DEMONSTRATIO MATHEMATICA
Vol. XLVI No 4 2013
10.1515/dema-2013-0487

Harun Karsli

SOME CONVERGENCE RESULTS FOR NONLINEAR SINGULAR INTEGRAL OPERATORS

Abstract. In this paper, we establish some pointwise convergence results for a family of certain nonlinear singular integral operators $T_\lambda f$ of the form

$$(T_\lambda f)(x) = \int_a^b K_\lambda(t - x, f(t))\,dt, \quad x \in (a, b),$$

acting on functions with bounded (Jordan) variation on an interval $[a, b]$, as $\lambda \to \lambda_0$. Here, the kernels $\mathbb{K} = \{K_\lambda\}_{\lambda \in \Lambda}$ satisfy some suitable singularity assumptions. We remark that the present study is a continuation and extension of the study of pointwise approximation of the family of nonlinear singular integral operators (1) begun in [18].

1. Introduction

Let $I \subset \mathbb{R}$ be a bounded or unbounded interval. As for the notation, throughout this paper, $V_I(f)$ stands for the total (Jordan) variation of the real-valued function defined on I. The class of all functions of bounded (Jordan) variation on I will be denoted by $BV(I)$. Especially, this kind of spaces and approximation problems via various positive linear operators were extensively studied in [10]–[14] and [23]. Let us observe that Shaw et al. [21] investigated this problem for the general family of positive linear operators which include Bernstein, Kantorovich and Durrmeyer operators as special cases. Later on, in 2003, Hua and Shaw [15] proved the approximation problems for the linear integral operators whose kernels are not necessarily positive.

The present paper concerns with pointwise convergence of certain families of nonlinear singular integral operators $T_\lambda f$ of the form

$$(1) \qquad (T_\lambda f)(x) = \int_a^b K_\lambda(t - x, f(t))\,dt, \quad x \in (a, b),$$

2010 *Mathematics Subject Classification*: 41A35, 41A25, 47G10.
Key words and phrases: nonlinear singular integral, bounded variation, Lipschitz condition, pointwise convergence.

acting on functions with bounded (Jordan) variation on an interval $[a,b]$, where K_λ satisfy some suitable assumptions. In particular, we obtain the rate of pointwise convergence for the nonlinear family of singular integral operators (1) to the point x, having a discontinuity of the first kind and at the Lebesgue points of f, as $\lambda \to \lambda_0$. We remark that the present study is a continuation and extension of the study of pointwise approximation of a family of nonlinear integral operators (1) begun in [18], where the kernel functions K_λ satisfy the strong Lipschitz condition, namely

$$|K_\lambda(t,u) - K_\lambda(t,v)| \leq L_\lambda(t)|u-v|,$$

for every $t,u,v \in \mathbb{R}$ and for any $\lambda \in \Lambda$.

We note that the approximation theory with nonlinear integral operators of convolution type was introduced by J. Musielak in [20] and widely developed in [4]. To the best of my knowledge, the approximation problem were limited to linear operators because the notion of singularity of an integral operator is closely connected with its linearity until the fundamental paper of Musielak [20]. In [20], the assumption of linearity of the singular integral operators was replaced by an assumption of a Lipschitz condition for the kernel function $K_\lambda(t,u)$ with respect to the second variable. After this approach, several mathematicians have undertaken the program of extending approximation by nonlinear integral operators in many ways, including pointwise and uniform convergence, Korovkin type theorems in abstract function spaces, sampling series and so on. Especially, operators of type (1) and its special cases were studied by Swiderski–Wachnicki [22], Karsli [16], [17] and Karsli–Ibikli [19] in some Lebesgue spaces.

Such developments delineated a theory which is nowadays referred to as the theory of approximation by nonlinear integral operators.

For further reading, we also refer the reader to [1]–[2], [5]–[9] as well as the monographs [12] and [4] where other kinds of convergence results of linear and nonlinear singular integral operators in the Lebesgue and Musielak–Orlicz spaces have been considered. Finally, in the very recent paper due to Angeloni and Vinti [3], some approximation properties with respect to the multidimensional φ-variation for the linear cases of the operators of type (1) have been studied.

An outline of the paper is as follows: The next section contains basic definitions and notations. In Section 3, the main approximation result of this study are given. In Section 4, we give some certain results which are necessary to prove the main result. The final section, that is Section 5, deals with the proof of the main result presented in Section 3.

