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# INJECTIVE COMPOSITION OPERATORS ON LORENTZ–BOCHNER SPACES

**Abstract.** In this paper, we extend the notion of essential range to vector-valued functions and present various equivalent conditions for the injectiveness of the composition operators alongwith a characterisation for measurable transformations inducing composition operators between Lorentz–Bochner spaces. Some aspects of the weighted composition operators on Lorentz–Bochner spaces, induced by a measurable transformation and an operator valued map, are also discussed.

## 1. Introduction

Let $f$ be a complex-valued measurable function defined on a $\sigma$-finite measure space $(\Omega, \mathcal{A}, \mu)$. For $s \geq 0$, define $\mu_f$ the *distribution function* of $f$ as

$$\mu_f(s) = \mu(\{\omega \in \Omega : |f(\omega)| > s\}).$$

By $f^*$ we mean the *non-increasing rearrangement* of $f$ given as

$$f^*(t) = \inf\{s > 0 : \mu_f(s) \leq t\},\ t \geq 0.$$

For $t > 0$, let

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds \text{ and } f^{**}(0) = f^*(0).$$

For $1 < p \leq \infty$, $1 \leq q \leq \infty$, and for a measurable function $f$ on $\Omega$ define $\|f\|_{pq}$ as

$$\|f\|_{pq} = \begin{cases} \{\frac{q}{p}\int_0^\infty (t^{1/p} f^{**}(t))^q \frac{dt}{t}\}^{1/q}, & 1 < p < \infty,\ 1 \leq q < \infty, \\ \sup_{t>0} t^{1/p} f^{**}(t), & 1 < p \leq \infty,\ q = \infty. \end{cases}$$

The *Lorentz space* $L_{pq}(\Omega)$ consists of those measurable functions $f$ on $\Omega$ such that $\|f\|_{pq} < \infty$. Also $\|\cdot\|_{pq}$ is a norm and $L_{pq}(\Omega)$ is a Banach

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space with respect to this norm. To be more specific, in case there are chances of confusion about the measure $\mu$ or the measure space $(\Omega, \mathcal{A}, \mu)$, we use the notations $f^{*,\mu}, f^{**,\mu}$ and $L_{pq}(\Omega, \mathcal{A}, \mu)$ in place of $f^*, f^{**}$ and $L_{pq}(\Omega)$, respectively. The $L^p$-spaces, for $1 < p \leq \infty$, are equivalent to the spaces $L_{pp}(\Omega)$. Let us recall that simple functions are dense in $L_{pq}(\Omega)$, for $q \neq \infty$ and also the duality results $L^*_{p1} = L_{p'\infty}$, for $1 < p < \infty$ as well as $L^*_{pq} = L_{p'q'}$ for $1 < p, q < \infty$, where $p', q'$ denote the conjugate exponent of $p, q$, respectively, that is, $\frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}$. The readers are referred to [6, 13 and 14] for these results and more details on Lorentz spaces.

We shall consider functions defined on a measure space whose values are in a general Banach space (the so-called abstract fuctions, see [13]). Let $X$ be a Banach space with norm $\|.\|$ and $(\Omega, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space. We use $\mathfrak{B}(X)$ to denote the class of all bounded operators on $X$. A mapping $f : \Omega \to X$ is said to be a simple function if there exist vectors $c_1, c_2, \ldots, c_n \in X$ and measurable subsets $B_1, B_2, \ldots, B_n$ of $\Omega$, $B_i \bigcap B_j = \emptyset$, for $i \neq j$ such that

$$f(\omega) = \sum_{i=1}^{n} \chi^{c_i}_{B_i}(\omega),$$

where $\chi^{c_i}_{B_i} : \Omega \to X$ is given by

$$\chi^{c_i}_{B_i}(\omega) = \begin{cases} c_i, & \text{if } \omega \in B_i, \\ 0, & \text{otherwise.} \end{cases}$$

A function $f : \Omega \to X$ is said to be strongly measurable if there exists a sequence $< f_n >$ of simple functions such that

$$\lim_{n\to\infty} \|f_n(\omega) - f(\omega)\| = 0 \quad \text{for almost all } \omega \in \Omega.$$

For a strongly measurable function $f : \Omega \to X$, define the function $\|f\|$ as

$$\|f\|(\omega) = \|f(\omega)\|,$$

for all $\omega \in \Omega$. All the notations make sense for $f$ by replacing the modulus by norm. This leads to the natural definition of the Lorentz–Bochner space $L^{\mathcal{A},\mu}_{pq}(\Omega, X)$ (or shortly $L_{pq}(\Omega, X)$), where the norm is defined by

$$\|f\|_{pq} = \begin{cases} \left\{ \frac{q}{p} \int_0^\infty \left( t^{1/p} \|f\|^{**}(t) \right)^q \frac{dt}{t} \right\}^{1/q}, & 1 < p < \infty,\ 1 \leq q < \infty, \\ \sup\limits_{t>0} t^{1/p} \|f\|^{**}(t), & 1 < p \leq \infty,\ q = \infty. \end{cases}$$

The Lorentz–Bochner space $L_{pq}(\Omega, X)$ is a Banach space. We still have the density of simple functions in it and its dual is

$$L^*_{pq}(\Omega, X) = L_{p'q'}(\Omega, X^*),$$

where $X^*$ has the Radon–Nikodym property. The particular case when $p = q$ is studied in [8], whereas for more general case for certain Banach lattices including $L_{pq}$ one can refer to [9]. The Lorentz–Bochner space $L_{pq}(\Omega, X)$ can also be viewed in terms of a space of vector measures (see [6]).

A measurable transformation $T : \Omega \to \Omega$, satisfying $\mu(T^{-1}(B)) = 0$ whenever $\mu(B) = 0$ for $B \in \mathcal{A}$, is said to be a non-singular measurable transformation. For a non-singular measurable transformation $T : \Omega \to \Omega$, the composition transformation $C_T : L_{pq}(\Omega, X) \to L(\Omega, X)$ is given by

$$(C_T f)(\omega) = f(T(\omega)), \quad \text{for all } \omega \in \Omega,$$

where $L(\Omega, X)$ is the space of all strongly measurable functions.

If $C_T$ is bounded with range in $L_{pq}(\Omega, X)$, then it is called a *composition operator* on $L_{pq}(\Omega, X)$ induced by $T$.

