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SOFT SET THEORY APPLIED TO GENERAL ALGEBRAS

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Abstract. The notions of a soft general algebra and a soft subalgebra are introduced and studied. The operations on them such as a restricted intersection, an extended intersection, a restricted union, a $\wedge$-intersection, a $\vee$-union and a cartesian product are established.

1. Introduction

Many of our real life problems in economics, engineering, environment, social and medical science, etc. involve uncertainties. As mathematical tools for dealing with uncerainties are considered probability, fuzzy sets, intuitionistic fuzzy sets, vague sets, interval mathematics and rough sets. However, all of these theories have their inherent difficulties as pointed out by Molodtsov in [13]. To avoid these difficulties, he proposed in [13] completely new concept of a soft set. In soft set theory, the problem of setting the membership function simply does not arise. This makes the theory easy to apply in practice. Some of applications of soft set theory in different fields are demonstrated by Molodtsov in [13]. Recently, the properties and applications of soft set theory have been studied increasingly (for example, see [3], [4], [7] and [15]).

There are many papers on applications of soft set theory in several different types of algebras, such as semigroups [9], groups [2], semirings [6], rings ([1], [5]), BCK/BCI-algebras [8], BCH-algebras [11], lattices [14], and Hilbert algebras [10]. However, many of the results can be generalized on general algebras, what is the purpose of this paper. We introduce the concept of soft general algebra and discuss and study some of their properties and structural characteristics. We also investigate relations between soft general algebras and soft subalgebras. The notions of restricted intersection, extended

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intersection, restricted union, $\wedge$-intersection, $\vee$-union and cartesian product of families of soft general algebras and soft subalgebras are established.

Proofs of some theorems given here are similar to those presented in earlier papers on soft algebras (e.g. on soft rings, soft lattices, etc.) and therefore will be omitted (then we refer to references).

2. Preliminaries

Molodtsov defined the notion of a soft set in the following way. Let U be an initial universe set and E a set of parameters. The power set of U is denoted by $\mathcal{P}(U)$ and A is a subset of E. A pair (F, A) is called a *soft set* over U, where F is a mapping $F : A \to \mathcal{P}(U)$. In other words, a soft set over U is a parametrized family of subsets of the universe U. For $x \in A$, $F(x)$ may be considered as the set of x-approximate elements of the soft set (F, A). Clearly, a soft set is not a set.

Now we give some properties and define some operations of soft sets (for these see also [3], [5], [6], [11], [12] and [13]).

Definition 2.1. Let (F, A) and (G, B) be two soft sets over a common universe U. Then (F, A) is called a *soft subset* of (G, B), denoted by $(F, A) \tilde{\subseteq} (G, B)$, if $A \subseteq B$ and $F(x) \subseteq G(x)$ for all $x \in A$.

Definition 2.2. The *extended intersection* of a nonempty family of soft sets $\{(F_t, A_t) : t \in T\}$ over a common universe U, denoted by $\tilde{\cap}_{t\in T}(F_t, A_t)$, is the soft set (G, B) such that $B = \bigcup_{t\in T} A_t$ and for every $x \in B$, $G(x) = \bigcap_{t\in T(x)} F_t(x)$, where $T(x) = \{t \in T : x \in A_t\}$.

Definition 2.3. The *restricted intersection* of a nonempty family of soft sets $\{(F_t, A_t) : t \in T\}$ over a common universe U such that $\bigcap_{t\in T} A_t \neq \emptyset$, denoted by $\tilde{\sqcap}_{t\in T}(F_t, A_t)$, is the soft set (G, B), where $B = \bigcap_{t\in T} A_t$ and $G(x) = \bigcap_{t\in T} F_t(x)$ for all $x \in B$.

Definition 2.4. The *restricted union* of a nonempty family of soft sets $\{(F_t, A_t) : t \in T\}$ over a common universe U such that $\bigcap_{t\in T} A_t \neq \emptyset$, denoted by $\tilde{\sqcup}_{t\in T}(F_t, A_t)$, is the soft set (G, B), where $B = \bigcap_{t\in T} A_t$ and $G(x) = \bigcup_{t\in T} F_t(x)$ for all $x \in B$.