2. Preliminaries

In this section, we assemble the main definitions and notations which will be used throughout the paper.

Let Λ be a nonempty set of positive indices with a topology and λ_0 be an accumulation point of Λ in this topology. Let X be the set of all Lebesgue measurable functions $f : [a,b] \to \mathbb{R}$.

Throughout this paper, we assume that $\mu : \Lambda \to \mathbb{R}^+$ is an increasing and continuous function such that $\lim_{\lambda\to\lambda_0} \mu(\lambda) = \infty$.

Let Ψ be the class of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that the function ψ is continuous and concave with $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$.

We now introduce a family of kernel functions. Let $\{K_\lambda\}_{\lambda\in\Lambda}$ be a family of Lebesgue measurable functions $K_\lambda : \mathbb{R}\,\mathrm{x}\,\mathbb{R} \to \mathbb{R}$ defined by

$$K_\lambda(t,u) = L_\lambda(t)H_\lambda(u),$$

for every $t, u \in \mathbb{R}$, where $H_\lambda : \mathbb{R} \to \mathbb{R}$ is such that $H_\lambda(0) = 0$ and $L_\lambda : \mathbb{R} \to \mathbb{R}_0^+$ is a Lebesgue integrable function, for every $\lambda \in \Lambda$.

First of all, we assume that the following conditions hold:

a) $H_\lambda : \mathbb{R} \to \mathbb{R}$ is such that

$$|H_\lambda(u) - H_\lambda(v)| \le \psi(|u - v|), \quad \psi \in \Psi,$$

holds for every $u, v \in \mathbb{R}$, for every $\lambda \in \Lambda$.

b) Let

$$\int_{\mathbb{R}} L_\lambda(t)dt = 1, \quad \text{for every } \lambda \in \Lambda.$$

We now set

$$B_\lambda(x) := \int_{x-(x-a)/\mu(\lambda)}^{x+(b-x)/\mu(\lambda)} L_\lambda(t)dt \quad \text{for any fixed } x \in (a,b).$$

We note that the use of the function $B_\lambda(x)$ concerns the behavior of the approximation near to the point x. Similar approach and some particular examples can be found in [15], [18] and [21].

c) For any fixed $\delta > 0$,

$$\lim_{\lambda\to\lambda_0} \int_{|t|\ge\delta} L_\lambda(t)\,dt = 0 \quad \text{and} \quad \lim_{\lambda\to\lambda_0} \Big[\sup_{|t|\ge\delta} L_\lambda(t)\Big] = 0.$$

d) Denoting by $r_\lambda(u) := H_\lambda(u) - u$, $u \in \mathbb{R}$ and $\lambda \in \Lambda$, such that

$$\lim_{\lambda\to\lambda_0} r_\lambda(u) = 0$$

uniformly with respect to u. In other words, for λ sufficiently close to the accumulation point λ_0

$$\sup_u |r_\lambda(u)| = \sup_u |H_\lambda(u) - u| \leq \frac{1}{\mu(\lambda)},$$

holds.

e) There exists a $\delta_0 > 0$, such that $L_\lambda(t)$ is non-decreasing on $(-\delta_0, 0]$ and non-increasing on $[0, \delta_0)$ as a function of t, for each $\lambda \in \Lambda$.

As in some previous papers [16], [17], we introduce a function $\widetilde{f}$ defined on $\mathbb{R}$ as

$$\widetilde{f}(t) := \begin{cases} f(t), & t \in [a, b], \\ 0, & t \notin [a, b]. \end{cases} \tag{2}$$

The symbol $[a]$ will denote the greatest integer not greater than a.

3. Convergence results

We will consider the following family of nonlinear integral operators,

$$(T_\lambda f)(x) = \int_a^b K_\lambda(t - x, f(t))\, dt, \quad x \in (a, b),\ \lambda \in \Lambda,$$

defined for every $f \in X$ for which $T_\lambda f$ is well-defined, where

$$K_\lambda(t, u) = L_\lambda(t) H_\lambda(u),$$

for every $t, u \in \mathbb{R}$. Some approximation properties, such as convergence and rate of convergence in the variation seminorm, were obtained by L. Angeloni and G. Vinti 2007 in [2].