If for a non-singular measurable transformation $T : \Omega \to \Omega$ and a function $u : \Omega \to \mathfrak{B}(X)$, the linear transformation $W_{u,T} : L_{pq}(\Omega, X) \to L(\Omega, X)$, given by

$$(W_{u,T} f)(\omega) = u(T(\omega))\big(f(T(\omega))\big), \quad \text{for all } \omega \in \Omega \text{ and } f \in L_{pq}(\Omega, X),$$

is continuous with range in $L_{pq}(\Omega, X)$, then it is called the *weighted composition operator* on $L_{pq}(\Omega, X)$. If $u \equiv I$ (i.e., $u(\omega) = I$, for all $\omega \in \Omega$) then $W_{u,T}$ becomes the *composition operator* $C_T$ and if $T$ is the identity mapping then it becomes the *multiplication operator* $M_u$. Publications are available regarding the study of operators $f \mapsto u.f \circ T$ and these operators, for the case $X = \mathbb{C}$, are discussed in [3]. It is shown in [3, Theorem 3.7] that the conditions $u \circ T \neq 0$ and surjectiveness of $T$ are necessary and sufficient for the injectivity of the operator $f \mapsto u.f \circ T$, whereas Example 3.8 of the paper verify that this result does not hold for the weighted composition operators on $L_{pq}(\Omega, X)$.

One of the applications of strongly measurable functions is the study of multipliers between spaces of vector-valued integrable functions (see [7], [10]). In [2], strongly measurable mappings are used to define multiplication operator between Lorentz–Bochner spaces. In [4], Blasco and Neerven discussed the notions, namely, strongly measurable, strongly $\mu$-normable, mapping with strong $\mu$-measurability of the orbits $\omega \to u(\omega)x$ and used these to define various spaces and multipliers between them. In this paper, we extend the notion of essential range to vector-valued functions and then generalize the injective composition operators in terms of these. Then pursue towards the applications of recently developed notions in inducing weighted composition operators and present some positive results in this direction. For applications of composition operators in ergodic theory, entropy theory, classical mechanics and in many more directions, we refer the reader to

[8, 11, 15] and the references therein. We use the symbols $N(A)$ and $R(A)$ to denote the kernel and the range respectively of the bounded operator $A$ on a Banach space.

## 2. Composition operators

In this section, our aim is to characterize those non-singular transformations which induce the injective composition operators on the Lorentz–Bochner spaces. However, we begin with a characterization of the non-singular measurable transformations $T : \Theta \to \Omega$, for which

$$C_T : L_{pq}^{\mathcal{A},\mu}(\Omega, X) \to L_{pq}^{\mathcal{B},\nu}(\Theta, X)$$

given by

$$C_T f = f \circ T,$$

is bounded. For any two non-zero vectors $c, d \in X$ and any measurable subset $B$ of $\Omega$, we observe the following:

(1) $\chi_B^c \in L_{pq}(\Omega, X)$ if and only if $\chi_B^d \in L_{pq}(\Omega, X)$. Moreover, $\|\chi_B^c\|_{pq} = \frac{\|c\|}{\|d\|}\|\chi_B^d\|_{pq}$.

(2) $C_T(\chi_B^c) = \chi_{T^{-1}(B)}^c$ so that $\|C_T(\chi_B^c)\|_{pq} = \frac{\|c\|}{\|d\|}\|C_T(\chi_B^d)\|_{pq}$.

With these observations, it will be convenient to formulate a characterization of composition operators from Lorentz–Bochner space $L_{pq}^{\mathcal{A},\mu}(\Omega, X)$ to $L_{pq}^{\mathcal{B},\nu}(\Theta, X)$, where $(\Omega, \mathcal{A}, \mu)$ and $(\Theta, \mathcal{B}, \nu)$ are two $\sigma$-finite measure spaces.

**Theorem 2.1.** *A measurable transformation $T : \Theta \to \Omega$ induces a composition operator*

$$C_T : L_{pq}^{\mathcal{A},\mu}(\Omega, X) \to L_{pq}^{\mathcal{B},\nu}(\Theta, X), 1 < p \le \infty, 1 \le q \le \infty$$

*if and only if*

$$(\nu \circ T^{-1})(E) \le b\mu(E), \quad \textit{for all } E \in \mathcal{A}, \textit{ for some } b > 0.$$

**Proof.** Suppose $C_T$ is a composition operator induced by $T$. If $E \in \mathcal{A}$ is such that $\mu(E) = \infty$, then the inequality is trivial. Let $E \in \mathcal{A}$, $\mu(E) < \infty$. Let $x_0$ be a fixed element of $X$ with $\|x_0\| = 1$. Then for measurable subset $E$ of $\Omega$, the non-increasing rearrangement of the characteristic function $\|\chi_E^{x_0}\|$ is given by

$$\|\chi_E^{x_0}\|^{*,\mu}(t) = \chi_{[0,\mu(E))}(t).$$

Along the lines of computations made in the proof of [1,Theorem 2.1], one can show that

$$\big(\nu \circ T^{-1}\big)(E) \le \|C_T\|^p \mu(E), \quad \text{for all E} \in \mathcal{A},$$

and also that, if the measurable transformation $T : \Theta \to \Omega$ satisfies

$$(\nu \circ T^{-1})(E) \le b\mu(E), \quad \text{for all E} \in \mathcal{A},$$

for some constant $b > 0$, then for $f$ in $L_{pq}^{\mathcal{A},\mu}(\Omega, X)$,

$$\|C_T f\|_{pq} \le b^{1/p}\|f\|_{pq},$$

so that $C_T$ is bounded. ∎

Some immediate consequences of this theorem are the following:

**Corollary 2.2.** . *Let $T : \Omega \to \Omega$ be a non-singular measurable transformation. Then $T$ induces a composition operator $C_T$ on the Lorentz–Bochner space $L_{pq}(\Omega, X)$, $1 < p < \infty$, $1 \le q \le \infty$, if and only if there exists some constant $b > 0$ such that*

$$(\mu \circ T^{-1})(E) \le b\mu(E), \quad \textit{for all } E \in \mathcal{A}.$$

This corollary is proved as an independent result in [2].

