DEMONSTRATIO MATHEMATICA
Vol. XLVI No 4 2013

10.1515/dema-2013-0489

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COMMON FIXED POINT THEOREMS UNDER CONTRACTIVE CONDITIONS OF INTEGRAL TYPE IN SYMMETRIC SPACES

Abstract. The purpose of this paper is to prove common fixed point theorems for a family of mappings in symmetric spaces using the property (E.A) and weak compatibility or occasionally weak compatibility. Our results extend some recent results.

1. Introduction and preliminaries

In this paper, we research some fixed point results in symmetric (or semi-metric) spaces. We begin by recalling some definitions.

A *symmetric* d on a set X is a nonnegative real valued function on $X \times X$ such that:

(i) $d\,(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$ and the pair (X, d) is called a *symmetric (or semi-metric) space.*

In the following, unless otherwise indicated, (X, d) is a symmetric space. Since d does not satisfy the triangular inequality, some supplementary axioms are often used. The following properties were defined by Wilson [33], Aamri and El Moutawakil [2] and Pathak, Tiwari and Khan [25].

Definition 1.1. [33]

(W.3): Given (x_n), x and y in X, if $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d\,(x_n, y) = 0$, then $x = y$.
(W.4): Given (x_n), (y_n) and x in X, if $\lim_{n\to\infty} d\,(x_n, x) = 0$ and $\lim_{n\to\infty} d\,(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(y_n, x) = 0$.

It is clear that (W.4) implies (W.3).

2010 *Mathematics Subject Classification*: 47H10, 54H25.
Key words and phrases: metric space, symmetric (semi-metric) space, weakly compatible mapping, occasionally weakly compatible mapping, common fixed point.

Definition 1.2. [2] (X,d) satisfies *property* (H.E) if and only if, given $(x_n),(y_n)$ and x in X, if $\lim_{n\to\infty} d(x_n,x)=0$ and $\lim_{n\to\infty} d(y_n,x)=0$, then $\lim_{n\to\infty} d(x_n,y_n)=0$.

Definition 1.3. [25] (X,d) satisfies *property* (CE.1) if and only if, given (x_n), x and y in X, $\lim_{n\to\infty} d(x_n,x)=0$ implies $\lim_{n\to\infty} d(x_n,y)=d(x,y)$. (X,d) satisfies *property* (CE.2) if and only if, given (x_n), (y_n) and (z_n) in X, $\lim_{n\to\infty} d(x_n,y_n)=0$ implies $\limsup_{n\to\to\infty} d(z_n,y_n)=\limsup_{n\to\infty} d(z_n,x_n)$.

On a symmetric space, d defines classically a topology called $t(d)$ for which $x_n\to x$ if and only if $\lim_{n\to\infty} d(x_n,x)=0$. So the notions of different compatibility or commutativity which are defined in metric spaces do not require extension or new definition in symmetric space. We recalled these which will be used in this paper.

Definition 1.4. [1] Let S and $T: X\to X$. The pair (S,T) satisfies *property* (E.A) if there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n=t\in X. \tag{1.1}$$

The property was defined before by Sastry and Murthy (2000) (see [29]) as the *tangential property.*

Definition 1.5. [20] S and T are said to be *R-weakly commuting* if there exists an $R>0$ such that, for every $x\in X$,

$$d(STx,TSx)\le Rd(Tx,Sx). \tag{$*$}$$

Definition 1.6. [15] S and T are said to be *compatible* if

$$\lim_{n\to\infty} d(STx_n,TSx_n)=0$$

whenever (x_n) is a sequence in X such that $\lim\limits_{n\to\infty} Sx_n=\lim\limits_{n\to\infty} Tx_n=t$ for some $t\in X$.

It is easy to show that weakly commuting implies compatible and there are examples in the literature showing that the inclusions are proper, see [15] and [30]. It is clear, from the definition of compatibility, that the pair (S,T) of a metric space (X,d) is noncompatible if there exists at least one sequence (x_n) in X such that (1.1) holds but, $\lim_{n\to\infty} d(STx_n,TSx_n)$ is either nonzero or does not exist. Therefore, a pair of two noncompatible mappings satisfies property (E.A), just like a pair of compatible mappings.

Definition 1.7. [15] S and T are said to be *weakly compatible* if they commute at their coincidence points; i.e.,

$$\{x\in X: Sx=Tx\}\subseteq\{x\in X: STx=TSx\}. \tag{$**$}$$

If S and T are compatible, then they are weakly compatible. The converse is not true in general (cf. [9]).

Definition 1.8. [3] Let S and T be self maps of a metric space X. Then S and T are called *occasionally weakly compatible* if

$$\{x \in X : Sx = Tx\} \cap \{x \in X : STx = TSx\} \neq \emptyset.$$

Remark 1.9. If the set $C(f,g)$ of coincidence points of f and g is empty, the pair (f,g) is trivially weakly compatible; but this situation is without interest for the research of common fixed points. If $C(f,g) \neq \emptyset$, the pair (f,g) is nontrivially weakly compatible and, with many authors, shortly called weakly compatible.

The following example shows that the nontrivially weakly compatible selfmaps form a proper subclass of the occasionally weakly compatible selfmaps. (See also [3]).

Example 1. Let $X = [0,4]$ with the usual metric, and let A and S be self mappings of X such that

$$Ax = \begin{cases} 2, & \text{if } x \in [0,2], \\ 1, & \text{if } x \in]2,4], \end{cases} \qquad Sx = \begin{cases} x, & \text{if } x \in [0,2], \\ 1, & \text{if } x \in]2,4]. \end{cases}$$

It is easy to see that $C(A,S) = [2,4]$, $AS2 = SA2$, but, for $x \in]2,4]$, we have $ASx = A(1) = 2 \neq SAx = S1 = 1$.

Therefore A and S are occasionally weakly compatible maps but not weakly compatible.

Definition 1.10. [21] S and T are *pointwise R-weakly commuting* if for every $x \in X$, there exists an $R > 0$ such that $(*)$ holds.

It was proved in [21] that R-weakly commutativity is equivalent to commutativity at coincidence points; i.e., S and T are pointwise R-weakly commuting if and only if they are weakly compatible.

Properties of weak compatibility or occasionally weak compatibility and property (E.A) are independent, as it is shown by the two following examples.

Example 2. Let $\mathcal{X} = [0,\infty[$ and f and g be two applications defined by:

$$f(x) = \begin{cases} 2-x, & \text{if } x \in [0,2[, \\ 4, & \text{if } x = 2, \\ 3, & \text{if } x \in]2,\infty[, \end{cases} \qquad g(x) = \begin{cases} 2+x, & \text{if } x \in [0,2[, \\ 0, & \text{if } x \in [2,\infty[. \end{cases}$$

$C(f,g) = \{0\}$. $f(0) = g(0) = 2$ but $fg(0) = f(2) = 4$ and $gf(0) = g(2) = 0$. So, f and g are not weakly compatible and not occasionally weakly compatible. (f,g) satisfy property (E.A) since, with $x_n = \frac{1}{n}$, $f(x_n) = 2 - \frac{1}{n} \to 2$ and $g(x_n) = 2 + \frac{1}{n} \to 2$. That is, property (E.A) does not imply occasionally weak compatibility.

Example 3. Let $\mathcal{X} = [0, \infty[$ and h and k be two applications defined by:

$$h(x) = \begin{cases} 2, & \text{if } x = 0, \\ 1 - \frac{1}{2}x, & \text{if } x \in \,]0, 2[, \\ 4, & \text{if } x = 2, \\ 3, & \text{if } x \in \,]2, \infty[, \end{cases} \qquad k(x) = \begin{cases} 2 + x, & \text{if } x \in [0, 2], \\ 0, & \text{if } x \in \,]2, \infty[. \end{cases}$$

$C(h, k) = \{0, 2\}$, $h(0) = k(0) = 2$ and $hk(0 = kh(0) = 4$. But, $h(2) = k(2) = 4$, $kh(2) = k(4) = 0$ and $hk(2) = h(4) = 3$. So, h and k are occasionally weakly compatible and are not weakly compatible.

As for property (E.A), it is easy to see that, for $x \in \,]0, 2[\cup]2, \infty[$, there is no sequence (x_n) such that $x_n \to x$, that satisfy $\lim h(x_n) = \lim k(x_n)$. If $x_n \to 0$, $\lim h(x_n) = 1$ and $\lim k(x_n) = 2$. If $x_n \to 2$ from below, $\lim h(x_n) = 0$ and $\lim k(x_n) = 4$. If $x_n \to 2$ from above, $\lim h(x_n) = 3$ and $\lim k(x_n) = 0$. And if $x_n \to 2$ on both sides of 2, there is no limit for $(h(x_n))$ and also for $(k(x_n))$. Then, (h, k) do not have property (E.A) and this example shows that occasionally weakly compatibility does not imply property (E.A).

Recently, Pathak et al. [24] proved the following common fixed point theorem in metric spaces.