We are now ready to establish the main results of this study:

Theorem 1. *Let $\psi \in \Psi$ and $f \in L_1([a, b])$ be such that $\psi o|f| \in BV([a, b])$. Suppose that $K_\lambda(t, u)$ satisfies conditions* a)–e). *Then for every $x \in (a, b)$, and for λ sufficiently close to the accumulation point λ_0 of Λ, we have*

$$\left|(T_\lambda f)(x) - \left[\psi\left(\left|\frac{f(x+) + f(x-)}{2}\right|\right) + \psi\left(\left|\frac{f(x+) - f(x-)}{2}\right|\right)\right]\right|$$

$$\leq \frac{B^*_\lambda(x)}{\mu(\lambda)} \left[\bigvee_a^b \psi(|f_x|) + \sum_{k=1}^{[\mu^\beta(\lambda)]} \bigvee_{x-(x-a)/k^{1/\beta}}^{x+(b-x)/k^{1/\beta}} \psi(|f_x|)\right] + B_\lambda(x) \bigvee_{x-(x-a)/\mu(\lambda)}^{x+(b-x)/\mu(\lambda)} \psi(|f_x|)$$

where $B^*_\lambda(x) = B_\lambda(x) \max\{(x-a)^{-\beta}, (b-x)^{-\beta}\}$, $(\beta > 0)$,

$$f_x(t) = \begin{cases} f(t) - f(x+), & x < t \leq b, \\ 0, & t = x, \\ f(t) - f(x-), & a \leq t < x \end{cases}$$

and $\bigvee_a^b \psi(|f_x|)$ is the total variation of $\psi(|f_x|)$ on $[a, b]$.

Proof. The proof of Theorem 1 is similar to that of the Theorem presented in [18]. So we omit it. ∎

The following result is a corollary of Theorem 1.

Corollary 1. *If we choose* $\mu(\lambda) := \lambda^{\gamma}$ $(\gamma \geq 1)$, $\lambda_0 := \infty$ *and* $\psi(t) = t$, (*i.e., strong Lipschitz condition*) *in Theorem* 1, *then we have the result given in* [18].

Definition 1. A point $x_0 \in \mathbb{R}$ is called a Lebesgue point of the function f, if

$$\lim_{h \to 0^+} \frac{1}{h} \int_0^h |f(x_0 + t) - f(x_0)| \, dt = 0, \tag{3}$$

holds.

Theorem 2. *Let* $\psi \in \Psi$ *and* $f \in L_1([a,b])$ *be such that* $\psi o |f| \in BV([a,b])$. *Suppose that* $K_\lambda(t,u)$ *satisfies conditions* a)–e). *Then at each point* $x \in (a,b)$ *for which* (3) *holds, we have for each* $\varepsilon > 0$ *and* λ *sufficiently close to* λ_0

$$\begin{aligned} |(T_\lambda f)(x) - f(x)| \leq \frac{\varepsilon}{\mu(\lambda)} &\left[(b-x) L_\lambda \left(\frac{b-x}{\mu(\lambda)} \right) + (x-a) L_\lambda \left(\frac{a-x}{\mu(\lambda)} \right) \right] \\ &+ \frac{B_\lambda^*(x)}{\mu(\lambda)} \left[\bigvee_a^b \psi(|f_x|) + \sum_{k=1}^{[\mu^\beta(\lambda)]} \bigvee_{x-(x-a)/k^{1/\beta}}^{x+(b-x)/k^{1/\beta}} \psi(|f_x|) \right] + \frac{1}{\mu(\lambda)}, \end{aligned}$$

where $B_\lambda^*(x) = B_\lambda(x) \max \left\{ \frac{1}{(x-a)}, \frac{1}{(b-x)} \right\}$.

Now we are ready to establish a convergence result.

Theorem 3. *Let* $\psi \in \Psi$ *and* $f \in L_1([a,b])$ *be such that* $\psi o |f| \in BV([a,b])$. *Suppose that the kernel function* $K_\lambda(t,u)$ *satisfies conditions* a)–e). *Then at each point* $x \in (a,b)$ *for which* (3) *holds, we have*

$$\lim_{\lambda \to \lambda_0} |(T_\lambda f)(x) - f(x)| = 0.$$

Proof. From Theorem 2 and *c*) we reach the result, by the arbitrariness of $\varepsilon > 0$. ∎

Corollary 2. *Let* $\psi \in \Psi$ *and* $f \in L_1([a,b])$ *be such that* $\psi o |f| \in BV([a,b])$. *Suppose that the kernel function* $K_\lambda(t,u)$ *satisfies conditions* a)–e). *Then*

$$\lim_{\lambda \to \lambda_0} |(T_\lambda f)(x) - f(x)| = 0$$

holds almost everywhere in (a,b).