**Corollary 2.3.** *Let $T : \Omega \to \Omega$ be a non-singular measurable transformation. Then the following are equivalent:*

(1) *$C_T$ on the Lorentz–Bochner space $L_{pq}(\Omega, X)$, $1 < p < \infty$, $1 \le q \le \infty$ satisfies $\|C_T\| \le 1$.*
(2) *$C_T$ is non-expansive on $L_{pq}(\Omega, X)$, $1 < p < \infty$, $1 \le q \le \infty$, i.e. $\|C_T f - C_T g\|_{pq} \le \|f - g\|_{pq}$, for all $f, g \in L_{pq}(\Omega, X)$.*
(3) *$\mu(E) \le \mu(T(E))$ for each $T(E), E \in \mathcal{A}$.*
(4) *$f_T \le 1$, where $f_T$ is the Radon–Nikodym derivative of $\mu T^{-1}$ with respect to $\mu$.*

Now to achieve the main task of the section, we set up few notations. For a non-singular measurable transformation $T : \Omega \to \Omega$ we denote by $f_T$ the Radon–Nikodym derivative of $\mu T^{-1}$ with respect to $\mu$. We denote the support of $f_T$, i.e. the set $\{\omega \in \Omega : f_T(\omega) \ne 0\}$ by $R_{f_T}$ and the set $\{\omega \in \Omega : f_T(\omega) = 0\}$ by $K_{f_T}$. Let $S$ be a measurable subset of $\Omega$ then we define the subspace $L_{pq}(S, X)$ of $L_{pq}(\Omega, X)$ as

$$\begin{aligned} L_{pq}(S, X) &= \{f \in L_{pq}(\Omega, X) : f \text{ vanishes outside } S\} \\ &= \{f \in L_{pq}(\Omega, X) : f(\omega) = 0 \text{ a.e. on } \Omega \setminus S\}. \end{aligned}$$

For $f : \Omega \to X$, define

$$R_f = \{\omega \in \Omega : f(\omega) \ne 0\}$$

and

$$K_f = \{\omega \in \Omega : f(\omega) = 0\}.$$

The following easy fact will be useful in our further work.

**Lemma 2.4.** *$L_{pq}(\Omega, X) = L_{pq}(K_{f_T}, X) \bigoplus L_{pq}(R_{f_T}, X)$, $1 < p < \infty$, $1 \le q \le \infty$.*

**Proof.** We have $\Omega = K_{f_T} \bigcup R_{f_T}$, $K_{f_T} \bigcap R_{f_T} = \emptyset$. It is easy to check that $L_{pq}(K_{f_T}, X) \bigcap L_{pq}(R_{f_T}, X) = \{0\}$. Define $f_1 : \Omega \to X$ and $f_2 : \Omega \to X$ corresponding to any function $f \in L_{pq}(\Omega, X)$ as

$$f_1(\omega) = \begin{cases} f(\omega), & \text{if } \omega \in K_{f_T}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_2(\omega) = \begin{cases} f(\omega), & \text{if } \omega \in R_{f_T}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f = f_1 + f_2$. Also, we find that $\mu_{\|f_1\|}(s) \le \mu_{\|f\|}(s)$, for each $s > 0$ and $\|f_1\|^*(t) \le \|f\|^*(t)$, for each $t > 0$. It follows that $\|f_1\|_{pq} \le \|f\|_{pq}$ and this implies $f_1 \in L_{pq}(K_{f_T}, X)$. Similarly, we check that $f_2 \in L_{pq}(R_{f_T}, X)$. This completes the proof. ■

**Lemma 2.5.** *$L_{pq}(K_{f_T}, X) = \{f \in L_{pq}(\Omega, X) : f_T(\omega) = 0$ for a.e. $\omega \in R_f\}$, $1 < p < \infty$, $1 \le q \le \infty$.*

**Proof.** Proof follows as $f \in L_{pq}(K_{f_T}, X)$ implies that $f \in L_{pq}(\Omega, X)$ and $R_{f_T} \subseteq K_f$ a.e. equivalently $R_f \subseteq K_{f_T}$ a.e. ■

**Theorem 2.6.** *Let $C_T \in \mathfrak{B}(L_{pq}(\Omega, X))$, $1 < p < \infty$, $1 \le q \le \infty$. Then $N(C_T) = L_{pq}(K_{f_T}, X)$.*

**Proof.** Let $f \in N(C_T)$, then $f \circ T = 0$ a.e.. Therefore

$$\mu(\{\omega \in \Omega : f(T(\omega)) \neq 0\}) = 0$$

and this gives

$$\mu T^{-1}(\{\omega \in \Omega : f(\omega) \neq 0\}) = 0$$

equivalently,

$$\int_{\{\omega \in \Omega : f(\omega) \neq 0\}} f_T \, d\mu = 0.$$

Hence $f_T = 0$ a.e. on $\{\omega \in \Omega : f(\omega) \neq 0\}$, so that $f \in L_{pq}(K_{f_T}, X)$. Conversely, if $f \in L_{pq}(K_{f_T}, X)$ then we have

$$\begin{aligned}\mu_{\|f \circ T\|}(s) &= \mu(\{\omega \in \Omega : \|f(T(\omega))\| > s\}) = \mu T^{-1}(\{\omega \in \Omega : \|f(\omega)\| > s\}) \\ &= \int_{\{\omega \in \Omega : \|f(\omega)\| > s\}} f_T \, d\mu = 0,\end{aligned}$$

for each $s > 0$. This gives $\|f \circ T\|^*(t) = 0$ for each $t > 0$. Hence $\|C_T f\|_{pq} = 0$. It follows that $f \in N(C_T)$. ■

As a corollary to this theorem we have the following:

**Corollary 2.7.** *If $(\Omega, \mathcal{A}, \mu)$ is a non-atomic measure space and $C_T \in \mathfrak{B}(L_{pq}(\Omega, X))$, $1 < p < \infty$, $1 \leq q \leq \infty$. Then $N(C_T)$ is zero dimensional or infinite dimensional.*

**Lemma 2.8.** *If $R_f \subseteq K_{f_T}$ a.e. implies $f = 0$ then $f_T \neq 0$ a.e.*

**Proof.** Suppose the condition holds. If the conclusion is not true then we can find a measurable subset $E$ of $K_{f_T}$ satisfying $0 < \mu(E) < \infty$. Now define $f : \Omega \to X$ as

$$f(\omega) = \begin{cases} x_0, & \text{if } \omega \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f \in L_{pq}(\Omega, X)$ and $R_f \subseteq K_{f_T}$, whereas $f \neq 0$. It follows that $f_T \neq 0$ a.e. ∎

By invoking the above lemmas and results, we now deduce some characterizations for injective composition operators on Lorentz–Bochner spaces.