Definition 2.5. The $\wedge$*-intersection* of a nonempty family of soft sets $\{(F_t, A_t) : t \in T\}$ over a common universe U, denoted by $\tilde{\wedge}_{t\in T}(F_t, A_t)$, is the soft set (G, B), where $B = \prod_{t\in T} A_t$ and $G(x) = \bigcap_{t\in T} F_t(x_t)$ for all $x = (x_t)_{t\in T} \in B$.

Definition 2.6. The $\vee$*-union* of a nonempty family of soft sets $\{(F_t, A_t) : t \in T\}$ over a common universe U, denoted by $\tilde{\vee}_{t\in T}(F_t, A_t)$, is the soft set (G, B), where $B = \prod_{t\in T} A_t$ and $G(x) = \bigcup_{t\in T} F_t(x_t)$ for all $x = (x_t)_{t\in T} \in B$.

Definition 2.7. Let $T \neq \emptyset$ and (F_t, A_t) *be a soft set over* U_t for all $t \in T$. *The cartesian product* of the family $\{(F_t, A_t) : t \in T\}$ over $\prod_{t\in T} U_t$, denoted by $\widetilde{\prod}_{t\in T}(F_t, A_t)$, is the soft set (G, B), where $B = \prod_{t\in T} A_t$ and $G(x) = \prod_{t\in T} F_t(x_t)$ for all $x = (x_t)_{t\in T} \in B$.

3. Soft general algebras

In this section, X always means a general algebra. We define

$$\mathrm{Sub}X = \{S \subseteq X : S = \emptyset \text{ or } S \text{ is a subalgebra of } X\}.$$

Definition 3.1. Let (F, A) be a soft set over X. We say that (F, A) is a *soft general algebra* over X if it satisfies:

$$(\forall x \in A)\,(F(x) \neq \emptyset \Longrightarrow F(x) \text{is a subalgebra of } X),$$

that is, $F(x) \in \mathrm{Sub}X$ for all $x \in A$.

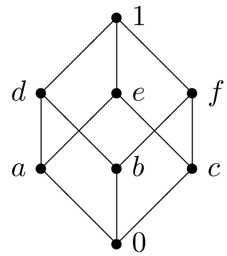
Remark. For a general algebra X and a nonempty set A, a set-valued function $F : A \to \mathcal{P}(X)$ can be defined by $F(x) = \{y \in X : (x, y) \in R\}$ for $x \in A$, where R is an arbitrary binary relation between elements of A and elements of X, that is, R is a subset of $A \times X$. The pair (F, A) is then a soft set over X. Thus appropriate choices of the relation R produces many examples of soft sets over X.

Example 3.2. [2] Consider the group $(\mathbb{Z}, +)$. Let (F, A) be a soft set over $\mathbb{Z}$, where $A = \mathbb{Z}$ and $F : A \to \mathcal{P}(\mathbb{Z})$ is a set-valued function defined by

$$F(x) = x\mathbb{Z} \quad (= \{kx : k \in \mathbb{Z}\}),$$

for all $x \in A$. Thus $F(x) = \{y \in \mathbb{Z} : (x, y) \in R\}$, where $R = \{(x, y) \in A \times \mathbb{Z} : y = kx \text{ for some } k \in \mathbb{Z}\}$. It is clear that $F(x)$ is a subgroup of $\mathbb{Z}$for all $x \in A$. Therefore, (F, A) is a soft group over $\mathbb{Z}$.

Example 3.3. [14] Let $L = \{0, a, b, c, d, e, f, 1\}$ be a lattice with the following Hasse diagram:



Let (F, A) be a soft set over L, where $A = \{a, b, d\}$ and $F : A \to \mathcal{P}(L)$ is a set-valued function defined by

$$F(x) = \{y \in L : x \vee y = 1\},$$

for all $x \in A$. Then $F(a) = \{f, 1\}$, $F(b) = \{e, 1\}$ and $F(d) = \{c, e, f, 1\}$ are sublattices of L. Hence (F, A) is a soft lattice over L.