Since almost all $x \in (a,b)$ are Lebesgue points of the function f, then the assertion follows by Theorem 3.

4. Auxiliary result

In this section, we give certain results, which are necessary to prove our theorems.

Lemma 1. *For all $x \in (a,b)$ and for each $\lambda \in \Lambda$, let*

$$\int_a^b L_\lambda(u-x)\,|u-x|^\beta\,du \le \frac{B_\lambda(x)}{\mu(\lambda)}, \qquad (\beta > 0) \tag{4}$$

holds, where $L_\lambda(t)$ and $B_\lambda(x)$ are as defined in Section 2. Then

$$m_\lambda(x,t) := \int_a^t L_\lambda(u-x)du \le \frac{1}{(x-t)^\beta}\frac{B_\lambda(x)}{\mu(\lambda)}, \qquad a \le t < x, \tag{5}$$

and

$$1 - m_\lambda(x,z) = \int_z^b L_\lambda(u-x)du \le \frac{1}{(z-x)^\beta}\frac{B_\lambda(x)}{\mu(\lambda)}, \qquad x < z < b. \tag{6}$$

Proof. We have

$$m_\lambda(x,t) \le \int_a^t L_\lambda(u-x)\left(\frac{x-u}{x-t}\right)^\beta du \le \frac{1}{(x-t)^\beta}\int_a^b L_\lambda(u-x)\,|u-x|^\beta\,du.$$

According to (4), we have

$$m_\lambda(x,t) \le \frac{1}{(x-t)^\beta}\frac{B_\lambda(x)}{\mu(\lambda)}.$$

Proof of (6) is analogous. ∎

The following lemma is a slight modification of the Lemma 1 in [8].

Lemma 2. *Let $\psi \in \Psi$. Then, if $x_0 \in \mathbb{R}$ is a Lebesgue point of the function f, we have*

$$\left|\int_0^h \psi\left(|f(x_0+t) - f(x_0)|\right)dt\right| = o(|h|) \quad \text{as} \quad h \to 0. \tag{7}$$

Proof. In order to prove our lemma we will show the following two statements:

$$\left|\int_0^h \psi\left(|f(x_0+t) - f(x_0)|\right)dt\right| = o(h) \quad \text{as} \quad h \to 0^+,$$

$$\left|\int_h^0 \psi\left(|f(x_0+t) - f(x_0)|\right)dt\right| = o(-h) \quad \text{as} \quad h \to 0^-.$$

Since ψ is concave, one has for $h<0$ and $h>0$, respectively,

$$\frac{1}{-h}\int_{h}^{0}\psi\left(\left|f(x_0+t)-f(x_0)\right|\right)dt \le \psi\left(\frac{1}{-h}\int_{h}^{0}\left|f(x_0+t)-f(x_0)\right|dt\right)$$

and

$$\frac{1}{h}\int_{0}^{h}\psi\left(\left|f(x_0+t)-f(x_0)\right|\right)dt \le \psi\left(\frac{1}{h}\int_{0}^{h}\left|f(x_0+t)-f(x_0)\right|dt\right).$$

Hence, by continuity of ψ and $\psi(0)=0$, we reach the desired result. ■

5. Proof of the main Theorem

Proof of Theorem 2. Suppose that

$$x+\delta<b,\ \ x-\delta>a, \tag{8}$$

for any $0<\delta$.

Let $|I(x,\lambda)| := |(T_\lambda f)(x)-f(x)|$, that is

$$|I(x,\lambda)| = \left|\int_a^b K_\lambda(t-x,f(t))dt\ -f(x)\right|.$$

From (2) and d), we can rewrite $|I(x,\lambda)|$ as follows:

$$\begin{aligned}|I(x,\lambda)| &= \left|\int_{\mathbb{R}} K_\lambda(t-x,\widetilde{f}(t))dt\ -f(x)\right| \\ &= \left|\int_{\mathbb{R}} K_\lambda(t-x,\widetilde{f}(t))dt - \int_{\mathbb{R}} K_\lambda(t-x,\widetilde{f}(x))dt\right| \\ &\quad + \left|\int_{\mathbb{R}} K_\lambda(t-x,\widetilde{f}(x))dt - f(x)\right|.\end{aligned}$$

From conditions b) and d), it is easy to see that the second term of the righthandside of the above inequality is less than or equal to $1/\mu(\lambda)$. Indeed;

$$\begin{aligned}\left|\int_{\mathbb{R}} K_\lambda(t-x,\widetilde{f}(x))dt - f(x)\right| &= \left|\int_{\mathbb{R}} L_\lambda(t-x)H_\lambda(\widetilde{f}(x))dt - f(x)\int_{\mathbb{R}} L_\lambda(t)dt\right| \\ &= |H_\lambda(f(x))-f(x)|\left|\int_{\mathbb{R}} L_\lambda(t)dt\right| \le \frac{1}{\mu(\lambda)}\end{aligned}$$

holds for λ sufficiently close to the accumulation point λ_0.