**Theorem 2.9.** *Let $C_T \in \mathfrak{B}(L_{pq}(\Omega, X))$, $1 < p < \infty$, $1 \leq q \leq \infty$. Then the following are equivalent:*

(1) *$C_T$ is injective.*
(2) *$L_{pq}(\Omega, X) = L_{pq}(R_{f_T}, X)$.*
(3) *$\mu(\Omega \setminus R_{f_T}) = 0$.*
(4) *$T$ is essentially surjective (i.e. $\mu(\Omega \setminus T(\Omega)) = 0$).*

**Proof.** (1) ⇔ (2) follow by using Lemma 2.4 together with Theorem 2.6. Now we show that (1) ⇒ (3) ⇒ (4) ⇒ (1).

(1) ⇒ (3): Suppose (1) holds i.e. $C_T$ is injective, then by using Theorem 3.3, $L_{pq}(K_{f_T}, X) = \{0\}$. Now in view of Lemma 2.5, if $f$ is such that $f_T(\omega) = 0$ for a.e. $\omega \in R_f$ then $f = 0$. Hence using Lemma 2.8, we find that $f_T \neq 0$ a.e. Therefore $\mu(\Omega \setminus R_{f_T}) = 0$ or $\mu(K_{f_T}) = 0$.

(3) ⇒ (4): Let $R_{f_T} = \Omega$ a.e., so $f_T \neq 0$ a.e. It implies that if $E \subseteq (\Omega \setminus T(\Omega))$ i.e. $\mu T^{-1}(E) = 0$ then $\mu(E) = 0$. We can conclude that $R_{f_T} \subseteq T(\Omega)$ a.e. Hence $\Omega = T(\Omega)$ a.e. or $\mu(\Omega \setminus T(\Omega)) = 0$.

(4) ⇒ (1): This is obvious. ∎

**Definition 2.10.** [13] The essential range of a complex-valued measurable function $f$ defined on measure space $(\Omega, \mathcal{A}, \mu)$ is given by the set

$$\big\{\lambda \in \mathbb{C} : \mu\big(\{\omega \in \Omega : |f(\omega) - \lambda| < \epsilon\}\big) > 0 \text{ for each } \epsilon > 0\big\}$$

and we denote this set by $EssR_f$.

For a strongly measurable function $f : \Omega \to X$, we observe that

(1) $EssR_{\|f\|} \subseteq R^+$, where $R^+$ denotes the set of non-negative real numbers.
(2) For any vector $c \in X$ and measurable subset $A$ with $\mu(A), \mu(\Omega \setminus A) > 0$,

$$EssR_{\|\chi_A^c\|} = \{0, \|c\|\}.$$

In [13, Theorem 2.2.2, p. 26], the injectiveness of the operator $C_T$ on $L^p$ spaces is related to the essential range of $f$ and $f \circ T$. The next result establishes a link of injective composition operators on Lorentz–Bochner spaces with the notion of essential range.

**Theorem 2.11.** *If $C_T$ is a composition operator on $L_{pq}(\Omega, X)$, $1 < p \leq \infty$, $1 \leq q < \infty$, then the following are equivalent:*

(1) *$C_T$ is injective.*
(2) *$EssR_{\|f\|} = EssR_{\|f \circ T\|}$, for every $f \in L_{pq}(\Omega, X)$.*
(3) *$\mu \ll \mu \circ T^{-1}$ i.e. $\mu(E) = 0$ whenever $\mu(T^{-1}(E)) = 0$.*
(4) *$f_T$ is different from zero a.e.*

We now extend the notion of essential range to vector-valued functions.

**Definition 2.12.** The essential range of a vector-valued function $f$, defined on measure space $(\Omega, \mathcal{A}, \mu)$ with values in Banach space $X$, is defined as the set

$$\big\{x \in X : \mu\big(\{\omega \in \Omega : \|f(\omega) - x\| < \epsilon\}\big) > 0, \text{ for each } \epsilon > 0\big\}$$

and we denote this set by $EssR^v_f$.

If $f$ is a simple function then its essential range is the same as its range. If $X = \mathbb{C}$, then $EssR^v_f$ coincides with $EssR_f$. Without any extra effort, we can proof the following:

**Theorem 2.13.** *If $C_T$ is a composition operator on $L_{pq}(\Omega, X)$, $1 < p \leq \infty$, $1 \leq q < \infty$, then the following are equivalent:*

(1) *$C_T$ is injective.*
(2) *$EssR^v_f = EssR^v_{f \circ T}$, for every $f \in L_{pq}(\Omega, X)$.*
(3) *$\mu \ll \mu \circ T^{-1}$.*
(4) *$f_T$ is different from zero a.e.*

**Remark.** If $X$ is a Banach algebra with the unit element $\mathbf{e}$ (i.e., $\mathbf{e}x = x\mathbf{e} = x$ $\forall x \in X$), then all the results (Theorem 2.1, Theorem 2.9 and Theorem 2.11) hold for the composition operator $C_T$ on $L_{pq}(\Omega, X)$.

**Definition 2.14.** If $X$ is a Banach algebra with the unit element $\mathbf{e}$, we define the generalized essential range of $f : \Omega \to X$, as the set

$$\big\{\lambda \in \mathbb{C} : \mu\big(\{\omega \in \Omega : \|f(\omega) - \lambda\mathbf{e}\| < \epsilon\}\big) > 0, \text{ for each } \epsilon > 0\big\}$$

and we denote this set by $GEssR_f$.

In case $X = \mathbb{C}$, we have $\mathbf{e} = 1$. Now, both the notions $GEssR_f$ and $EssR^v_f$ coincide with the notion of the essential range $EssR_f$.

