $\delta^2 = 0$ and $\partial^2 = 0$, but otherwise the δ's and the ∂'s are independent.

By 4.8 in [2], the category of duchain complexes $\mathbf{R}(\partial, \delta)$ has a Quillen model structure (see [8] for the original reference or [4] for a more recent account). A map $f : U \to V$ is

- a weak equivalence, if it induces isomorphisms $f_i : \mathbf{H}_i(U) \to \mathbf{H}_i(V)$ and $f^i : \mathbf{H}^i(U) \to \mathbf{H}^i(V)$ for $i \geq 0$ on the homology groups and the cohomology groups,
- a fibration, if it is an onto map in positive dimensions,
- a cofibration, if it is a retract of the (possibly transfinite) compositions of cobase extensions (see [4], 2.16) of the sphere into ball inclusions described in the subsection below.

2.1. Spheres and balls in $\mathbf{R}(\partial, \delta)$. Let $\mathbf{0}$ denote the trivial duchain complex.

- For $n \geq 0$, let D^n (the n-disc) denote the free object with one generator x_n in dimension n. Let D^{-1} be the zero complex 0.
- For $n > 0$, let S^n_∂ (the homology sphere) be the object with one generator y_n in dimension n and one relation $\partial y_n = 0$. Let $S^0_\partial = D^0$ and let $S^{-1}_\partial = 0$.
- For $n \geq 0$, let S^n_δ (the cohomology sphere) be the object with one generator z_n in dimension n and one relation $\delta z_n = 0$.

There are natural maps between the spheres and balls, which play an important role in what follows. They are all injective except the map i^δ_{-1}.

- For $n \geq 0$, let $j_n : 0 \to D^n$ (there is only one such map).
- For $n > 0$, let $i^\partial_n : S^{n-1}_\partial \to D^n$ be the map sending $y_{n-1} \mapsto \partial x_n$. Let $i^\partial_0 : S^{-1}_\partial \to D^0$ be the unique map $0 \to D^0$.
- For $n \geq 0$, let $i^\delta_n : S^{n+1}_\delta \to D^n$ be the map sending $z_{n+1} \mapsto \delta x_n$. Let $i^\delta_{-1} : S^0_\delta \to D^{-1}$ be the map sending $z_0 \mapsto 0$.

If a lift exists in every commutative diagram of the form

$$\begin{array}{ccc} K & \longrightarrow & U \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ L & \longrightarrow & V \end{array}$$

then we say that p has the right lifting property (RLP) with respect to i, and that i has the left lifting property (LLP) with respect to p.

Proposition 1. *A map $p : U \to V$ in $\boldsymbol{R}(\partial, \delta)$ is a fibration iff it has the right lifting property with respect to the maps $j_n : 0 \to D^n$ for $n > 0$.*

Proposition 2. *A map $p : U \to V$ in $\boldsymbol{R}(\partial, \delta)$ is an acyclic fibration iff it has the right lifting property with respect to the maps $i^\partial_n : S^{n-1}_\partial \to D^n$ for $n \geq 0$, the maps $i^\delta_n : S^{n+1}_\delta \to D^n$ for $n \geq -1$ and the map $j_0 : 0 \to D^0$.*

3. The factorization and the lifting-extension theorems

We start with two lemmas, which form a more detailed version of the last proposition in the previous section. The lengthy proofs are contained in the last section. In both these lemmas, for the smallest choice of dimension, part of the statement refers to the (nonexistent) homology or cohomology group in dimension -1 - of course, this part of the statement should be treated as vacuous.

Lemma 1. *Let n be a nonnegative integer and $p : U \to V$ be a fibration in $\boldsymbol{R}(\partial, \delta)$. A lift exists in the diagram*

$$
\begin{array}{ccc}
S_\partial^{n-1} & \xrightarrow{f} & U \\
{\scriptstyle i_n^\partial}\downarrow & & \downarrow{\scriptstyle p} \\
D^n & \xrightarrow{g} & V
\end{array} \tag{1}
$$

iff p induces a monomorphism on H_{n-1} and an epimorphism on H_n.

Lemma 2. *Let $n \geq -1$ be an integer and $p : U \to V$ be a fibration in $\boldsymbol{R}(\partial, \delta)$ such that p is onto in dimension 0. A lift exists in the diagram*

$$
\begin{array}{ccc}
S_\delta^{n+1} & \xrightarrow{f} & U \\
{\scriptstyle i_n^\delta}\downarrow & & \downarrow{\scriptstyle p} \\
D^n & \xrightarrow{g} & V
\end{array} \tag{2}
$$

iff p induces a monomorphism on H^{n+1} and an epimorphism on H^n.