Theorem 1.11. [24] *Let A, B, S and T be self-maps of a metric space (X, d) satisfying*

(i) $$A(X) \subseteq T(X), \quad B(X) \subseteq S(X),$$

(ii) *there exists a continuous function: $F : \mathbb{R}^6_+ \to \mathbb{R}$ such that*

$$F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(By, Sx), d(Ax, Ty)) \leq 0,$$

for all $x, y \in X$, where F satisfies the following conditions:
(F_1): $F(u, 0, u, 0, 0, u) \leq 0 \Rightarrow u = 0$,
(F_2): $F(u, 0, 0, u, u, 0) \leq 0 \Rightarrow u = 0$,
(F_3): $\forall u > 0$ $F(u, u, 0, 0, u, u) \geq 0$,

(iii) *(A, S) and (B, T) are weakly compatible self-mappings of (X, d),*
(iv) *(A, S) or (B, T) satisfies the property* (E.A).
Assuming that one the following conditions holds:
(v) *$\{By_n\}$ is a bounded sequence for every $\{y_n\} \subseteq X$ such that $\{Ty_n\}$ is convergent (in case (A, S) satisfies the property* (E.A)*), and $\{Ay_n\}$ is a bounded sequence for every $\{y_n\} \subseteq X$ such that $\{Sy_n\}$ is convergent (in case (B, T) satisfies the property* (E.A)*),*
(vi) *if $\{z_n\}, \{r_n\}$ and $\{w_n\}$ are nonnegative sequences such that $\{z_n\} \to \infty$, $\{w_n\} \to \infty$, as $n \to \infty$ and $F(z_n, r_n, r_n, z_n, w_n, 0) \leq 0, n \in N$, (in*

case (A, S) *satisfies* (E.A)), $F(z_n, r_n, z_n, r_n, 0, w_n) \leq 0, n \in N$, (*in case* (B, T) *satisfies* (E.A)), *then* $\{r_n\} \to \infty$, *as* $n \to \infty$.

If the range of one of the mappings is a complete subspace of X, *then* A, B, S *and* T *have a unique common fixed point.*

2. Main results

In this part, we give four results which extend the previous theorem and the others. We begin with the definition of implicit relations.

Let $\mathcal{F}$ be the set of all continuous functions $F : \mathbb{R}_+^6 \longrightarrow \mathbb{R}$ satisfying the following conditions:

(φ_1): $F\left(\int_0^u \varphi(t)\,dt, \int_0^u \varphi(t)\,dt, 0, 0, \int_0^u \varphi(t)\,dt, \int_0^u \varphi(t)\,dt\right) > 0$ for all $u > 0$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally integrable function which satisfies

$$\int_0^\varepsilon \varphi(t)\,dt > 0, \forall \varepsilon > 0,$$

(φ_2): there exists $0 < \alpha < 1$ such that for all $u, v \geq 0$, if (F_a) or (F_b) is satisfied,

$$(F_a)\quad F\left(\int_0^u \varphi(t)\,dt, \int_0^v \varphi(t)\,dt, \int_0^u \varphi(t)\,dt, \int_0^v \varphi(t)\,dt, \int_0^u \varphi(t)\,dt + \int_0^v \varphi(t)\,dt, 0\right) \leq 0,$$

$$(F_b)\quad F\left(\int_0^u \varphi(t)\,dt, \int_0^v \varphi(t)\,dt, \int_0^v \varphi(t)\,dt, \int_0^u \varphi(t)\,dt, 0, \int_0^u \varphi(t)\,dt + \int_0^v \varphi(t)\,dt\right) \leq 0,$$

we have $\int_0^u \varphi(t)\,dt \leq \alpha \int_0^v \varphi(t)\,dt$.

As an example, we can give: $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - c\max\{t_2, t_3, t_4, \frac{1}{2}[t_5 + t_6]\}$, with $c \in [0, 1[$ and $\varphi(t) = 3t^2$.

*(φ_1) $\int_0^u \varphi(t)dt = u^3$. So, (φ_1) is satisfied since $F(u^3, u^3, 0, 0, u^3, u^3) = (1 - c)u^3$.

*(φ_2) For (F_a) let $u, v \geq 0$ and

$$F\left(\int_0^u \varphi(t)\,dt, \int_0^v \varphi(t)\,dt, \int_0^u \varphi(t)\,dt, \int_0^v \varphi(t)\,dt, \int_0^u \varphi(t)\,dt + \int_0^v \varphi(t)\,dt, 0\right)$$
$$= u^3 - c\max\{u^3, v^3, \tfrac{1}{2}(u^3 + v^3)\}$$

which is ≤ 0 only if $u \leq v$ and then it is equal to $u^3 - cv^3$. So (φ_2) is satisfied with $\alpha = c$.

Similarly we show (F_b).

We can give also:

* $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - c\max\{t_2, t_3, t_4, t_5, t_6\}$, with $c \in [0, 1[$ and $\varphi(t) = 3t^2$.

* $F(t_1,t_2,t_3,t_4,t_5,t_6) = \min\{t_1,t_2\} - c\max\{t_3,t_4,t_5,t_6\}$, with $c \in [0,1[$ and $\varphi(t) = 3t^2$.
* $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - c\max\{t_2,t_3,t_4,t_5,t_6\}$, with $c \in [0,1[$ and $\varphi(t) = 1/(1+t)$.

Now, we give the first common fixed point result for a family of maps. In this and in the following theorems, I is an arbitrary set.

Theorem 2.1. *Let d be a symmetric on X which satisfies* (W_4), (HE), (CE_1) *and* (CE_2), *and let $(A_i)_{i\in I}$, A, S and T be self-mappings of (X,d) satisfying*

$$AX \subset TX, \text{ and } A_iX \subset SX \text{ for every } i \in I \tag{1.2}$$

and

$$F\left(\int_0^{d(Ax,A_iy)} \varphi(t)\,dt, \int_0^{d(Sx,Ty)} \varphi(t)\,dt, \int_0^{d(Ax,Sx)} \varphi(t)\,dt, \int_0^{d(A_iy,Ty)} \varphi(t)\,dt, \int_0^{d(Ax,Ty)} \varphi(t)\,dt, \int_0^{d(Sx,A_iy)} \varphi(t)\,dt\right) \le 0, \tag{1.3}$$

for every $i \in I$, for all x and y in X, where $F \in \mathcal{F}$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally integrable function which satisfies $\int_0^{\varepsilon} \varphi(t)\,dt > 0$, $\forall \varepsilon > 0$.

Suppose that:

(i) *(A,S) satisfies the property* (E.A),
(ii) *(A,S) and, for some $k \in I$, (A_k,T) are weakly compatible.*

If one of the subspaces AX, SX, A_iX and TX of X is closed, then A, S, T and A_i, for all $i \in I$, have a unique common fixed point in X.

Proof. Since (A,S) satisfies the property (E.A), there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} d(Ax_n,z) = \lim_{n\to\infty} d(Sx_n,z) = 0$, for some z in X.

By the property (HE), we have

$$\lim_{n\to\infty} d(Ax_n,Sx_n) = 0. \tag{1.4}$$

As $AX \subset TX$, there exists a sequence (y_n) in X such that $Ax_n = Ty_n$, for all $n \in \mathbb{N}$. By (CE_2) and (1.4), we have, for every $i \in I$,

$$\limsup_{n\to\infty} d(Ax_n, A_iy_n) = \limsup_{n\to\infty} d(Sx_n, A_iy_n).$$

By (CE_2), we have, for every $i \in I$,

$$\limsup_{n\to\infty} d(Ax_n, A_iy_n) = \limsup_{n\to\infty} d(A_iy_n, Ty_n).$$

Now, we show that for every $i \in I$, $\lim_{n\to\infty} A_iy_n = z$. Assume that there exists $j \in I$ such that $\limsup_{\infty} d(Ax_n, A_j y_n) = \alpha_j \neq 0$. Using (1.3) with $x = x_n$ and $y = y_n$, we get

$$F\Big(\int_0^{d(Ax_n,A_jy_n)} \varphi(t)\,dt, \int_0^{d(Sx_n,Ty_n)} \varphi(t)\,dt, \int_0^{d(Ax_n,Sx_n)} \varphi(t)\,dt, \int_0^{d(A_jy_n,Ty_n)} \varphi(t)\,dt, \int_0^{d(Ax_n,Ty_n)} \varphi(t)\,dt, \int_0^{d(Sx_n,A_jy_n)} \varphi(t)\,dt\Big) \leq 0.$$

Letting $n \to \infty$, we find

$$F\Big(\int_0^{\alpha_j} \varphi(t)\,dt, 0, 0, \int_0^{\alpha_j} \varphi(t)\,dt, 0, \int_0^{\alpha_j} \varphi(t)\,dt\Big) \leq 0.$$

Thanks to (F_b) of (φ_2), $\alpha_j = 0$. So, for all $i \in I$, $\limsup_{n\to\infty} d(Ax_n, A_i y_n) = 0$ thus, $\lim_{n\to\infty} d(Ax_n, A_i y_n) = 0$.