As to the first term, by a), we have the following inequality,

$$|I(x,\lambda)| \le \int_{\mathbb{R}} L_\lambda(t-x)\psi(|\widetilde{f}(t)-\widetilde{f}(x)|)dt.$$

According to b), we can split the last integral in three terms as follows:

$$\begin{aligned}|I(x,\lambda)| &\leq \left(\int\limits_{a}^{x-(x-a)/\mu(\lambda)} + \int\limits_{x-(x-a)/\mu(\lambda)}^{x+(b-x)/\mu(\lambda)} + \int\limits_{x+(b-x)/\mu(\lambda)}^{b} \right) \\ &\qquad \times L_\lambda(t-x)\psi\left(|f(t)-f(x)|\right)dt \\ &:= I_1(\lambda,x)+I_2(\lambda,x)+I_3(\lambda,x).\end{aligned}$$

We shall evaluate $I_1(\lambda,x), I_2(\lambda,x)$ and $I_3(\lambda,x)$. To do this, we first observe that $I_1(\lambda,x), I_2(\lambda,x)$ and $I_3(\lambda,x)$ can be written as a Lebesgue–Stieltjes integral as follows:

$$|I_1(\lambda,x)| = \int\limits_{a}^{x-(x-a)/\mu(\lambda)} \psi\left(|f(t)-f(x)|\right) d_t\left(m_\lambda(x,t)\right),$$

$$|I_2(\lambda,x)| = \int\limits_{x-(x-a)/\mu(\lambda)}^{x+(b-x)/\mu(\lambda)} \psi\left(|f(t)-f(x)|\right) L_\lambda(t-x)dt$$

and

$$|I_3(\lambda,x)| = \int\limits_{x+(b-x)/\mu(\lambda)}^{b} \psi\left(|f(t)-f(x)|\right) d_t\left(m_\lambda(x,t)\right).$$

First, we estimate $I_2(\lambda,x)$. We have, for $t\in[x-(x-a)/\mu(\lambda), x+(b-x)/\mu(\lambda)]$

$$\begin{aligned}|I_2(\lambda,x)| &= \int\limits_{x-(x-a)/\mu(\lambda)}^{x+(b-x)/\mu(\lambda)} \psi\left(|f(t)-f(x)|\right) L_\lambda(t-x)dt \\ &\leq \int\limits_{x-(x-a)/\mu(\lambda)}^{x} \psi\left(|f(t)-f(x)|\right) L_\lambda(t-x)dt \\ &\quad + \int\limits_{x}^{x+(b-x)/\mu(\lambda)} \psi\left(|f(t)-f(x)|\right) L_\lambda(t-x)dt \\ &= I_{2,1}(\lambda,x)+I_{2,2}(\lambda,x).\end{aligned}$$

Setting

$$F(t) := \int\limits_{t}^{x} \psi\left(|f(y)-f(x)|\right) dy,$$

then, according to (7), for each $\varepsilon>0$ there exists $\delta>0$ such that

$$F(t)\leq \varepsilon\,(x-t), \tag{9}$$

for all $0<x-t\leq\delta$.

We now fix this δ and estimate $I_{2,1}(\lambda,x)$ and $I_{2,2}(\lambda,x)$ respectively.