It is easy to observe the following:

(1) For any vector $c \in X$ and measurable subset $A$ with $\mu(A), \mu(\Omega \setminus A) > 0$, $0 \in GEssR_{\chi_A^c}$.
(2) $|GEssR_f| \subseteq EssR_{\|f\|}$.
(3) $(GEssR_f)\mathbf{e} \subseteq EssR_f^v$, where $(GEssR_f)\mathbf{e} = \{\lambda\mathbf{e} : \lambda \in GEssR_f\}$.

Only (2) needs a proof. Let $\lambda$ be a complex number in the generalized essential range of $f$ i.e. $\lambda \in GEssR_f$. As for each $\epsilon > 0$,

$$f^{-1}\big(S_\epsilon(\lambda\mathbf{e})\big) \subseteq \|f\|^{-1}\big(S_\epsilon(|\lambda|)\big)$$

where $S_\epsilon(\lambda\mathbf{e}) = \{x \in X : \|x - \lambda e\| < \epsilon\}$ and $S_\epsilon(|\lambda|) = \{z \in \mathbb{C} : |z - |\lambda|| < \epsilon\}$. This yields that

$$\mu(\|f\|^{-1}(S_\epsilon(|\lambda|))) > 0,$$

for each $\epsilon > 0$. This proves that $|\lambda| \in EssR_{\|f\|}$.

We verify the notions of $EssR_{\|f\|}$, $EssR_f^v$ and $GEssR_f$, for $f : \Omega \to X$, where $X$ is a Banach algebra with unit $\mathbf{e}$, with the help of the following examples:

**Example 2.15.** For a measurable subset $A$ with $\mu(A), \mu(\Omega \setminus A) > 0$,

$$GEssR_{\chi_A^e} = EssR_{\|\chi_A^e\|} = \{0, 1\} \quad \text{and} \quad EssR_{\chi_A^e}^v = \{0, \mathbf{e}\}.$$

**Example 2.16.** For a measurable subset $A$ with $\mu(A), \mu(\Omega \setminus A) > 0$,

$$GEssR_f = \{1, -1\}, \quad EssR_f^v = \{\mathbf{e}, -\mathbf{e}\} \quad \text{and} \quad EssR_{\|f\|} = \{1\}$$

where the function $f : \Omega \to X$ is given by

$$f(\omega) = \begin{cases} \mathbf{e}, & \text{if } \omega \in A, \\ -\mathbf{e}, & \text{otherwise.} \end{cases}$$

**Example 2.17.** Let $\Omega$ be any measure space and $X$ be any Banach algebra with unit $e$. Let $z_1, z_2$ be two complex numbers satisfying $|z_1| = |z_2| = k$ and $A$ be a measurable subset of $\Omega$ satisfying $\mu(A), \mu(\Omega \setminus A) > 0$. Consider the function $f : \Omega \to X$ given by

$$f(\omega) = \begin{cases} z_1\mathbf{e}, & \text{if } \omega \in A, \\ z_2\mathbf{e}, & \text{otherwise.} \end{cases}$$

Then $EssR_{\|f\|} = \{k\}$, whereas $GEssR_f = \{z_1, z_2\}$.

**Example 2.18.** Let $X$ be a Banach algebra with identity $e$ of dimension more than 1 and $A$ be a measurable subset of $\Omega$ with $0 < \mu(A) < \mu(\Omega)$. Let $f$ be given by

$$f(\omega) = \begin{cases} \mathbf{x}, & \text{if } \omega \in A, \\ -\mathbf{0}, & \text{otherwise,} \end{cases}$$

where $x \in X$ is such that $x \neq \lambda e$ for any $\lambda \in \mathbb{C}$. Then $EssR_f^v = \{0, x\}$ and $GEssR_f = \{0\}$.

**Example 2.19.** Let $\Omega = \{\omega \in \mathcal{R} : a < \omega < b\}$, for real numbers a, b and $\mu =$ Lebesgue measure. Let $X = \mathcal{B}(\mathcal{L}_2)$, where $\mathcal{L}_2$ is the space of all square summable sequences and $T : \mathcal{L}_2 \to \mathcal{L}_2$, be an element of $X$ given by

$$T(x) = (x_1, 0, 0, ., ., ., ., .),$$

for all $x = (x_1, x_2, x_3, ) \in \mathcal{L}_2$. Then for $\chi_A^T$ given by

$$\chi_A^T(\omega) = \begin{cases} T, & \text{if } \omega \in A, \\ 0, & \text{otherwise}, \end{cases}$$

where $A = (a, c)$ for any $c \in (a, b)$, we have $\|T\| = 1$ and $EssR_{\|\chi_A^T\|} = \{0, 1\}$. Also, $0 \in GEssR_{\chi_A^T}$ and if $\lambda$ is a complex number for which $|\lambda| \neq \|T\| = 1$ then it does not belong to $GEssR_{\chi_A}$. However, for any complex number $\lambda = \alpha + \iota\beta$ with $|\lambda| = 1$

$$\|T - \lambda I\| = \begin{cases} |\lambda| = 1, & \text{if } \alpha \geq 0, \\ (2(1-\alpha))^{\frac{1}{2}}, & \text{if } \alpha < 0. \end{cases}$$

Therefore,

$$\mu\big((\chi_A^T)^{-1}(S_\epsilon(\lambda \mathbf{e})\big) = 0,$$

for $\epsilon \leq 1$ and, as a consequence of this, $\lambda$ does not belong to $\mathfrak{ER}_{\chi_A}$ for $\lambda = \alpha + \iota\beta$ with $|\lambda| = 1$. This shows that

$$GEssR_{\chi_A^T} = \{0\}.$$

It is easy to prove the following characterization for the injectiveness of the composition operator $C_T$ on $L_{pq}(\Omega, X)$, $1 < p \leq \infty$, $1 \leq q < \infty$, where $X$ is a Banach algebra with the unit element $e$.