We will often need the phrase "the map is an isomorphism on H_n". This will be abbreviated to "the map is an H_n-isomorphism". A similar convention will be used with "epimorphism", "monomorphism" and "H^n".

Let $\overline{N}$ be the ordered set $\overline{N} = \{n \in \mathbb{Z} | n \geq -1\} \cup \{\infty\}$.

Definition 2. For $(n, m) \in \overline{N} \times \overline{N}$, a fibration $p : U \to V$ of duchain complexes is an (n, m)-acyclic fibration if the map p is

- an H_k-isomorphism for $k > n$ and an H_n-monomorphism,
- an H^k-isomorphism for $k < m$ and an H^m-monomorphism.

Definition 3. For $(n, m) \in \overline{N} \times \overline{N}$, a cofibration $i : U \to V$ of duchain complexes is an (n, m)-acyclic cofibration if the map i is

- an H_k-isomorphism for $k < n$ and an H_n-epimorphism,
- an H^k-isomorphism for $k > m$ and an H^m-epimorphism.

Theorem 1. *Let $(n, m) \in \overline{N} \times \overline{N}$. Every map $f : U \to V$ in $\boldsymbol{R}(\partial, \delta)$, which is onto in dimension 0, can be factored as*

$$U \overset{i_\infty}{\to} W \overset{p_\infty}{\to} V,$$

with i_∞ an (n, m)-acyclic cofibration and p_∞ an (n, m)-acyclic fibration. Moreover, every (n, m)-acyclic fibration has the right lifting property with respect to i.

Proof. In order to prove this result, we need D. Quillen's small object argument (Lemma 3, §3, Chapter 2 in [8]). This construction can in fact be traced back to R. Baer's construction embedding an R-module into an injective one (see Cartan–Eilenberg [1], proof of Theorem 3.3). A more detailed version of the small object argument is given in Dwyer–S. [4], §7.12. It can be summarized as follows.

Assume that $\mathbf{C}$ is a category with all small colimits. Let $\mathbf{Z}_+$ be the category $\{0 \to 1 \to 2 \to \dots\}$. An object A of $\mathbf{C}$ is called sequentially small if for every functor $B : \mathbf{Z}_+ \to \mathbf{C}$, the canonical map

$$\mathtt{colim}\,\mathtt{Hom}_{\mathbf{C}}(A, B(n)) \to \mathtt{Hom}_{\mathbf{C}}(A, \mathtt{colim}\, B(n))$$

is an isomorphism. A set is sequentially small if it is finite, and an R-module is sequentially small if it has a finite presentation.

Given a set of maps $\mathcal{G} = \{\gamma_k : A_k \to B_k\}$ with the domains sequentially small, every map $f : X \to Y$ in $\mathbf{C}$ can be factored as

$$X \overset{i_\infty}{\to} X' \overset{p_\infty}{\to} Y$$

in such a way that p_∞ has the RLP with respect to each map in $\mathcal{G}$. Moreover, the map i_∞ is the inclusion of X into an object obtained from X by repeatedly attaching copies of B_k along maps A_k to X at the beginning, or to the previous stages of the "infinite gluing construction" at successive stages.

We now apply the above construction to the map $f : U \to V$ with the set of maps $\mathcal{G}$ given by

$$\left\{j_k : 0 \to D^k\right\}_{k=1}^{\infty} \cup \left\{i_\partial^k : S_\partial^{k-1} \to D^k\right\}_{k=n+1}^{\infty} \cup \left\{i_\delta^k : S_\delta^{k+1} \to D^k\right\}_{k=-1}^{m-1}$$

Since the domain of each of the above maps is finitely generated, it is sequentially small. Of course the maps in the first set in the union above induce trivial maps on homology and cohomology. The maps in the second set induce isomorphisms on cohomology, and isomorphisms in homology up to dimension $n-1$ and n-epimorphisms. The maps in the third set induce isomorphisms in homology and isomorphisms in cohomology in dimensions greater than m and epimorphisms in dimension m.

The construction of i_∞ is such that if all the maps γ_k are isomorphisms (epimorphisms, monomorphisms) on homology (cohomology) in some dimension, the map i_∞ has the same property. Hence i_∞ is an (n,m)-acyclic cofibration.