Theorem 3.4. *Let $\{(F_t, A_t) : t \in T\}$ be a nonempty family of soft general algebras over X. Then:*

(1) *the extended intersection $\widetilde{\cap}_{t\in T}(F_t, A_t)$ is a soft general algebra over X;*
(2) *if $\bigcap_{t\in T} A_t \neq \emptyset$, then the restricted intersection $\widetilde{\sqcap}_{t\in T}(F_t, A_t)$ is a soft general algebra over X;*
(3) *the $\wedge$-intersection $\widetilde{\wedge}_{t\in T}(F_t, A_t)$ is a soft general algebra over X.*

Proof. It is well-known that the intersection of any number of subalgebras of an algebra is either a subalgebra or the empty set. Therefore, (1), (2) and (3) are results of Definition 2.2, 2.3 and 2.5, respectively. ■

Theorem 3.5. *Let $\{(F_t, A_t) : t \in T\}$ be a nonempty family of soft general algebras over X such that $B := \bigcap_{t\in T} A_t \neq \emptyset$. Suppose that for every $x \in B$, the collection $S_x = \{F_t(x) : t \in T\}$ is directed (that is, for all $R, S \in S_x$ there is $W \in S_x$ such that $R \subseteq W$ and $S \subseteq W$). Then the restricted union $\widetilde{\sqcup}_{t\in T}(F_t, A_t)$ is a soft general algebra over X.*

Proof. Similar to the proof of Theorem 3.7 in [11]. ■

Theorem 3.6. *Let $\{(F_t, A_t) : t \in T\}$ be a nonempty family of soft general algebras over X. If for every $x = (x_t)_{t\in T} \in \prod_{t\in T} A_t$, the collection $S_x = \{F_t(x_t) : t \in T\}$ is directed, then the $\vee$-union $\widetilde{\vee}_{t\in T}(F_t, A_t)$ is a soft general algebra over X.*

Proof. Similar to the proof of Theorem 3.16 in [5]. ■

Theorem 3.7. *Let $T \neq \emptyset$ and X_t be a general algebra for all $t \in T$. If (F_t, A_t) is a soft general algebra over X_t for $t \in T$, then the cartesian product $\widetilde{\prod}_{t\in T}(F_t, A_t)$ is a soft general algebra over $\prod_{t\in T} X_t$.*

Proof. Let (F_t, A_t) be a soft general algebra over X_t for all $t \in T$. We have $\widetilde{\prod}_{t\in T}(F_t, A_t) = (G, B)$, where $B = \prod_{t\in T} A_t$ and $G(x) = \prod_{t\in T} F_t(x_t)$ for all $x = (x_t)_{t\in T} \in B$.

Let $x = (x_t)_{t\in T} \in B$. By assumption, $F_t(x_t) \in \mathrm{Sub}(X_t)$ for all $t \in T$. It is easy to see that $\prod_{t\in T} F_t(x_t) \in \mathrm{Sub}\left(\prod_{t\in T} X_t\right)$. Thus $\widetilde{\prod}_{t\in T}(F_t, A_t)$ is a soft general algebra over $\prod_{t\in T} X_t$. ■

Definition 3.8. A soft general algebra (F, A) over X is called *whole* if $F(x) = X$ for all $x \in A$.

Definition 3.9. Let X, Y be two similar general algebras and $f : X \to Y$ a mapping. If (F, A) and (G, B) are soft sets over X and Y, respectively, then $(f(F), A)$ is a soft set over Y, where $f(F) : A \to \mathcal{P}(Y)$ is defined by

$f(F)(x) = f(F(x))$ for all $x \in A$, and $(f^{-1}(G), B)$ is a soft set over X, where $f^{-1}(G) : B \to \mathcal{P}(X)$ is defined by $f^{-1}(G)(y) = f^{-1}(G(y))$ for all $y \in B$.

Theorem 3.10. *Let $f : X \to Y$ be a homomorphism of similar general algebras X and Y. Then:*

(1) *if (F, A) is a soft general algebra over X, then $(f(F), A)$ is a soft general algebra over Y;*
(2) *if (G, B) is a soft general algebra over Y, then $(f^{-1}(G), B)$ is a soft general algebra over X.*

Proof. (1) Similar to the proof of Lemma 4.13 in [8].