By (W_4), since $\lim_{n\to\infty} d(Ax_n, z) = 0$ and $\lim_{n\to\infty} d(Ax_n, A_i y_n) = 0$, we obtain that $\lim_{n\to\infty} d(A_i y_n, z) = 0$, i.e. $\lim_{n\to\infty} A_i y_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for all $i \in I$. Suppose that $T(X)$ is closed. Then, $z = Tu$ for some $u \in X$. If there exists j such that $A_j u \neq z$, by (1.3) with $x = x_n$ and $y = u$, we get

$$F\Big(\int_0^{d(Ax_n,A_ju)} \varphi(t)\,dt, \int_0^{d(Sx_n,Tu)} \varphi(t)\,dt, \int_0^{d(Ax_n,Sx_n)} \varphi(t)\,dt, \int_0^{d(A_ju,Tu)} \varphi(t)\,dt, \int_0^{d(Ax_n,Tu)} \varphi(t)\,dt, \int_0^{d(Sx_n,A_ju)} \varphi(t)\,dt\Big) \leq 0.$$

Taking limit as $n \to \infty$ and using (CE_2), we have:

$$F\Big(\int_0^{d(A_ju,z)} \varphi(t)\,dt, 0, 0, \int_0^{d(A_ju,z)} \varphi(t)\,dt, 0, \int_0^{d(A_ju,z)} \varphi(t)\,dt\Big) \leq 0$$

which implies by (CE_2) and (F_b) $A_j u = z$ and, for all $i \in I$, $A_i u = Tu = z$. As $A_i X \subset SX$, there exists $v \in X$ such that $z = A_i u = Sv$.

If $Av \neq z$, applying (1.3), we obtain

$$F\Big(\int_0^{d(Av,A_iu)} \varphi(t)dt, \int_0^{d(Sv,Tu)} \varphi(t)dt, \int_0^{d(Av,Sv)} \varphi(t)dt, \int_0^{d(A_iu,Tu)} \varphi(t)dt, \int_0^{d(Av,Tu)} \varphi(t)dt, \int_0^{d(Sv,A_iu)} \varphi(t)dt\Big) \leq 0$$

and, by taking limit as $n \to \infty$

$$F\left(\int_0^{d(Av,z)}\varphi(t)\,dt,0,\int_0^{d(Av,z)}\varphi(t)\,dt,0,\int_0^{d(Av,z)}\varphi(t)\,dt,0\right)\leq 0$$

which implies that $Av = Sv = z$ by using the condition (F_a). Since the pair (A, S) is weakly compatible, it follows that $Az = Sz$.

If $z \neq Az$, using (1.3), we get

$$F\left(\int_0^{d(Az,A_iv)}\varphi(t)\,dt,\int_0^{d(Sz,Tv)}\varphi(t)\,dt,\int_0^{d(Az,Sz)}\varphi(t)\,dt,\right.$$
$$\left.\int_0^{d(A_iv,Tv)}\varphi(t)\,dt,\int_0^{d(Az,Tv)}\varphi(t)\,dt,\int_0^{d(Sz,A_iv)}\varphi(t)\,dt\right)$$
$$=F\left(\int_0^{d(Az,z)}\varphi(t)\,dt,\int_0^{d(Az,z)}\varphi(t)\,dt,0,0,\int_0^{d(Az,z)}\varphi(t)\,dt,\int_0^{d(Az,z)}\varphi(t)\,dt\right)\leq 0$$

which is a contradiction of (φ_1). So $Az = z$. By the weak compatibility of A_k and T, we have $A_kz = ATu = TA_ku = Tz$, and applying (1.3) with $x = y = z$, we get

$$F\left(\int_0^{d(A_kz,z)}\varphi(t)\,dt,\int_0^{d(A_kz,z)}\varphi(t)\,dt,0,0,\int_0^{d(A_kz,z)}\varphi(t)\,dt,\int_0^{d(A_kz,z)}\varphi(t)\,dt\right)\leq 0$$

which is a contradiction of (φ_1) and so $A_kz = Tz = z$.

Also, for every i, we have

$$F\left(\int_0^{d(Az,A_iz)}\varphi(t)\,dt,\int_0^{d(Sz,Tz)}\varphi(t)\,dt,\int_0^{d(Az,A_iz)}\varphi(t)\,dt,\right.$$
$$\left.\int_0^{d(A_iz,Tz)}\varphi(t)\,dt,\int_0^{d(Az,Tz)}\varphi(t)\,dt,\int_0^{d(Sz,A_iz)}\varphi(t)\,dt\right)$$
$$=F\left(\int_0^{d(z,A_iz)}\varphi(t)\,dt,0,0,\int_0^{d(z,A_iz)}\varphi(t)\,dt,0,\int_0^{d(z,A_iz)}\varphi(t)\,dt\right)\leq 0.$$

Thanks to (F_b), we obtain $A_iz = z$ for $i \in I$ and z is a common fixed point of A, S, T and A_i, for every i.

Now, we show the unicity of the common fixed point. If $\bar{z}$ is another common fixed point, from (1.3) appling to $x = \bar{z}$ and $y = z$, we get:

$$F\left(\int_0^{d(A\bar{z},A_iz)}\varphi(t)dt,\int_0^{d(S\bar{z},Tz)}\varphi(t)dt,\int_0^{d(A\bar{z},S\bar{z})}\varphi(t)dt,\right.$$
$$\left.\int_0^{d(A_iz,Tz)}\varphi(t)dt,\int_0^{d(A\bar{z},Tz)}\varphi(t)dt,\int_0^{d(S\bar{z},A_iz)}\varphi(t)dt\right)$$

$$= F\Bigg(\int_0^{d(\bar{z},z)} \varphi(t)dt, \int_0^{d(\bar{z},z)} \varphi(t)dt, \int_0^{d(\bar{z},\bar{z})} \varphi(t)dt, \int_0^{d(z,z)} \varphi(t)dt, \int_0^{d(\bar{z},z)} \varphi(t)dt, \int_0^{d(\bar{z},z)} \varphi(t)dt\Bigg) \leq 0.$$

From (φ_1), we obtain $\bar{z} = z$ and z is the unique common fixed point of A, S, T and A_i, for every i.

When $S(X)$ is assumed to be closed subspace of X, then the proof is similar. On the other hand, the cases in which $A(X)$ or $A_i(X)$ is a closed subspace of X are, respectively, similar to the cases in which $T(X)$ or $S(X)$ is closed. ∎

In the following theorem, we change the hypothesis (i). The methods of its proof are similar except the necessity of many sequences indexed by $i \in I$.

Theorem 2.2. *Let d be a symmetric on X which satisfies* (W$_4$), (HE), (CE$_1$) *and* (CE$_2$) *and let $(A_i)_{i\in I}$, A, S and T be self-mappings of (X,d) satisfying*

$$AX \subset TX, \text{ and } A_iX \subset SX \text{ for every } i \in I \tag{1.2}$$

and

$$F\Bigg(\int_0^{d(Ax,A_iy)} \varphi(t)\,dt, \int_0^{d(Sx,Ty)} \varphi(t)\,dt, \int_0^{d(Ax,Sx)} \varphi(t)\,dt, \int_0^{d(A_iy,Ty)} \varphi(t)\,dt, \int_0^{d(Ax,Ty)} \varphi(t)\,dt, \int_0^{d(Sx,A_iy)} \varphi(t)\,dt\Bigg) \leq 0, \tag{1.3}$$

for every $i \in I$, for all x and y in X, where $F \in \mathcal{F}$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally integrable function which satisfies $\int_0^{\varepsilon} \varphi(t)dt > 0$, $\forall \varepsilon > 0$.

Suppose that:

(i) *for every i, (A_i,T) satisfies the property* (E.A),
(ii) *(A,S) and, for some $k \in I$, (A_k,T) are weakly compatible.*

If one of the subspaces AX, SX, A_iX and TX of X is closed, then A, S, T and A_i, for all $i \in I$, have a unique common fixed point in X.

Proof. Since (A_i,T) satisfies the property (E.A), there exists a sequence $(x_n^i)_n$ in X such that $\lim_{n\to\infty} d(A_ix_n^i, z_i) = \lim_{n\to\infty} d(Tx_n^i, z_i) = 0$, for some z_i in X.