Since we assume that f is onto in dimension 0, so is p_∞. Hence, by Lemmas 1 and 2, the map p_∞ is an (n,m)-acyclic fibration. ∎

Theorem 2. *Let $(n,m) \in \overline{N} \times \overline{N}$. If $i : K \to L$ is an (n,m)-acyclic cofibration which is onto in dimension* 0 *and $p : U \to V$ is an (n,m)-acyclic fibration, then a lift (i.e. $h : L \to U$ with $ph = g$ and $hi = f$) exists in every* *commutative diagram of the form:*

$$\begin{array}{ccc} K & \xrightarrow{f} & U \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ L & \xrightarrow{g} & V \end{array}$$

Proof. By Theorem 1, we can factor $i : K \to L$ as

$$K \overset{i_\infty}{\to} L' \overset{p_\infty}{\to} L,$$

where i_∞ is an (n, m)-acyclic cofibration and p_∞ is an (n, m)-acyclic fibration. Since both i and i_∞ are isomorphisms in homology for $i < n$, so is p_∞. Moreover, since i is an epimorphism on homology in dimension n, so is p_∞. But p_∞ is also a monomorphism in homology in dimension n by Theorem 1 above, and an isomorphism in homology for $i > n$. We conclude that p_∞ induces isomorphisms in homology in every dimension.

Moreover, since i and i_∞ are isomorphisms on cohomology in dimensions greater than m, so is p_∞. Moreover, since i is an epimorphism in dimension m on cohomology, so is p_∞. However, p_∞ is an isomorphism in dimensions less than m and a monomorphism in dimension m in cohomology by Theorem 1 above, hence it is an isomorphism in cohomology in every dimension.

We conclude that p_∞ is a weak equivalence of duchain complexes in the model category structure described in Section 2.

Consider the diagram

$$\begin{array}{ccc} K & \xrightarrow{i_\infty} & L' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_\infty} \\ L & \xrightarrow{\texttt{id}} & L \end{array}$$

By the model category structure for duchain complexes, since i is a cofibration and p_∞ is an acyclic fibration, we conclude there is a lift $h : L \to L'$. The following diagram shows that i is a retract of i_∞.

$$\begin{array}{ccccc} K & \xrightarrow{\texttt{id}} & K & \xrightarrow{\texttt{id}} & K \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_\infty} & & \downarrow{\scriptstyle i} \\ L & \xrightarrow{h} & L' & \xrightarrow{p_\infty} & L \end{array}$$

Since by Theorem 1, every n-acyclic fibration has the right lifting property with respect to i_∞, it has the right lifting property with respect to i (a straightforward calculation). ∎

4. Proofs of Lemmas 1 and 2

Proof of Lemma 1. For $n = 0$ we need to show that a lifting exists in the diagram (1) if and only if p is an H^0-epimorphism. Assume that the lift exists, $\sigma \in H_0(V)$, and let v_0 be a representative of σ. The assignment $x_0 \mapsto v_0$ determines a map $D^0 \to V$ (since D^0 is free on x_0), and hence a diagram of the form (1). By assumption, there exists a lift $h : D^0 \to U$. The element $u_0 = h(x_0)$ has the property that $p(u_0) = v_0$ and being in dimension 0 is a cycle. We conclude that p is an H_0-epimorphism.

Conversely, assume that p is an H_0-epimorphism and let a diagram (1) with $n = 0$ be given. Let $v_0 = g(x_0)$. Since every element of V_0 is a cycle, there exists an element $\bar{u}_0$ such that $p_*([\bar{u}_0]) = [v_0]$. Hence, there exists an element $v_1 \in V_1$ such that $p(\bar{u}_0) = v_0 + \partial v_1$. Since p is a fibration, there exists a $u_1 \in U_1$ such that $p(u_1) = v_1$. Let $u_0 = \bar{u}_0 - \partial u_1$. We have

$$p(u_0) = p(\bar{u}_0) - p(\partial u_1) = v_0 + \partial v_1 - \partial p(u_1) = v_0.$$

Hence we can define the lift h by setting $h(x_0) = u_0$.

Now assume $n \geq 1$.

Note that the diagrams (1) are in one to one correspondence with sets of elements

$$\begin{array}{ccccc} 0 & \xleftarrow{\partial} & u_{n-1} & & \\ & & \big\downarrow{\scriptstyle p} & & \\ & & v_{n-1} & \xleftarrow{\partial} & v_n \end{array} \tag{3}$$

Finding a lift amounts to finding an element $u_n \in U_n$ such that $\partial u_n = u_{n-1}$ and $p(u_n) = v_n$.

First, we show that if a lift exists, then p is an H_n-epimorphism. Let $\sigma \in H_n(V)$. Hence $\sigma = [v_n]$ for some cycle $v_n \in V_n$. Consider the set of elements (3) with $u_{n-1} = 0$ and $v_{n-1} = 0$. The lift gives a cycle $u_n \in U_n$ such that $p(u_n) = v_n$.