(2) Let $y \in B$. If $G(y) \neq \emptyset$, then $G(y)$ is a subalgebra of Y. Hence $f^{-1}(G(y))$ is a subalgebra of X. Therefore, $(f^{-1}(G), B)$ is a soft general algebra over X. ■

Theorem 3.11. *Let $f : X \to Y$ be a homomorphism of similar general algebras X and Y. Let (F, A) and (G, B) be two soft general algebras over X and Y, respectively. Then:*

(1) *if $f(X) = Y$ and (F, A) is whole, then $(f(F), A)$ is a whole soft general algebra over Y;*
(2) *if $G(y) = f(X)$ for all $y \in B$, then $(f^{-1}(G), B)$ is a whole soft general algebra over X.*

Proof. Straightforward. ■

Definition 3.12. Let (F, A) and (G, B) be two soft general algebras over X. Then (F, A) is called a *soft subalgebra* of (G, B), denoted by $(F, A) \tilde{\lessgtr} (G, B)$, if it satisfies:

(i) $A \subseteq B$,
(ii) $F(x) \in \mathrm{Sub}(G(x))$ for all $x \in A$.

Example 3.13. Let (F, A) be the soft lattice over L given in Example 3.3. Let $B = \{a, d\}$ be a subset of A and let $G : B \to \mathcal{P}(L)$ be a set-valued function defined by

$$G(x) = \{y \in L : x \vee y = 1\},$$

for all $x \in B$. Then $G(a) = \{f, 1\}$ and $G(d) = \{c, e, f, 1\}$ are sublattices of $F(a)$ and $F(d)$, respectively. Hence (G, B) is a soft sublattice of (F, A).

Theorem 3.14. *Let (F, A) and (G, B) be two soft general algebras over X. Then (F, A) is a soft subalgebra of (G, B) if and only if $A \subseteq B$ and $F(x) \subseteq G(x)$ for all $x \in A$.*

Proof. Similar to the proof of Proposition 3.5 in [14]. ■

Corollary 3.15. *Let (F, A) and (G, A) be two soft general algebras over X. Then (F, A) is a soft subalgebra of (G, A) if and only if $F(x) \subseteq G(x)$ for all $x \in A$.*

Corollary 3.16. *If (F, A) is a soft general algebra over X, then (F, A) is a soft subalgebra of (F, A).*

Theorem 3.17. *Let (F, A) be a soft general algebra over X and let $\{(G_t, A_t) : t \in T\}$ be a nonempty family of soft subalgebras of (F, A). Then:*

(1) *the extended intersection $\widetilde{\cap}_{t\in T}(G_t, A_t)$ is a soft subalgebra of (F, A);*

(2) *if $\bigcap_{t\in T} A_t \neq \emptyset$, then the restricted intersection $\widetilde{\sqcap}_{t\in T}(G_t, A_t)$ is a soft subalgebra of (F, A);*

(3) *the $\wedge$-intersection $\widetilde{\wedge}_{t\in T}(G_t, A_t)$ is a soft subalgebra of $\widetilde{\wedge}_{t\in T}(F, A)$.*

Proof. (1) Let $B = \bigcup_{t\in T} A_t$ and $G(x) = \bigcap_{t\in T(x)} G_t(x)$ for all $x \in B$, where $T(x) = \{t \in T : x \in A_t\}$, that is, $\widetilde{\cap}_{t\in T}(G_t, A_t) = (G, B)$. From Theorem 3.4 (1), we see that (G, B) is a soft general algebra over X. We have $B \subseteq A$, because $A_t \subseteq A$ for all $t \in T$. Let $x \in B$. Then $x \in A_t$ for some $t \in T$. Since $G_t(x) \subseteq F(x)$ for $t \in T(x)$, we conclude that $G(x) \subseteq F(x)$. By Theorem 3.14, $(G, B) \,\widetilde{\leqslant}\, (F, A)$, that is, $\widetilde{\cap}_{t\in T}(G_t, A_t)$ is a soft subalgebra of (F, A).

(2) Similar to the proof of Theorem 3.7 (3) in [6].

(3) By Theorem 3.4 (3), it follows that $\widetilde{\wedge}_{t\in T}(G_t, A_t)$ and $\widetilde{\wedge}_{t\in T}(F, A)$ are soft general algebras over X. Let $B = \prod_{t\in T} A_t$, $\widetilde{\wedge}_{t\in T}(G_t, A_t) = (G, B)$, and $\widetilde{\wedge}_{t\in T}(F, A) = (H, A^T)$. Then $G(x) = \bigcap_{t\in T} G_t(x_t)$ and $H(x) = \bigcap_{t\in T} F(x_t)$ for all $x = (x_t)_{t\in T} \in B$. Obviously, $B \subseteq A^T$.