By property (HE), we have

$$\lim_{n\to\infty} d\left(A_ix_n^i, Tx_n^i\right) = 0. \tag{1.4}$$

Since $A_iX \subset SX$, there exists a sequence $(y_n^i)_n$ in X such that $A_ix_n^i = Sy_n^i$, for all $n \in \mathbb{N}$ and all i. Furthermore, we get

$$\forall i > 1 \qquad \lim_{n\to\infty} d\left(A_ix_n^i, Sy_n^i\right) = 0. \tag{1.5}$$

By (CE$_2$) and (1.4), we have

$$\limsup_{n\to\infty} d(Ay_n^i, Tx_n^i) = \limsup_{n\to\infty} d(Ay_n^i, A_ix_n^i).$$

Now, we show that $\lim_{n\to\infty} Ay_n^i = z_i$. We denote $\limsup_{n\to\infty} d(Ay_n^i, A_ix_n^i) = \alpha_i$. Using (1.3) with $x = y_n^i$ and $y = x_n^i$, we get

$$F\Bigg(\int_0^{d(Ay_n^i, A_ix_n^i)} \varphi(t)dt, \int_0^{d(Sy_n^i, Tx_n^i)} \varphi(t)dt, \int_0^{d(Ay_n^i, Sy_n^i,)} \varphi(t)dt,$$
$$\int_0^{d(A_ix_n^i, Tx_n^i)} \varphi(t)dt, \int_0^{d(Ay_n^i, Tx_n^i)} \varphi(t)dt, \int_0^{d(Sy_n^i, A_ix_n^i)} \varphi(t)dt\Bigg) \leq 0.$$

Letting $n \to \infty$, we find

$$F\left(\int_0^{\alpha_i} \varphi(t)\,dt, 0, \int_0^{\alpha_i} \varphi(t)\,dt, 0, \int_0^{\alpha_i} \varphi(t)\,dt, 0\right) \leq 0.$$

By (F_a), $\alpha_i = 0$, i.e. $\limsup_{n\to\infty} d(Ay_n^i, A_ix_n^i) = 0$, thus, $\lim_{n\to\infty} d(Ay_n^i, A_ix_n^i) = 0$.

By (W$_4$), since $\lim_{n\to\infty} d(A_ix_n^i, z_i) = 0$ and $\lim_{n\to\infty} d\left(Ay_n^i, A_ix_n^i\right) = 0$, we have $\lim_{n\to\infty} d(Ay_n^i, z_i) = 0$, i.e.

$$\forall i \qquad \lim_{n\to\infty} A_ix_n^i = \lim_{n\to\infty} Tx_n^i = \lim_{n\to\infty} Ay_n^i = \lim_{n\to\infty} Sy_n^i = z_i.$$

Suppose that $S(X)$ is closed. Then, $z_i \in S(X)$ and there exists $u_i \in X$ such that $z_i = Su_i$. By (1.3), we get

$$F\Bigg(\int_0^{d(Au_i, A_ix_n^i)} \varphi(t)dt, \int_0^{d(Su_i, Tx_n^i)} \varphi(t)dt, \int_0^{d(Au_i, Su_i)} \varphi(t)dt,$$
$$\int_0^{d(A_ix_n^i, Tx_n^i)} \varphi(t)dt, \int_0^{d(Au_i, Tx_n^i)} \varphi(t)dt, \int_0^{d(Su_i, A_ix_n^i)} \varphi(t)dt\Bigg) \leq 0.$$

Taking limit as $n \to \infty$ and using (CE$_1$), we get:

$$F\left(\int_0^{d(Au_i, z_i)} \varphi(t)dt, 0, \int_0^{d(Au_i, z_i)} \varphi(t)dt, 0, \int_0^{d(Au_i, z_i)} \varphi(t)dt, 0\right) \leq 0$$

which implies, by (CE$_1$) and (F_a), $Au_i = Su_i = z_i$. As $AX \subset TX$, there exists $v_i \in X$ such that $z_i = Au_i = Tv_i$. Applying (1.3), we have

$$F\Big(\int_0^{d(Au_i,A_iv_i)} \varphi(t)dt, \int_0^{d(Su_i,Tv_i)} \varphi(t)dt, \int_0^{d(Au_i,Su_i)} \varphi(t)dt, \int_0^{d(A_iv_i,Tv_i)} \varphi(t)dt, \int_0^{d(Au_i,Tv_i)} \varphi(t)dt, \int_0^{d(Su_i,A_iv_i)} \varphi(t)dt\Big)$$
$$= F\Big(\int_0^{d(z_i,A_iv_i)} \varphi(t)dt, 0, 0, \int_0^{d(A_iv_i,z_i)} \varphi(t)dt, 0, \int_0^{d(z_i,A_iv_i)} \varphi(t)dt\Big) \le 0,$$

which implies that $A_iv_i = Tv_i = z_i$ by using the condition (F_b). Since the pair (A, S) is weakly compatible, $Az_i = Sz_i$. Using (1.3) with $x = z_i$ and $y = v_i$, we get

$$F\Big(\int_0^{d(Az_i,A_iv_i)} \varphi(t)dt, \int_0^{d(Sz_i,Tv_i)} \varphi(t)dt, \int_0^{d(Az_i,Sz_i)} \varphi(t)dt, \int_0^{d(A_iv_i,Tv_i)} \varphi(t)dt, \int_0^{d(Az_i,Tv_i)} \varphi(t)dt, \int_0^{d(Sz_i,A_iv_i)} \varphi(t)dt\Big)$$
$$= F\Big(\int_0^{d(Az_i,z_i)} \varphi(t)dt, \int_0^{d(Az_i,z_i)} \varphi(t)dt, 0, 0, \int_0^{d(Az_i,z_i)} \varphi(t)dt, \int_0^{d(Az_i,z_i)} \varphi(t)dt\Big) \le 0.$$

So by (φ_1),

$$z_i = Az_i = Sz_i, \text{ for every } i \in I. \tag{1.6}$$

By the weak compatibility of A_k and T, we obtain $A_kz_k = Tz_k$ and applying (1.3) with $x = y = z_k$, we have

$$F\Big(\int_0^{d(Az_k,A_kz_k)} \varphi(t)dt, \int_0^{d(Sz_k,Tz_k)} \varphi(t)dt, \int_0^{d(Az_k,Sz_k)} \varphi(t)dt, \int_0^{d(A_kz_k,Tz_k)} \varphi(t)dt, \int_0^{d(Az_k,Tz_k)} \varphi(t)dt, \int_0^{d(Sz_k,A_kz_k)} \varphi(t)dt\Big)$$
$$= F\Big(\int_0^{d(z_k,A_kz_k)} \varphi(t)dt, \int_0^{d(z_k,A_kz_k)} \varphi(t)dt, 0, 0, \int_0^{d(z_k,A_kz_k)} \varphi(t)dt, \int_0^{d(z_k,A_kz_k)} \varphi(t)dt\Big) \le 0.$$

From (φ_1) and (1.6), it follows $A_kz_k = Tz_k = Sz_k = z_k$. So, z_k is a common fixed point of A, S, T and A_k. But, for every i, we have also:

$$F\Big(\int_0^{d(Az_k,A_iz_k)}\varphi(t)dt,\int_0^{d(Sz_k,Tz_k)}\varphi(t)dt,\int_0^{d(Az_k,Sz_k)}\varphi(t)dt,$$
$$\int_0^{d(A_iz_k,Tz_k)}\varphi(t)dt,\int_0^{d(Az_k,Tz_k)}\varphi(t)dt,\int_0^{d(Sz_k,A_iz_k)}\varphi(t)dt\Big)$$
$$=F\Big(\int_0^{d(z_k,A_iz_k)}\varphi(t)dt,\int_0^{d(z_k,A_iz_k)}\varphi(t)dt,0,$$
$$0,\int_0^{d(z_k,A_iz_k)}\varphi(t)dt,\int_0^{d(z_k,A_iz_k)}\varphi(t)dt\Big)\le 0.$$

From (φ_1), we obtain $A_iz_k = z_k$ and z_k is a common fixed point of A, S, T and A_i, for every i.

The proof of the unicity of the common fixed point is analogous to that of Theorem 2.1. Then, z_k is the unique common fixed point of S, T, A and A_i for all i.

When TX is assumed to be closed subspace of X, then the proof is similar. On the other hand, the cases in which AX or A_iX is a closed subspace of X are similar to the cases in which TX or SX is closed, respectively. ∎

Theorem 2.3. *Let (X,d) be a symmetric space and $(A_i)_{i\in I}$, S and T be self-mappings of (X,d) satisfying the following conditions:*

(i) *for some $k\in I$, the pair (A_k,S) is occasionally weakly compatible,*

(ii) *there exists $v\in\bigcap_{i\in I\setminus\{k\}}C(A_i,T)$ such that $A_iTv=TA_iv$, for all $i\in I\setminus\{k\}$, where $C(A_i,T)$ is the set of coincidence points of A_i and T,*

(iii)

$$F\Big(\int_0^{d(A_kx,A_iy)}\varphi(t)\,dt,\int_0^{d(Sx,Ty)}\varphi(t)\,dt,\int_0^{d(A_kx,Sx)}\varphi(t)\,dt,$$
$$\int_0^{d(A_iy,Ty)}\varphi(t)\,dt,\int_0^{d(A_kx,Ty)}\varphi(t)\,dt,\int_0^{d(Sx,A_iy)}\varphi(t)\,dt\Big)\le 0, \tag{2.1}$$

for every $i\in I\setminus\{k\}$, for all x and y in X, where $F:\mathbb{R}_+^6\longrightarrow\mathbb{R}$ satisfies condition (φ_1) and $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a locally integrable function satisfying $\int_0^\varepsilon\varphi(t)\,dt>0,\forall\varepsilon>0$, then S, T and A_i, for all $i\in I$, have a unique common fixed point in X.