Next, we show that if a lift exists, p is an H_{n-1}-monomorphism. Let $\tau = [u_{n-1}] \in H_{n-1}(U)$ and $p_*(\tau) = 0$. Since $p_*(\tau) = 0$, $v_{n-1} = p(u_{n-1}) = \partial v_n$. Hence, we have a diagram of elements (3) and the lift gives an element u_n with $\partial u_n = u_{n-1}$ and therefore τ must equal zero.

We now assume that p is an H_n-epimorphism on and an H_{n-1}-monomorphism and wish to show that a lift exists in every square diagram (1). Such a diagram determines a set of elements (3). Notice that u_{n-1} is a cycle and $p_*([u_{n-1}]) = 0$. Hence, there exists an element $\tilde{u}_n \in U_n$ such that $\partial \tilde{u}_n = u_{n-1}$ (since p_* is a H_{n-1}-monomorphism). It need not be true that $p(\tilde{u}_n) = v_n$.

However, we have

$$\partial(v_n - p(\tilde{u}_n)) = \partial v_n - \partial p(\tilde{u}_n) = v_{n-1} - p(\partial \tilde{u}_n) = v_{n-1} - v_{n-1} = 0.$$

Hence $v_n - p(\tilde{u}_n)$ is a cycle of dimension n. Since p is an H_n-epimorphism, there exists a cycle $\bar{u}_n \in U_n$, with $p_*[\bar{u}_n] = [v_n - p(\tilde{u}_n)]$, i.e. $p(\bar{u}_n) = v_n - p(\tilde{u}_n) + \partial v_{n+1}$ for some $v_{n+1} \in V_{n+1}$. Since p is a fibration, there exists a $u_{n+1} \in U_{n+1}$ such that $p(u_{n+1}) = v_{n+1}$. Consider the element

$$u_n = \tilde{u}_n + \bar{u}_n - \partial u_{n+1}.$$

We have

$$\partial u_n = \partial \tilde{u}_n + \partial \bar{u}_n = u_{n-1},$$

$$p(u_n) = p(\tilde{u}_n) + p(\bar{u}_n) - p(\partial u_{n+1}) = p(\tilde{u}_n) + v_n - p(\tilde{u}_n) + \partial v_{n+1} - \partial v_{n+1} = v_n.$$

Hence u_n determines the desired lift. ∎

Proof of Lemma 2. Assume first $n = -1$ and that a lift exists in every square diagram (2). Suppose that u_0 is a cocycle and that $p_*[u_0] = 0$. Hence $p(u_0) = 0$. This data determines a diagram of the form (2) with the map f determined by the map $z_0 \mapsto u_0$. The existance of a lift implies that the map f factors through the zero complex, hence f equals zero and u_0 equals zero. We conclude that p is an H^0-monomorphism.

Conversely, assume that p is an H^0-monomorphism and suppose that a diagram of the form (2) is given. Let $u_0 = f(z_0)$. By the commutativity of the diagram, $p(u_0) = pf(z_0) = 0$. Since u_0 is a cocycle which maps to zero, and p is an H^0-monomorphism, we conclude that $u_0 = 0$ and hence that f equals the zero map. We conclude that the zero map is a lift in this diagram.

Next, assume that $n = 0$ and the given lifting exists. Let $\sigma \in H^0(V)$ and $v_0 \in V_0$ be a generator of σ, i.e. $[v_0] = \sigma$ and $\delta v_0 = 0$. Consider the square diagram (2) given by sending the generator x_0 of D^0 to v_0 and the generator z_1 of S^1_δ to 0. The existence of a lift shows that there exists a cohomology class u_0 such that $p(u_0) = v_0$. Hence p is an epimorphism on H^0.

Now, assume ($n = 1$) that $\tau \in H^1(U)$ is such that $p_*(\tau) = 0$. Let u_1 be a generator. Hence, there exists an element $v_0 \in V_0$ such that $\delta v_0 = p(u_1)$. This data determines a square diagram (2) and the lifting gives an element $u_0 \in U_0$ such that $\delta u_0 = u_1$. Hence p is an H^1-monomorphism.