**Theorem 2.20.** *If $C_T$ is a composition operator on $L_{pq}(\Omega, X)$, $1 < p \leq \infty$, $1 \leq q < \infty$, then the following are equivalent:*

(1) *$C_T$ is injective.*
(2) *$GEssR_f = GEssR_{f \circ T}$, for every $f \in L_{pq}(\Omega, X)$.*
(3) *$\mu \ll \mu \circ T^{-1}$.*
(4) *$f_T$ is different from zero a.e.*
(5) *$\mu(\Omega \setminus T(\Omega)) = 0$.*

## 3. Weighted composition operators

Let $X$ be a Banach space and $(\Omega, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space. In this section, the assumption of measurability on $u$ refers to the norm of $\mathfrak{B}(X)$ as a Banach space i.e. $u : \Omega \to \mathfrak{B}(X)$ is called measurable if $f^{-1}(G)$ is measurable for each open subset $G$ of $\mathfrak{B}(X)$ with respect to the topology generated by the metric induced by the norm. $L_\infty(\Omega, \mathfrak{B}(X))$ represents the class of all essentially bounded measurable functions from $\Omega$ into $\mathfrak{B}(X)$. We denote the collection of all measurable functions from $\Omega$ into $\mathfrak{B}(X)$ which are strongly measurable by $\mathfrak{U}_0$.

Blasco and Neerven [5] introduced the notion of strongly $\mu$-normable by which $u : \Omega \to \mathfrak{B}(X)$ is said to be strongly $\mu$-normable if for each $\epsilon > 0$ there exists strongly measurable function $f_\epsilon : \Omega \to X$ such that, for almost every $\omega \in \Omega$, $\|f_\epsilon(\omega)\| \le 1$ and

$$\|u(\omega)\| \le \|u(\omega)f_\epsilon(\omega)\| + \epsilon.$$

We denote the collection of all measurable functions that are strongly $\mu$-normable by $\mathfrak{U}_1$.

A function $u : \Omega \to \mathfrak{B}(X)$ is said to have strong $\mu$-measurability of the orbits if for each $x \in X$, $\omega \to \|u(\omega)x\|$ is a measurable mapping. Collection of all measurable functions having strong $\mu$-measurability of the orbits is denoted by $\mathfrak{U}_2$.

Using [7, Lemma 1.1], it is clear that $\mathfrak{U}_0 \subseteq \mathfrak{U}_1$. However, if $X$ is a separable Banach space, then using [5, Corollary 2.3(1)], we find that $\mathfrak{U}_2 \subseteq \mathfrak{U}_1$.

Let $\mathfrak{U}_3$ denote the collection of measurable functions $u : \Omega \to \mathfrak{B}(X)$ satisfying the property that "if $E \subseteq S_u$, the support of u with $\mu(E) > 0$, then there exists a measurable subset F of E such that $\mu(F) > 0$ and u is constant over F". Clearly, $\mathfrak{U}_3$ contains all the simple functions.

**Proposition 3.1.** *If $u \in \mathfrak{U}_3$ is such that the set $\{\omega \in \Omega : \|u(\omega)\| > \delta\}$ has positive measure for some $\delta > 0$ then there exists a measurable subset $F$ of $\{\omega \in \Omega : \|u(\omega)\| > \delta\}$ with $\mu(F) > 0$ and some vector $x \in X$ with $\|x\| = 1$ and $\|u(\omega)x\| > \delta$, for all $\omega \in F$.*

**Proof.** Replace the set E by the set $\{\omega \in \Omega : \|u(\omega)\| > \delta\}$ and hence we find a subset $F$ of $E$ with $\mu(F) > 0$ such that for each $\omega \in F$, $u(\omega) = u(\omega_0)$ for some $\omega_0 \in F$. then we can easily find some vector $x_0 \in X$ with $\|x_0\| = 1$ and $\|u(\omega)x_0\| = \|u(\omega_0)x_0\| > \delta$, for each $\omega \in F$. ∎

**Proposition 3.2.** *If $u \in \mathfrak{U}_3$ is such that the set $\{\omega \in \Omega : \|u(\omega)\| > \delta\}$ has positive measure, for some $\delta > 0$ then there exists a measurable subset $F$ of $\{\omega \in \Omega : \|u(\omega)\| > \delta\}$ such that $\mu(F) > 0$ and a strongly measurable function $f : \Omega \to X$ such that $\|f(\omega)\| = 1$ and for all $\omega \in F$*

$$\|u(\omega)f(\omega)\| > \delta.$$

**Proof.** Let F be a measurable subset of E with $\mu(F) > 0$ and let $x \in X$ be such that $\|x\| = 1$ and $\|u(\omega)x\| > \delta$, for all $\omega \in F$. Thus the function

$f : \Omega \to X$ given by

$$f(\omega) = \begin{cases} x, & \text{if } \omega \in F, \\ 0, & \text{otherwise,} \end{cases}$$

is the desired function. ■

We refer the reader to [5] for more details on the collections $\mathfrak{U}_0, \mathfrak{U}_1$ and $\mathfrak{U}_2$. With the relations known, it is enough to extend the study for the cases $u \in \mathfrak{U}_1$ and $u \in \mathfrak{U}_3$. Proposition 3.1 of [5] can be simply stated as every $u : \Omega \to \mathfrak{B}(X)$ for which each mapping $\omega \to u(\omega)x$ corresponding to $x \in X$, is strongly $\mu$-measurable, induces a multiplication transformation $M_u : L_{pq}(\Omega, X) \to L_{pq}(\Omega, X)$, $1 < p \le \infty$, $1 \le q \le \infty$. However, in the coming results, we discuss the boundedness of the multiplication transformation $M_u$ induced by $u$ under various situations.

**Theorem 3.3.** *Let $u : \Omega \to \mathfrak{B}(X)$ be in $\mathfrak{U}_1$. Then $M_u : L_{pq}(\Omega, X) \mapsto L_{pq}(\Omega, X)$, $1 < p \le \infty$, $1 \le q \le \infty$, given by*

$$M_u f(\omega) = u(\omega)\big(f(\omega)\big),$$

*for all $\omega \in \Omega$ and $f \in L_{pq}(\Omega, X)$, is bounded if and only if $u \in L_\infty(\Omega, \mathfrak{B}(X))$.*

**Proof.** If $u \in L^\infty(\Omega, \mathfrak{B}(X))$ then simple computations give

$$\|M_u f\|^*(t) \le \|u\|_\infty \|f\|^*(t), \quad \text{for all } f \in L_{pq}(\Omega, X).$$