Proof. By (i) and (ii), there exist u and $v \in X$ such that, for every $i \neq k$,

$$A_k u = Su, \quad A_k S u = S A_k u, \quad A_i v = Tv, \quad A_i T v = T A_i v. \tag{2.2}$$

Using (2.1), we have

$$\begin{aligned} &F\Bigg(\int_0^{d(A_k u, A_i v)} \varphi(t)dt, \int_0^{d(Su,Tv)} \varphi(t)dt, \int_0^{d(A_k u, Su)} \varphi(t)dt, \\ &\qquad\qquad \int_0^{d(A_i v, Tv)} \varphi(t)dt, \int_0^{d(A_k u, Tv)} \varphi(t)dt, \int_0^{d(Su, A_i v)} \varphi(t)dt\Bigg) \\ &\quad = F\Bigg(\int_0^{d(A_k u, Tv)} \varphi(t)dt, \int_0^{d(A_k u, Tv)} \varphi(t)dt, \int_0^{d(A_k u, A_k u)} \varphi(t)dt, \\ &\qquad\qquad \int_0^{d(Tv, Tv)} \varphi(t)dt, \int_0^{d(A_k u, Tv)} \varphi(t)dt, \int_0^{d(A_k u, Tv)} \varphi(t)dt\Bigg) \leq 0 \end{aligned}$$

then, by (φ_1), $A_k u = Tv$, we have

$$A_k u = Su = A_i v = Tv, \quad \text{for every } i \neq k. \tag{2.3}$$

From (2.2), we can write: for all $i \neq k$

$$A_i A_k u = A_i T v = T A_i v = T A_k u. \tag{2.4}$$

Using (2.1) again, we obtain with (2.4)

$$\begin{aligned} &F\Bigg(\int_0^{d(A_k u, A_i A_k u)} \varphi(t)\,dt, \int_0^{d(Su, TA_k u)} \varphi(t)\,dt, \int_0^{d(A_k u, Su)} \varphi(t)\,dt, \\ &\qquad \int_0^{d(A_i A_k u, TA_k u)} \varphi(t)\,dt, \int_0^{d(A_k u, TA_k u)} \varphi(t)\,dt, \int_0^{d(Su, A_i A_k u)} \varphi(t)\,dt\Bigg) \\ &= F\Bigg(\int_0^{d(A_k u, TA_k u)} \varphi(t)\,dt, \int_0^{d(A_k u, TA_k u)} \varphi(t)\,dt, 0, \\ &\qquad 0, \int_0^{d(A_k u, TA_k u)} \varphi(t)\,dt, \int_0^{d(A_k u, TA_k u)} \varphi(t)\,dt\Bigg) \leq 0. \end{aligned}$$

So, thanks to (φ_1), $A_k u = T A_k u$. Therefore, by (2.2), (2.3) and (2.4), for every $i \neq k$, we have

$$A_i(A_k u) = T(A_k u) = A_k u \tag{2.5}$$

and

$$A_k(A_k u) = A_k(Su) = S(A_k u). \tag{2.6}$$

Using (2.1) again, we get

$$F\Big(\int_0^{d(A_kA_ku,A_iA_ku)}\varphi(t)dt, \int_0^{d(SA_ku,TA_ku)}\varphi(t)dt, \int_0^{d(A_kA_ku,SA_ku)}\varphi(t)dt,$$
$$\int_0^{d(A_iA_ku,TA_ku)}\varphi(t)dt, \int_0^{d(A_kA_ku,TA_ku)}\varphi(t)dt, \int_0^{d(SAku,A_iA_ku)}\varphi(t)dt\Big)$$
$$= F\Big(\int_0^{d(A_kA_ku,A_ku)}\varphi(t)dt, \int_0^{d(SA_ku,A_ku)}\varphi(t)dt, 0,$$
$$0, \int_0^{d(A_kA_ku,A_ku)}\varphi(t)dt, \int_0^{d(SA_ku,A_ku)}\varphi(t)dt\Big) \le 0.$$

Hence, by (φ_1), (2.5) and (2.6), for all $i \neq k$, we have

$$A_k(A_ku) = S(A_ku) = A_i(A_ku) = T(A_ku) = A_ku.$$

So A_ku is a fixed point of S, T and A_i, for all $i \in I$. The uniqueness of the common fixed point is proved as in the previous theorem by using (2.1). And the proof is finished. ■

As a particular case, we get the following theorem which generalizes Theorem 4.1 of [24].

Theorem 2.4. *Let (X, d) be a symmetric space, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a locally integrable function such that $\int_0^\varepsilon \varphi(t)dt > 0, \forall \varepsilon > 0$ and A, B, S and T be self-mappings of (X, d) satisfying, for all x and y in X,*

$$\text{(3.1)} \quad F\Big(\int_0^{d(Ax,By)}\varphi(t)\,dt, \int_0^{d(Sx,Ty)}\varphi(t)\,dt, \int_0^{d(Ax,Sx)}\varphi(t)\,dt,$$
$$\int_0^{d(By,Ty)}\varphi(t)\,dt, \int_0^{d(Ax,Ty)}\varphi(t)\,dt, \int_0^{d(Sx,By)}\varphi(t)\,dt\Big) \le 0,$$

where $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfies condition (φ_1).

If the pairs (A, S) and (B, T) are occasionally weakly compatible, then A, B, S and T have a unique common fixed point in X.

3. Examples and applications

Before explaining several already published results which can be obtained as particular cases of our previous theorems, we give some examples that illustrate our theorems.

Example 4. For Theorem 2.1 and Theorem 2.2, we present the following example.

Let $X = [0,4]$ be endowed with the symmetric $d(x,y) = (|x-y|)^{\frac{1}{3}}$ and let $A_{i\in\mathbb{N}}, S$ and T be self mappings of X such that

$$Ax = \begin{cases} \frac{7}{3}, & \text{if } x \in [0,2[, \\ 2, & \text{if } x \in [2,3[, \\ \frac{3}{2}, & \text{if } x \in [3,4], \end{cases} \qquad Sx = \begin{cases} 4, & \text{if } x \in [0,2[, \\ x, & \text{if } x \in [2,3[, \\ \frac{7}{2}, & \text{if } x \in [3,4], \end{cases}$$

for all $i \geq 2$,

$$A_i y = \begin{cases} 2, & \text{if } y \in [0,3[, \\ 2+\frac{1}{i}, & \text{if } y \in [3,4], \end{cases} \qquad Ty = \begin{cases} 4-y, & \text{if } y \in [0,3[, \\ \frac{3}{4}, & \text{if } y \in [3,4]. \end{cases}$$

Putting $x_n = 2+\frac{1}{n}$, it is clear that (A,S) and, for every $i \geq 2$, (A_i,T) satisfy the property E.A.

Since $C(A,S) = \{2\}$ and $AS2 = SA2$, (A,S) are weakly compatible. $C(A_2,T) = \{2\}$ and $A_2T2 = TA_22 = 2$; hence hypotheses (i) and (ii) of Theorem 2.2 are satisfied. $AX = \left\{\frac{3}{2}, 2, \frac{7}{3}\right\} \subset TX = \left\{\frac{3}{4}\right\} \cup\,]1,4]$ and $A_iX = \left\{2, 2+\frac{1}{i}\right\} \subset SX = [2,3[\cup \left\{\frac{7}{2}, 4\right\}$. Now, we verify the condition (1.3) of Theorem 2.2.

Let $F(t_1,\ldots,t_6) = t_1 - h\max\{t_2,t_3,t_4,t_5,t_6\}$ and $\varphi(t) = 3t^2$. So $\int_0^{d(x,y)} \varphi(t)dt = |x-y|$.

It is easy to verify that $F \in \mathcal{F}$. If

$$R(x,y) = F\Bigg(\int_0^{d(Ax,A_iy)} \varphi(t)\,dt, \int_0^{d(Sx,Ty)} \varphi(t)\,dt, \int_0^{d(Ax,Sx)} \varphi(t)\,dt,$$
$$\int_0^{d(A_iy,Ty)} \varphi(t)\,dt, \int_0^{d(Ax,Ty)} \varphi(t)\,dt, \int_0^{d(Sx,A_iy)} \varphi(t)\,dt\Bigg)$$

then $R(x,y) = |Ax - A_iy| - hM(x,y)$ where

$$M(x,y) = \max\{|Sx-Ty|, |Ax-Sx|, |A_iy-Ty|, |Ax-Ty|, |Sx-A_iy|\}.$$

We have to prove that there exists $h \in [0,1[$, such that, for each x and $y \in X$, $R(x,y) \leq 0$. It is evident that $R(x,y) \leq 0$ when $x \in [2,3[$ and $y \in [0,3[$ since, in this case, $|Ax - A_iy| = 0$. We have to study the other following cases.