Now suppose that p is an H^0-epimorphism and an H^1-monomorphism. We wish to show that a lift exists in the diagram (2). Such a diagram determines (and is determined by) the following set of elements and maps:

$$\begin{array}{ccccc} & & u_1 & \xrightarrow{\delta} & 0 \\ & & {\scriptstyle p}\downarrow & & \\ v_0 & \xrightarrow{\delta} & v_1 & & \end{array}$$

Hence u_1 is a cocycle with $p_*[u_1] = 0$. Since p is an H^1-monomorphism, there exists an element $\bar{u}_0$ such that $\delta\bar{u}_0 = u_1$. Consider the element $v_0 - p\bar{u}_0$. We have

$$\delta(v_0 - p\bar{u}_0) = \delta v_0 - p\delta\bar{u}_0 = v_1 - pu_1 = 0.$$

Hence $v_0 - p\bar{u}_0$ is a 0-cycle. Since p is an H^0-epimorphism, there exists a 0-cocycle $\tilde{u}_0$ such that $p_*[\tilde{u}_0] = [v_0 - p\bar{u}_0]$. Since the 0-coboundaries are the trivial group, we have $p(\tilde{u}_0) = v_0 - p\bar{u}_0$. Set $u_0 = \bar{u}_0 + \tilde{u}_0$. We have

$$\begin{aligned} \delta u_0 &= \delta\bar{u}_0 + \delta\tilde{u}_0 = u_1, \\ p(u_0) &= p(\bar{u}_0) + p(\tilde{u}_0) = p(\bar{u}_0) + v_0 - p(\bar{u}_0) = v_0. \end{aligned}$$

Hence a lift exists.

Now suppose that $n \geq 1$. Assume that a lift exists in every diagram of the form (2). This is equivalent to saying that for every set of elements and maps

$$\begin{array}{ccc} & & u_{n+1} & \xrightarrow{\delta} & 0 \\ & & \Big\downarrow{\scriptstyle p} & & \\ v_n & \xrightarrow{\delta} & v_{n+1} & & \end{array} \tag{4}$$

there exists an element $u_n \in U_n$ such that $\delta u_n = u_{n+1}$ and $p(u_n) = v_n$.

Suppose that $u_{n+1} \in U_{n+1}$ is a cocycle and $p_*([u_{n+1}]) = 0$. This data determines a diagram of elements as above and the lift determines the element u_n such that $\delta u_n = u_{n+1}$ and hence $[u_{n+1}] = 0$, i.e. p is an H^{n+1}-monomorphism.

Next, suppose that $v_n \in V_n$ is a cocycle. The diagram of elements

$$\begin{array}{ccc} & & 0 \\ & & \Big\downarrow \\ v_n & \longrightarrow & 0 \end{array}$$

determines a commutative square of maps (2). The lift provides an element u_n such that $\delta u_n = 0$ and $p(u_n) = v_n$. We conclude that p is an H^n-epimorphism.

Finally, we show that if p is an H^n-epimorphism and an H^{n+1}-monomorphism, then a lift exists in every diagram of the form (2). This is equivalent to saying that every set of elements (4) can be completed to a commutative square with an element u_n such that $\delta u_n = u_{n+1}$ and $p(u_n) = v_n$.

Since $p_*[u_{n+1}] = 0$, and p is an H^{n+1}-monomorphism, there exists $\bar{u}_n \in U_n$ such that $\delta\bar{u}_n = u_{n+1}$. Consider the element $v_n - p(\bar{u}_n)$. We have

$$\delta(v_n - p(\bar{u}_n)) = \delta v_n - p(\delta\bar{u}_n) = v_{n+1} - pu_{n+1} = 0.$$

Hence $v_n - p(\bar{u}_n)$ is a cocycle. Since p_* is an epimorphism in the appropriate dimension, there exists a cycle $\tilde{u}_n$ such that

$$p_*[\tilde{u}_n] = [v_n - p(\bar{u}_n)].$$

Hence the elements on the left and right hand sides of the equation differ by a coboundary, i.e. there exists an element $v_{n-1} \in V_{n-1}$ such that

$$p(\tilde{u}_n) = v_n - p(\bar{u}_n) + \delta v_{n-1}.$$

Since p is a fibration which is onto in dimension 0 (i.e. it is an onto map in every dimension), there exists and element $u_{n-1} \in U_{n-1}$ such that $p(u_{n-1}) = v_{n-1}$. Set

$$u_n = \bar{u}_n + \tilde{u}_n - \delta u_{n-1}.$$

We have

$$\begin{aligned} \delta u_n &= \delta \bar{u}_n + \delta \tilde{u}_n = u_{n+1}, \\ p(u_n) &= p(\bar{u}_n) + p(\tilde{u}_n) - p\delta u_{n-1} \\ &= p(\bar{u}_n) + (v_n - p(\bar{u}_n) + \delta v_{n-1}) - \delta p(u_{n-1}) = v_n. \ \blacksquare \end{aligned}$$

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