This implies that

$$\|M_u f\|_{pq} \le \|u\|_\infty \|f\|_{pq}.$$

Conversely, suppose that $M_u$ is a bounded operator on $L_{pq}(\Omega, X)$ induced by some $u \in \mathfrak{U}_1$. If possible $u$ is not in $L^\infty(\Omega, \mathfrak{B}(X))$. Then for each $n$ in $\mathbb{N}$, the set $E_n = \{\omega \in \Omega : \|u(\omega)\| > n\}$ has positive measure. By the definition of strong $\mu$-normable, we can find a strongly measurable function $f : \Omega \to X$ such that for almost every $\omega \in \Omega$, $\|f(\omega)\| \le 1$ and

$$\|u(\omega)\| \le \|u(\omega)f(\omega)\| + 1.$$

Now for each $n$ in $\mathbb{N}$, take $F_n$ as a measurable subset of $E_{n+1}$ with $0 < \mu(F_n) < \infty$ and define $f_n : \Omega \to X$ as

$$f_n(\omega) = \begin{cases} f(\omega), & \text{if } \omega \in F_n, \\ 0, & \text{otherwise.} \end{cases}$$

Then each $f_n$ is strongly measurable with $\|f_n\|_{pq} \le p'^{1/q}\big(\mu(F_n)\big)^{1/p}$ where $1/p + 1/p' = 1$. Also we have, for almost every $\omega$ in $F_n$,

$$\|u(\omega)f_n(\omega)\| + 1 \ge \|u(\omega)\| > (n+1).$$

This yields that

$$\|M_u f_n\|^*(t) \ge n\|f_n\|^*(t).$$

Therefore

$$\|M_u f_n\|_{pq} \geq n\|f_n\|_{pq}.$$

This contradicts the boundedness of $M_u$. This completes the proof. ∎

**Theorem 3.4.** *Let $u : \Omega \to \mathfrak{B}(X)$ be in $\mathfrak{U}_3$. Then $M_u : L_{pq}(\Omega, X) \mapsto L_{pq}(\Omega, X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$, given by*

$$M_u f(\omega) = u(\omega)\big(f(\omega)\big),$$

*for all $\omega \in \Omega$ and $f \in L_{pq}(\Omega, X)$, is bounded if and only if $u \in L_\infty(\Omega, \mathfrak{B}(X))$.*

**Proof.** Proof is almost along the same lines as in Theorem 3.3. However, in this case, in the proof of the necessary part, we use Proposition 3.2 to obtain the required strongly measurable function $f_n : \Omega \to X$ and measurable subset $F_n$ of $E_n$ with $0 < \mu(F_n) < \infty$ satisfying $\|f(\omega)\| = 1$ and for all $\omega \in F_n$

$$\|u(\omega)f_n(\omega)\| > n\|f_n(\omega)\|. \ ∎$$

Theorem 3.3 is independently proved in [2] when u is a strongly measurable mapping, which follows from Theorem 3.3 as $\mathfrak{U}_0 \subseteq \mathfrak{U}_1$. If the space $X$ under consideration is a separable Banach space then $\mathfrak{U}_2 \subseteq \mathfrak{U}_1$ and hence Theorem 3.2 holds even when $u \in \mathfrak{U}_2$.

With these observations, we proceed towards the applications of these mappings in the study of weighted composition operator $W_{u,T}$ on $L_{pq}(\Omega, X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ given by

$$(W_{u,T}f)(\omega) = u(T(\omega))\big(f(T(\omega))\big),$$

for all $\omega \in \Omega$ and $f \in L_{pq}(\Omega, X)$. We are moving ahead with the assumption that $u \in \mathfrak{U}_1 \cup \mathfrak{U}_3$.

Although $W_{u,T} = M_{u \circ T} C_T$, one can still find $u$ and $T$ inducing a bounded operator $W_{u,T}$ and not inducing $C_T$. For, if $u \equiv 0$ and $T$ is such that $f_T$ does not belong to $L_\infty(\mu)$ then $C_T$ can not be a well defined bounded operator on $L_{pq}(\Omega, X)$ where as $W_{u,T} \equiv 0$ is such.

**Theorem 3.5.** *Let $u \in \mathfrak{U}_1 \cup \mathfrak{U}_3$ be a mapping inducing the multiplication transformation $M_u : L_{pq}(\Omega, X) \mapsto L_{pq}(\Omega, X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ and $T : \Omega \mapsto \Omega$ be a non-singular measurable transformation such that the Radon–Nikodym derivative $f_T = d(\mu T^{-1})/d\mu \in L_\infty(\mu)$. Then the linear transformation $W_{u,T} : L_{pq}(\Omega, X) \mapsto L_{pq}(\Omega, X)$ given by*

$$(W_{u,T}f)(\omega) = u(T(\omega))\big(f(T(\omega))\big),$$

*for all $\omega \in \Omega$ and $f \in L_{pq}(\Omega, X)$ is bounded if $u \in L_\infty(\Omega, \mathfrak{B}(X))$. However, in case $f_T \geq 1$ almost everywhere on $S_u$, the support of $u$, then the converse also holds.*

**Proof.** Let $u \in L_\infty(\Omega, \mathfrak{B}(X))$. If $f_T \equiv 0$ then $W_{u,T} = 0$. If $0 \neq f_T \in L_\infty(\mu)$, then for each $f \in L_{pq}(\Omega, X)$,

$$\|W_{u,T}f\|^*(t) \leq \|u\|_\infty \|f\|^* \left(\frac{t}{f_T}\right),$$

which provides

$$\|W_{u,T}f\|_{pq}^q \leq \|u\|_\infty^q \|f_T\|_\infty^{q/p} \|f\|_{pq}^q$$

so that

$$\|W_{u,T}\| \leq \|u\|_\infty \|f_T\|_\infty^{1/p}.$$

However, if $f_T \geq 1$ almost everywhere on $S_u$, the support of u and $W_{u,T}$ is bounded on $L_{pq}(\Omega, X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$, then assuming that $u \notin L_\infty(\Omega, \mathfrak{B}(X))$, for each natural number $n$, $E_n = \{\omega \in \Omega : \|u(\omega)\| > n\}$ has positive measure. Since $u \in \mathfrak{U}_1 \cup \mathfrak{U}_3$, by applying the Theorem 3.3 or Theorem 3.4, we can assume that we have a measurable subset $F_n$ of $E_n$ with $0 < \mu(F_n) < \infty$ and a strongly measurable function $f_n \in L_{pq}(\Omega, X)$ such that $\|u(\omega)f_n(\omega)\| \geq n\|f_n(\omega)\|$, for all $\omega \in \Omega$. Thus

$$\|u(T(\omega))f_n(T(\omega))\| \geq n\|f_n(T(\omega))\|,$$

for all $\omega \in \Omega$. As $f_T \geq 1$, we have for $t > 0$,

$$\|W_{u,T}f_n\|^*(t) \geq n\|f_n\|^*(t)$$

and hence

$$\|W_{u,T}f_n\|_{pq} \geq n\|f_n\|_{pq}.$$

This contradicts the boundedness of $W_{u,T}$. Hence for the boundedness of $W_{u,T}$, $u$ must be in $L_\infty(\Omega, \mathfrak{B}(X))$. ∎

Without any extra efforts, we can further improve the last theorem as follows.