	(1) $x \in [0,2[$, $y \in [0,2[$	(2) $x \in [0,2[$, $y \in [2,3[$	(3) $x \in [0,2[$, $y \in [3,4]$	(4) $x \in [2,3[$, $y \in [3,4]$
$\|Ax - A_iy\|$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3} - \frac{1}{i}$	$\frac{1}{i}$
$\|Sx - Ty\|$	y	y	$\frac{13}{4}$	$x - \frac{3}{4}$
$\|Ax - Sx\|$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$	$x - 2$
$\|A_iy - Ty\|$	$2 - y$	$y - 2$	$\frac{5}{4} + \frac{1}{i}$	$\frac{5}{4} + \frac{1}{i}$
$\|Ax - Ty\|$	$\|y - \frac{5}{3}\|$	$y - \frac{5}{3}$	$\frac{19}{12}$	$\frac{5}{4}$
$\|Sx - A_iy\|$	2	2	$\frac{3}{2} - \frac{1}{i}$	$\|x - 2 - \frac{1}{i}\|$
$M(x,y)$	2	y	$\frac{13}{4}$	$\frac{5}{4} + \frac{1}{i}$
$R(x,y)$	$\frac{1}{3} - 2h$	$\frac{1}{3} - yh$ $< \frac{1}{3} - 2h$	$(\frac{1}{3} - \frac{1}{i}) - \frac{13}{4}h$ $< \frac{1}{3} - \frac{13}{4}h$	$\frac{1}{i} - h(\frac{5}{4} + \frac{1}{i})$ $< \frac{1}{2} - \frac{5}{4}h$

	(5) $x \in [3,4]$, $y \in [0,1[$	(6) $x \in [3,4]$, $y \in [1,\frac{5}{2}[$	(7) $x \in [3,4]$, $y \in [\frac{5}{2},3[$	(8) $x \in [3,4]$, $y \in [3,4]$
$\|Ax - A_iy\|$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2} + \frac{1}{i}$
$\|Sx - Ty\|\|$	$\frac{1}{2} - y$	$y - \frac{1}{2}$	$y - \frac{1}{2}$	$\frac{11}{4}$
$\|Ax - Sx\|$	2	2	2	$\frac{5}{2}$
$\|A_iy - Ty\|$	$2 - y$	$\|2 - y\|$	$2 - y$	$\frac{5}{4} + \frac{1}{i}$
$\|Ax - Ty\|$	$3 - y$	$3 - y$	$3 - y$	$\frac{3}{4}$
$\|Sx - A_iy\|$	$\frac{5}{2} - y$	$\|y - \frac{5}{2}\|$	$\frac{5}{2} - y$	$\frac{3}{2} - \frac{1}{i}$
$M(x,y)$	$3 - y$	2	$y - \frac{1}{2}$	$\frac{11}{4}$
$R(x,y)$	$\frac{1}{2} - h(3 - y)$ $< \frac{1}{2} - \frac{5}{2}h$	$\frac{1}{2} - 2h$	$\frac{1}{2} - h(y - \frac{1}{2})$ $< \frac{1}{2} - 2h$	$1 + \frac{1}{i} - \frac{11}{4}h$ $< \frac{3}{2} - \frac{11}{4}h$

Then, if $h_0 = \max\{\frac{1}{6}, \frac{4}{39}, \frac{2}{5}, \frac{1}{5}, \frac{1}{4}, \frac{6}{11}\} = \frac{6}{11}$, for each $h \in [h_0, 1[$, $R(x,y) \leq 0$ for each x and each $y \in X$ and all conditions of Theorems 2.1 and 2.2 are satisfied.

Example 5. This example illustrates Theorem 2.3. Let $X = [0,4]$ be endowed with the symmetric $d(x,y) = e^{|x-y|} - 1$ and let $A_{i\in\mathbb{N}}, S$ and T be self mappings of X such that

$$A_1x = \begin{cases} \frac{17}{8}, & \text{if } x \in [0,2[, \\ 2, & \text{if } x \in [2,3[\\ 1, & \text{if } x \in [3,4], \end{cases} \qquad Sx = \begin{cases} 4, & \text{if } x \in [0,2[, \\ x, & \text{if } x \in [2,3[, \\ x-2, & \text{if } x \in [3,4], \end{cases}$$

for all $i \geq 2$,

$$A_i y = \begin{cases} 2, & \text{if } y \in [0,3[, \\ 2 + \frac{1}{i}, & \text{if } y \in [3,4], \end{cases} \qquad Ty = \begin{cases} 2, & \text{if } y \in [0,3[, \\ \frac{3}{4}, & \text{if } y \in [3,4]. \end{cases}$$

We note that

$$C(A_1, S) = \{2, 3\}, \quad A_1 S2 = SA_1 2, \quad A_1 S3 = A1 = \frac{17}{8} \neq SA_1 3 = S1 = 4.$$

So (A_1, S) are occasionally weakly compatible and non weakly compatible. Concerning the sequence $(A_i)_{i\geq 2}$, there exists $v = 2 \in \bigcap_{i\geq 2} C(A_i, T)$ such that $A_i T v = T A_i v = 2$ hence hypothesis (ii) of Theorem 2.3 is satisfied.

Now, we verify other conditions of Theorem 2.3.

Let $F(t_1, \ldots, t_6) = t_1 - h \max\{t_2, t_3, t_4, t_5, t_6\}$ and $\varphi(t) = 1/(1+t)$. So $\int_0^{d(x,y)} \varphi(t)dt = |x - y|$. It is easy to verify that F satisfies the property (φ_1). If

$$R(x,y) = F\Bigg(\int_0^{d(Ax,By)} \varphi(t)\,dt, \int_0^{d(Sx,Ty)} \varphi(t)\,dt, \int_0^{d(Ax,Sx)} \varphi(t)\,dt, \int_0^{d(By,Ty)} \varphi(t)\,dt, \int_0^{d(Ax,Ty)} \varphi(t)\,dt, \int_0^{d(Sx,By)} \varphi(t)\,dt \Bigg),$$

$$R(x,y) = |Ax - By| - hM(x,y),$$

where $M(x,y) = \max\{|Sx - Ty|, |Ax - Sx|, |By - Ty|, |Ax - Ty|, |Sx - By|\}$.

We have to prove that there exists $h \in]0,1[$, such that, for each x and $y \in X$, $R(x,y) \leq 0$. If $x \in [2,3]$ and $y \in [0,3[$, $R(x,y) \leq 0$. We have the other following cases.

	(1) $x \in [0,2[$, $y \in [0,3[$	(2) $x \in [0,2[$, $y \in [3,4]$	(3) $x \in [2, 2+\frac{1}{i}]$, $y \in [0,3[$
$\lvert A_1x - A_iy\rvert$	$\frac{1}{8}$	$\frac{1}{8} - \frac{1}{i}$	$\frac{1}{i}$
$\lvert Sx - Ty\rvert$	2	$\frac{13}{4}$	$x - \frac{3}{4}$
$\lvert A_1x - Sx\rvert$	$\frac{15}{8}$	$\frac{15}{8}$	$x - 2$
$\lvert A_iy - Ty\rvert$	0	$\frac{1}{i}$	$\frac{5}{4} + \frac{1}{i}$
$\lvert A_1x - Ty\rvert$	$\frac{1}{8}$	$\frac{5}{4}$	
$\lvert Sx - A_iy\rvert$	2	2	$2 + \frac{1}{i} - x$
$M(x,y)$	2	$\frac{13}{4}$	$\frac{5}{4} + \frac{1}{i}$
$R(x,y)$	$\frac{1}{8} - 2h$	$\frac{1}{8} - \frac{1}{i} - \frac{13}{4}h < \frac{1}{8} - 2h$	$\frac{1}{i} - h(\frac{5}{4} + \frac{1}{i}) < \frac{1}{2} - \frac{5}{4}h$

	(4) $x \in [2+\frac{1}{i}, 3[$ $y \in [3,4]$	(5) $x \in [3,4]$ $y \in [0,3[$	(6) $x \in [3,4]$ $y \in [3,4]$
$\lvert A_1x - A_iy\rvert$	$\frac{1}{i}$	1	$1+\frac{1}{i}$
$\lvert Sx - Ty\rVert$	$x-\frac{3}{4}$	$4-x$	$x-\frac{11}{4}$
$\lvert A_1x - Sx\rvert$	$x-2$	$x-1$	$x-1$
$\lvert A_iy - Ty\rvert$	$\frac{5}{4}+\frac{1}{i}$	0	$\frac{5}{4}+\frac{1}{i}$
$\lvert A_1x - Ty\rvert$	$\frac{5}{4}$	1	$\frac{1}{4}$
$\lvert Sx - A_iy\rvert$	$x-2-\frac{1}{i}$	$4+\frac{1}{i}-x$	$4+\frac{1}{i}-x$
$M(x,y)$	$x-\frac{3}{4}$	$x-1$	$x-1$
$R(x,y)$	$\frac{1}{i}-(x-\frac{3}{4})h$	$1-h(x-1)$	$1-\frac{1}{i}-h(x-1)$
	$< \frac{1}{2}-\frac{5}{4}h$	$< 1-2h$	$< \frac{3}{2}-2h$

Then, if $h_0 = \max\{\frac{1}{16}, \frac{2}{5}, \frac{1}{2}, \frac{3}{4}\} = \frac{3}{4}$, for each $h \in [h_0, 1[$, $R(x,y) \leq 0$ for each x and each $y \in X$, and Theorem 2.3 can be used to resolve this example.