Let $x = (x_t)_{t\in T}$, where $x_t \in A_t$. Since $(G_t, A_t) \,\widetilde{\leqslant}\, (F, A)$, we get $G_t(x_t) \subseteq F(x_t)$. Hence

$$G(x) = \bigcap_{t\in T} G_t(x_t) \subseteq \bigcap_{t\in T} F(x_t) = H(x),$$

that is, $G(x) \subseteq H(x)$ for all $x \in B$. By Theorem 3.14, $(G, B) \,\widetilde{\leqslant}\, (H, A^T)$. Thus $\widetilde{\wedge}_{t\in T}(G_t, A_t)$ is a soft subalgebra of $\widetilde{\wedge}_{t\in T}(F, A)$. ∎

Theorem 3.18. *Let (F, A) be a soft general algebra over X and let $\{(G_t, A_t) : t \in T\}$ be a nonempty family of soft subalgebras of (F, A) such that $B := \bigcap_{t\in T} A_t \neq \emptyset$. If for every $x \in B$, the collection $S_x = \{G_t(x) : t \in T\}$ is directed, then the restricted union $\widetilde{\sqcup}_{t\in T}(G_t, A_t)$ is a soft subalgebra of (F, A).*

Proof. We can write $\widetilde{\sqcup}_{t\in T}(G_t, A_t) = (G, B)$, where $G(x) = \bigcup_{t\in T} G_t(x)$ for all $x \in B$. From Theorem 3.5, we see that (G, B) is a soft general algebra over X. Let $t \in T$. By assumption, $G_t(x) \subseteq F(x)$ for all $x \in A_t$. Hence $G(x) = \bigcup_{t\in T} G_t(x) \subseteq F(x)$ for all $x \in B$. Theorem 3.14 now shows that $(G, B) \,\widetilde{\leqslant}\, (F, A)$, that is, $\widetilde{\sqcup}_{t\in T}(G_t, A_t)$ is a soft subalgebra of (F, A). ∎

Theorem 3.19. *Let (F,A) be a soft general algebra over X and let $\{(G_t,A_t):t\in T\}$ be a nonempty family of soft subalgebras of (F,A). If for every $x=(x_t)_{t\in T}\in\prod_{t\in T}A_t$, the collection $S_x=\{F_t(x_t):t\in T\}$ is directed, then the $\vee$-union $\widetilde{\vee}_{t\in T}(G_t,A_t)$ is a soft subalgebra of $\widetilde{\vee}_{t\in T}(F,A)$.*

Proof. Similar to the proof of Theorem 3.17 (3) (use Theorem 3.6). ∎

Theorem 3.20. *Let (F,A) be a soft general algebra over X. If $\{(G_t,A_t): t\in T\}$ is a nonempty family of soft subalgebras of (F,A), then the cartesian product $\widetilde{\prod}_{t\in T}(G_t,A_t)$ is a soft subalgebra of $\widetilde{\prod}_{t\in T}(F,A)$.*

Proof. Similar to the proof of Theorem 3.17 (3) (apply Theorem 3.7). ∎

Theorem 3.21. *Let $f:X\to Y$ be a homomorphism of similar general algebras X and Y. Let (F,A) and (G,B) be two soft general algebras over X. If $(F,A)\widetilde{\lessgtr}(G,B)$ then $(f(F),A)\widetilde{\lessgtr}(f(G),B)$.*

Proof. Similar to the proof of Theorem 4.19 in [8]. ∎

Theorem 3.22. *Let $f:X\to Y$ be a homomorphism of similar general algebras X and Y. Let (F,A) and (G,B) be two soft general algebras over Y. If $(F,A)\widetilde{\lessgtr}(G,B)$, then $(f^{-1}(F),A)\widetilde{\lessgtr}(f^{-1}(G),B)$.*

Proof. Let $(F,A)\widetilde{\lessgtr}(G,B)$. Then $A\subseteq B$ and $F(y)\subseteq G(y)$ for all $y\in A$. By Theorem 3.10, $(f^{-1}(F),A)$ and $(f^{-1}(G),B)$ are soft general algebras over X. We obtain $f^{-1}(F)(y)=f^{-1}(F(y))\subseteq f^{-1}(G(y))=f^{-1}(G)(y)$ for all $y\in A$. Thus $(f^{-1}(F),A)\widetilde{\lessgtr}(f^{-1}(G),B)$. ∎

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