**Theorem 3.6.** *Let $u : \Omega \mapsto \mathfrak{B}(X)$ be in $\mathfrak{U}_1 \cup \mathfrak{U}_3$ and $T : \Omega \mapsto \Omega$ be a non-singular measurable transformation such that the Randon–Nikodym derivative $f_T = d(\mu T^{-1})/d\mu \in L_\infty(\mu)$ and $f_T \geq \delta$ a.e. on the support of $u$, for some $\delta > 0$. Then $W_{u,T}$ on $L_{pq}(\Omega, X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$, is bounded if and only if $u \in L_\infty(\Omega, \mathfrak{B}(X))$.*

**Proof.** Under the hypothesis, we obtain that for each $f \in L_{pq}(\Omega, X)$,

$$\delta^{\frac{1}{p}} \|M_u f\|_{pq} \leq \|W_{u,T}f\|_{pq} \leq \|f_T\|^{\frac{1}{p}} \|M_u f\|_{pq},$$

which, on applying Theorem 3.3, yields the result. ∎

**Theorem 3.7.** *Let $\mu$ be a non-atomic measure. Let $u : \Omega \to \mathcal{B}(X)$ and $T : \Omega \to \Omega$ are such that $W_{u,T} \in \mathfrak{B}(L_{pq}(\Omega, X))$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. Then $T$ is surjective and $u(T(\omega)) \neq 0$, for almost every $\omega \in \Omega$ if $W_{u,T}$ is injective.*

**Proof.** Suppose $W_{u,T}$ is injective. Let $x_0 \in X$ be fixed such that $\|x_0\| = 1$. If $T$ is not surjective, then we can find a measurable subset $E$ of $\Omega \setminus T(\Omega)$ such that $0 < \mu(E) < \infty$. Now define $f_E : \Omega \to X$ as

$$f_E(\omega) = \begin{cases} x_0, & \text{if } \omega \in E, \\ 0, & \text{if } \omega \notin E, \end{cases}$$

then $0 \neq f_E \in L_{pq}(\Omega, X)$ and $W_{u,T} f_E = 0$. This contradicts the injectiveness of $W_{u,T}$ and hence $T$ is surjective. Further if $E = \{\omega \in \Omega : u(T(\omega)) = 0\}$ has positive measure then we can find a measurable set $A$ such that $T^{-1}(A) \subset E$ and $0 < \mu(A) < \infty$. Then define $f_A : \Omega \to X$ as

$$f_A(\omega) = \begin{cases} x_0, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A. \end{cases}$$

Then $0 \neq f_A \in L_{pq}(\Omega, X)$ and for $t > 0$ we have $\|W_{u,T} f_A\|^*(t) = 0$, so that $W_{u,T} f_A = 0$. This is a contradiction, hence $u(T(\omega)) \neq 0$, for almost every $\omega \in \Omega$. ∎

Converse of the Theorem 3.7 is not true and can be verified through the following example:

**Example 3.8.** Let $\Omega = (0, 1)$, $\mu$ is Lebesgue measure. $X = \mathbb{R}^2$ and let $P$ be the operator defined on $\mathbb{R}^2$ as $P(x, y) = (x, 0)$, for all $(x, y) \in \mathbb{R}^2$. Then $P \in \mathcal{B}(X)$ and $\text{Kernel}(P) = \{(0, y) : y \in \mathbb{R}^2\}$.

Define $u : \Omega \to \mathcal{B}(X)$ as $u(\omega) = P \ \forall \omega \in \Omega$ and $T : \Omega \to \Omega$ as $T(\omega) = \omega \forall \omega \in \Omega$. Then $\mu$ is non-atomic, $T$ is non-singular, $f_T (\equiv 1) \in L_\infty(\mu)$, $u \in L_\infty(\Omega, \mathcal{B}(X))$ is strongly measurable mapping.

Hence $W_{u,T} \in \mathcal{B}(L_{pq}(\Omega, \mathbb{R}^2))$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. Also, $T$ is surjective and $u(T(\omega)) \neq 0$, for all $\omega \in \Omega$.

For each $\omega \in \Omega$, we define $f_\omega : \Omega \to \mathbb{R}^2$ as $f_\omega(x) = (0, \omega) \ \forall x \in \Omega$. Then each $f_\omega \in L_{pq}(\Omega, \mathbb{R}^2)$ with $\|f_\omega\|_{pq} = \omega (p')^{1/q}$ where $\frac{1}{p} + \frac{1}{p'} = 1$, but $W_{u,T} f_\omega = 0$. Hence $W_{u,T}$ is not injective.

**Theorem 3.9.** *Let $u : \Omega \to \mathcal{B}(X)$ and $T : \Omega \to \Omega$ are such that $W_{u,T} \in \mathcal{B}(L_{pq}(\Omega, X))$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. If $T$ is surjective and for almost every $\omega \in \Omega$ there exists $k_\omega > 0$ such that*

$$\|u(T(\omega))x\| \geq k_\omega \|x\|, \quad \textit{for all } x \in X$$

*then $W_{u,T}$ is surjective.*

**Proof.** If $W_{u,T} f = 0$ for $f \in L_{pq}(\Omega, X)$ then $u(T(\omega))\big(f(T(\omega))\big) = 0$ for almost every $\omega \in \Omega$. Under the hypothesis this gives $f(T(\omega)) = 0$ for almost every $\omega \in \Omega$. $T$ being surjective, we find that $f = 0$ so that $W_{u,T}$ is injective. ∎

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