Example 6. The following example illustrates Theorem 2.4. We choose $X = [0, \frac{\pi}{2}]$, endowed with the symmetric $d(x,y) = 1 - \cos|x-y|$ and A, B, S and T self mappings of X such that

$$Ax = \begin{cases} \frac{\pi}{4}, & \text{if } x \in \left[0, \frac{\pi}{4}\right], \\ \frac{2\pi}{7}, & \text{if } x \in \left]\frac{\pi}{4}, \frac{\pi}{2}\right], \end{cases} \qquad Sx = \begin{cases} x, & \text{if } x \in \left[0, \frac{\pi}{4}\right], \\ \frac{\pi}{2}, & \text{if } x \in \left]\frac{\pi}{4}, \frac{\pi}{2}\right], \end{cases}$$

$$Ax = \begin{cases} \frac{\pi}{4}, & \text{if } x \in \left[0, \frac{\pi}{4}\right], \\ \frac{\pi}{5}, & \text{if } x \in \left]\frac{\pi}{4}, \frac{\pi}{2}\right], \end{cases} \qquad Tx = \begin{cases} \frac{\pi}{2} - x, & \text{if } x \in \left[0, \frac{\pi}{4}\right], \\ \frac{\pi}{6}, & \text{if } x \in \left]\frac{\pi}{4}, \frac{\pi}{2}\right], \end{cases}$$

$C(A,S) = \left\{\frac{\pi}{4}\right\}$ and $AS\frac{\pi}{4} = SA\frac{\pi}{4}$. Then A and S are weakly compatible, and therefore occasionally weakly compatible selfmaps. On the other hand, $C(B,T) = \left\{\frac{\pi}{4}\right\}$ and $BT\frac{\pi}{4} = TB\frac{\pi}{4}$. Therefore B and T are occasionally weakly compatible selfmaps.

Let $F(t_1, \ldots, t_6) = \min\{t_1, t_2\} - h\max\{t_3, t_4, t_5, t_6\}$ such that $0 \leq h < 1$ and $\varphi(t) = \frac{1}{1+t}$. So $\int_0^u \varphi(t)dt = ln(1+u)$. It is clear that F satisfies the property (φ_1). Now, we begin to verify the other conditions of Theorem 2.3. We have to prove that, for suitably chosen h, for every x and every y in X,

$$R(x,y) = F\Bigg(\int_0^{d(Ax,By)} \varphi(t)\,dt, \int_0^{d(Sx,Ty)} \varphi(t)\,dt, \int_0^{d(Ax,Sx)} \varphi(t)\,dt,$$
$$\int_0^{d(By,Ty)} \varphi(t)\,dt, \int_0^{d(Ax,Ty)} \varphi(t)\,dt, \int_0^{d(Sx,By)} \varphi(t)\,dt\Bigg) \leq 0,$$

$$\int_0^{d(x,y)} \varphi(t)\,dt = \ln(2-\cos|x-y|)$$

and $R(x,y) = \min\{\ln(2-\cos|Ax-By|), \ln(2-\cos|Sx-Ty|)\} - hM(x,y)$ where $M(x,y) = \max\{\ln(2-\cos|Ax-Sy|), \ln(2-\cos|Bx-Ty|, \ln(2-\cos|Ax-Ty|), \ln(2-\cos|Sx-By|)\}$.

We have to consider the following cases:

	$x \in [0, \frac{\pi}{4}]$, $y \in [0, \frac{\pi}{4}]$	$x \in [0, \frac{\pi}{4}]$, $y \in]\frac{\pi}{4}, \frac{\pi}{2}]$
$\ln(2-\cos|Ax-By|)$	0	$\ln\left(2-\cos\frac{\pi}{20}\right) = 0,01\,22$
$\ln(2-\cos|Sx-Ty|)$	$\ln\left(2-\cos\left|x+y-\frac{\pi}{2}\right|\right)$	$\ln\left(2-\cos\left|x-\frac{\pi}{6}\right|\right)$
$\ln(2-\cos|Ax-Sx|)$	$\ln\left(2-\cos\left|\frac{\pi}{4}-x\right|\right)$	$\ln\left(2-\cos\left|\frac{\pi}{4}-x\right|\right)$
$\ln(2-\cos|By-Ty|)$	$\ln\left(2-\cos\left|y-\frac{\pi}{4}\right|\right)$	$\ln\left(2-\cos\frac{\pi}{30}\right) = 0,0054$
$\ln(2-\cos(Ax-Ty))$	$\ln\left(2-\cos\left|y-\frac{\pi}{4}\right|\right)$	$\ln\left(2-\cos\frac{\pi}{12}\right) = 0,033$
$\ln(2-\cos|Sx-By|)$	$\ln\left(2-\cos\left|x-\frac{\pi}{4}\right|\right)$	$\ln\left(2-\cos\left|\frac{\pi}{5}-x\right|\right)$

	$x \in]\frac{\pi}{4}, \frac{\pi}{2}]$, $y \in [0, \frac{\pi}{4}]$	$x \in]\frac{\pi}{4}, \frac{\pi}{2}]$, $y \in]\frac{\pi}{4}, \frac{\pi}{2}]$
$\ln(2-\cos|Ax-By|)$	$\ln\left(2-\cos\frac{\pi}{28}\right) = 0,006$	$\ln\left(2-\cos\frac{3\pi}{35}\right) = 0,035$
$\ln(2-\cos|Sx-Ty|)$	$\ln(2-\cos y)$	$\ln\left(2-\cos\frac{2\pi}{3}\right) = 0.916$
$\ln(2-\cos|Ax-Sx|)$	$\ln\left(2-\cos\frac{3\pi}{14}\right) = 0.197$	$\ln\left(2-\cos\frac{3\pi}{14}\right) = 0.197$
$\ln(2-\cos|By-Ty|)$	$\ln\left(2-\cos\left|y-\frac{\pi}{4}\right|\right)$	$\ln\left(2-\cos\frac{\pi}{30}\right) = 0.0046$
$\ln(2-\cos(Ax-Ty))$	$\ln\left(2-\cos\left|y-\frac{3\pi}{14}\right|\right)$	$\ln\left(2-\cos\frac{5\pi}{42}\right) = 0.0668$
$\ln(2-\cos|Sx-By|)$	$\ln\left(2-\cos\frac{\pi}{4}\right) = 0.256\,88$	$\ln\left(2-\cos\frac{3\pi}{10}\right) = 0.345$

Finally, in each of previous four cases, we can notice that the value of

$$\min\{\ln(2-\cos|Ax-By|), \ln(2-\cos|Sx-Ty|)\}$$

is smaller than one of other values in the same column. Consequently, there exists an $h \in]0,1[$ such that the inequality (3.1) is satisfied. This example illustrates Theorem 2.4.

Now, we mention some already published results which can be obtained as particular cases of our previous theorems.

The next corollary improves the Theorem 3 of [9]. Since this result is concerned with symmetric spaces, no conditions are taken about the ranges of A, B, S and T, the hypotheses of upper semicontinuity and nondecrease of ψ are dropped and the weakly compatibility of the pairs (A,S) and (B,T) is replaced with the occasionally weak compatibility.

Corollary 3.1. *Let (X,d) be a symmetric space and A, B, S and T be self-mappings of (X,d) satisfying, for all x and y in X,*

$$\left(\int_0^{d(Ax,By)} \varphi(t)dt\right)^p \leq$$

$$\psi\left[a\left(\int_0^{d(Sx,Ty)} \varphi(t)dt\right)^p + (1-a)\max\left\{\int_0^{d(Ax,Sx)} \varphi(t)dt, \int_0^{d(By,Ty)} \varphi(t)dt,\right.\right.$$

$$\left.\left.\left(\int_0^{d(Ax,Sx)} \varphi(t)dt\right)^{\frac{1}{2}}\left(\int_0^{d(Ax,Ty)} \varphi(t)dt\right)^{\frac{1}{2}}, \left(\int_0^{d(Ax,Ty)} \varphi(t)dt\right)^{\frac{1}{2}}\left(\int_0^{d(Sx,By)} \varphi(t)dt\right)^{\frac{1}{2}}\right\}\right],$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally integrable mapping such that $\int_0^\varepsilon \varphi(t)\, dt > 0$, for every $\varepsilon > 0$, $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\psi(t) < t$, for every $t > 0$, $a \in]0,1[$ and $p \geq 1$.

If the pairs (A,S) and (B,T) are occasionally weakly compatible, then A, B, S and T have a unique common fixed point in X.

Proof. We define $F : \mathbb{R}_+^6 \to \mathbb{R}$ by $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^p - \psi(at_2^p + (1-a)\max\{t_3, t_4, \sqrt{t_3t_5}, \sqrt{t_5t_6}\})$. It is easy to verify that

$$F\left(\int_0^u \varphi(t)\, dt, \int_0^u \varphi(t)\, dt, 0, 0, \int_0^u \varphi(t)\, dt, \int_0^u \varphi(t)\, dt\right)$$

$$= \left(\int_0^u \varphi(t)\, dt\right)^p - \psi\left(\left(\int_0^u \varphi(t)\, dt\right)^p\right) > 0,$$

for every $u > 0$. So by Theorem 2.4, the proof is finished. ∎

The following corollary generalizes the Theorem 1 of [4] since, it involves, there is no hypotheses of inclusion about the ranges of the maps A, B, S and T and also the property E.A is not required.

Corollary 3.2. *Let (X,d) be a symmetric space and A, B, S and T be self-mappings of (X,d) satisfying, for all x and y in X,*

$$\int_0^{d(Ax,By)} \varphi(t)\, dt \leq \psi\left(\int_0^{\max\{d(Sx,Ty),d(Sx,By),d(By,Ty)\}} \varphi(t)\, dt\right)$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally integrable function such that $\int_0^\varepsilon \varphi(t)\, dt > 0$, for every $\varepsilon > 0$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\psi(t) < t$, for every $t > 0$.

If the pairs (A,S) and (B,T) are occasionally weakly compatible, then A, B, S and T have a unique common fixed point in X.

Proof. We use Theorem 2.4 with $F:\mathbb{R}_+^6\to\mathbb{R}$ which is defined by

$$F\left(\int_0^{d(Ax,By)}\varphi(t)\,dt,\int_0^{d(Sx,Ty)}\varphi(t)\,dt,\int_0^{d(Ax,Sx)}\varphi(t)\,dt,\right.$$
$$\left.\int_0^{d(By,Ty)}\varphi(t)\,dt,\int_0^{d(Ax,Ty)}\varphi(t)\,dt,\int_0^{d(Sx,By)}\varphi(t)\,dt\right)$$
$$=\int_0^{d(Ax,By)}\varphi(t)\,dt-\psi\left(\int_0^{\max\{d(Sx,Ty),d(Sx,By),d(By,Ty)\}}\varphi(t)\,dt\right).$$

It is easy to verify that $F\left(\int_0^u\varphi(t)\,dt,\int_0^u\varphi(t)\,dt,0,0,\int_0^u\varphi(t)\,dt,\int_0^u\varphi(t)\,dt\right)=\int_0^u\varphi(t)\,dt-\psi(\int_0^u\varphi(t)\,dt)>0$, for every $u>0$. ∎

If we choose $\varphi(t)=1$, for all $t>0$ in the previous theorems, and if we denote by $\mathcal{G}$ the set of all continuous functions $G:\mathbb{R}_+^6\to\mathbb{R}$ satisfying the following conditions:

(ϕ_1): $G(u,u,0,0,u,u)>0$, for all $u>0$,
(ϕ_2): there exists $0<c<1$ such that for all $u,v\geq 0$, if (G_a) or (G_b) is satisfied, we have $u\leq cv$,

$$(G_a)\quad G(u,v,u,v,u+v,0)\leq 0,\qquad (G_b)\quad G(u,v,v,u,0,u+v)\leq 0,$$

we get the following results:

Theorem 3.3. *Let d be a symmetric on X which satisfies the properties* (W_4), (HE), (CE_1) *and* (CE_2) *and let $(A_i)_{i\in I}$, A, S and T be self-mappings of (X,d) satisfying $AX\subset TX$ and $A_iX\subset SX$, for every $i\in I$ and*

$$G(d(Ax,A_iy),d(Sx,Ty),d(Ax,Sx),d(A_iy,Ty),d(Ax,Ty),d(Sx,A_iy))\leq 0,$$

for every $i\in I$, for all x and y in X, where $G\in\mathcal{G}$. Suppose that:

(i) *(A,S) and, for all i, (A_i,T) satisfies the property* (E.A),
(ii) *(A,S) and, for some $k\in I$, (A_k,T) are weakly compatible.*

If one of the subspaces AX, SX, A_iX and TX of X is closed, then A, S, T and A_i, for all $i\in I$, have a unique common fixed point in X.

Theorem 3.4. *Let (X,d) be a symmetric space and $(A_i)_{i\in I}$, S and T be self-mappings of (X,d) satisfying the following conditions:*

(i) *for some $k\in I$, the pair (A_k,S) is occasionally weakly compatible,*
(ii) *there exists $v\in\bigcap_{i\in I}C(A_i,T)$ such that $A_iTv=TA_iv$, for all $i\in I-\{k\}$, where $C(A_i,T)$ is the set of coincidence points of A_i and T,*
(iii) *$G(d(A_kx,A_iy),d(Sx,Ty),d(A_kx,Sx),d(A_iy,Ty),d(A_kx,Ty),d(Sx,A_iy))\leq 0$, for every $i\in I$, for all x and y in X, where $G:\mathbb{R}_+^6\to\mathbb{R}$ satisfies*

the condition (ϕ_1)*. Then* S*,* T *and* A_i*, for all* $i \in I$*, have a unique common fixed point in* X*.*

Theorem 3.5. *Let* (X, d) *be a symmetric space and* A*,* B*,* S *and* T *be self-mappings of* (X, d) *satisfying, for all* x *and* y *in* X*,*

$$G(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(Sx, By)) \leq 0$$

where $G : \mathbb{R}_+^6 \to \mathbb{R}$*, satisfies condition* (ϕ_1)*.*

If the pairs (A, S) *and* (B, T) *are occasionally weakly compatible, then* A*,* B*,* S *and* T *have a unique common fixed point in* X*.*

Corollary 3.6. *Let* (X, d) *be a symmetric space,* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfies* $\psi(t) < t$*, for every* $t > 0$ *and* A*,* B*,* S *and* T *be self-mappings of* (X, d) *satisfying, for all* x *and* y *in* X*, one of following conditions:*

(i) $d(Ax, By) \leq \psi\left(\max\{d(Sx, Ty), d(Sx, By), d(By, Ty)\}\right)$,

(ii) $d(Ax, By) \leq \psi(m(x, y)$ *where* $m(x, y) = \max\big\{d(Sx, Ty), \frac{k}{2}(d(Ax, Sx) + d(By, Ty)), \frac{1}{2}(d(Ax, Ty) + d(By, Sx))\big\}$ *with* $k \in [1, 2[$*,*

(iii) $(d(Ax, By))^p \leq \psi\left(a(d(Ax, Ty))^p + (1 - a)n(x, y)\right)$ *where* $n(x, y) = \max\{\alpha(d(Ax, Sx))^p,\ \beta(d(By, Ty))^p,\ (d(Ax, Sx))^{\frac{p}{2}}(d(Ax, Ty))^{\frac{p}{2}}, (d(Ax, Ty))^{\frac{p}{2}}\ (d(Sx, By))^{\frac{p}{2}}, \frac{1}{2}((d(Ax, Sx))^p + (d(By, Ty))^p)\}$ *and* $\{a, \alpha, \beta\} \subset\ ]0, 1]$ *and* $p \geq 1$*.*

If the pairs (A, S) *and* (B, T) *are occasionally weakly compatible, then* A*,* B*,* S *and* T *have a unique common fixed point in* X*.*

Proof. 1. With the condition (i), this corollary improves Theorem 2.2 of [2]. In this case, the proof results from Theorem 3.5 with $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \psi(\max\{t_2, t_4, t_6\})$.

2. With the condition (ii), this corollary improves Theorem 2.5 of [12]. For the proof, we use Theorem 3.5 with $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \psi(\max\{t_2, \frac{k}{2}(t_3 + t_4), \frac{1}{2}(t_5 + t_6)$.

3. The case (iii) improves Theorem 3 of [18]. If we take $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^p - \psi(at_5^p + (1 - a)\max\{\alpha t_3^p, \beta t_4^p, t_3^{\frac{p}{2}} t_5^{\frac{p}{2}}, t_5^{\frac{p}{2}} t_6^{\frac{p}{2}}, \frac{1}{2}(t_3^p + t_4^p)\})$ the result follows from the previous theorem. ∎

In the context of metric spaces, we can give the next corollary of Theorem 3.5.

Corollary 3.7. *Let* (X, d) *be a metric space and* A*,* B*,* S *and* T *be self-mappings of* (X, d) *satisfying, for all* x *and* y *in* X*,*

$$G(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(Sx, By)) \leq 0$$

where $G : \mathbb{R}_+^6 \to \mathbb{R}$ *satisfies condition* (ϕ_1)*.*

If the pairs (A, S) *and* (B, T) *are occasionally weakly compatible, then* A*,* B*,* S *and* T *have a unique common fixed point in* X*.*

This corollary generalizes many already published results, since hypotheses on A, B, S, T, ψ or G are weaker, and for some results, these results are given with particular functions G. For examples, we can cite Theorem 1 of [31] for the part of fixed point, Theorem 2 of [1], Theorem 5 of [27], Theorem 3.1 of [13] for the part of fixed point, Theorem 4.1 of [26] and Theorem 3.1 of [5].

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Received April 24, 2010; revised version October 24, 2010.