Now, we recall the Lebesgue–Stieltjes integral representation, so we can write $I_{2,1}(\lambda, x)$ as

$$I_{2,1}(\lambda) = \int_{x-(x-a)/\mu(\lambda)}^{x} L_\lambda(t-x)dF(t). \tag{10}$$

Applying partial Lebesgue–Stieltjes integration (10) and using (9), we obtain

$$\begin{aligned}
I_{2,1}(\lambda) &= -F(x-(x-a)/\mu(\lambda))\, L_\lambda(-(x-a)/\mu(\lambda)) \\
&\quad + \int_{x-(x-a)/\mu(\lambda)}^{x} F(t)\,\frac{\partial}{\partial t}L_\lambda(t-x)\,dt \\
&\le -F(x-(x-a)/\mu(\lambda))L_\lambda(-(x-a)/\mu(\lambda)) \\
&\quad + \int_{x-(x-a)/\mu(\lambda)}^{x} F(t)\frac{\partial}{\partial t}L_\lambda(t-x)\,dt \\
&\le \varepsilon\ (x-a)/\mu(\lambda)\, L_\lambda(-(x-a)/\mu(\lambda)) \\
&\quad + \varepsilon \int_{x-(x-a)/\mu(\lambda)}^{x} (x-t)\frac{\partial}{\partial t}L_\lambda(t-x)\,dt.
\end{aligned}$$

Integration by parts again gives

$$\begin{aligned}
I_{2,1}(\lambda) &\le \varepsilon\ (x-a)/\mu(\lambda)\, L_\lambda(-(x-a)/\mu(\lambda)) \\
&\quad + \varepsilon\left\{-(x-a)/\mu(\lambda)L_\lambda(-(x-a)/\mu(\lambda)) - \int_{x-(x-a)/\mu(\lambda)}^{x} L_\lambda(t-x)\,dt\right\} \\
&= \varepsilon \int_{-(x-a)/\mu(\lambda)}^{0} L_\lambda(t)\,dt.
\end{aligned}$$

Setting

$$I_{2,1,1}(x,\lambda) := \int_{-(x-a)/\mu(\lambda)}^{0} L_\lambda(t)\,dt,$$

according to e) and (8), we can now obtain the following estimate:

$$\begin{aligned}
I_{2,1}(x,\lambda) = \varepsilon I_{2,1,1}(x,\lambda) &= \varepsilon L_\lambda(-(x-a)/\mu(\lambda)) \int_{-(x-a)/\mu(\lambda)}^{0} dt \\
&= \varepsilon L_\lambda\left(-\frac{x-a}{\mu(\lambda)}\right)\frac{x-a}{\mu(\lambda)}.
\end{aligned}$$

We can use a similar method for $I_{2,2}(x,\lambda)$. Then, we find the following inequality

$$I_{2,2}(\lambda) \le \varepsilon L_\lambda \left(\frac{b-x}{\mu(\lambda)}\right) \frac{b-x}{\mu(\lambda)}.$$

Next, we estimate $I_1(\lambda, x)$. Using partial Lebesgue–Stieltjes integration, we obtain

$$\begin{aligned}|I_1(\lambda,x)| &= \int_a^{x-(x-a)/\mu(\lambda)} \psi\left(|f_x(t)|\right) d_t\left(m_\lambda(x,t)\right)\\ &= \psi\left(\left|f_x\left(x-\frac{(x-a)}{\mu(\lambda)}\right)\right|\right) m_\lambda\left(x, x-\frac{(x-a)}{\mu(\lambda)}\right)\\ &\quad - \int_a^{x-(x-a)/\mu(\lambda)} m_\lambda(x,t) d_t(\psi(|f_x(t)|)).\end{aligned}$$

Let $y = x - (x-a)/\mu(\lambda)$. By (5), it is clear that

$$m_\lambda(x,y) \le B_\lambda(x)\,(x-a)^{-\beta}\,\mu^{\beta-1}(\lambda). \tag{11}$$

Here we note that

$$\begin{aligned}\psi\left(\left|f_x\left(x-\frac{(x-a)}{\mu(\lambda)}\right)\right|\right) &= \left|\psi\left(\left|f_x\left(x-\frac{(x-a)}{\mu(\lambda)}\right)\right|\right) - \psi\left(|f_x(x)|\right)\right|\\ &\le \bigvee_{x-(x-a)/\mu(\lambda)}^{x} \psi(|f_x|).\end{aligned}$$

Using partial integration and applying (11), we obtain

$$\begin{aligned}|I_1(\lambda,x)| \le &\bigvee_{x-(x-a)/\mu(\lambda)}^{x} \psi(|f_x|)\left|m_\lambda\left(x, x-\frac{(x-a)}{\mu(\lambda)}\right)\right|\\ &+ \int_a^{x-(x-a)/\mu(\lambda)} m_\lambda(x,t) d_t\left(-\bigvee_t^x \psi(|f_x|)\right)\\ \le &\bigvee_{x-(x-a)/\mu(\lambda)}^{x} \psi(|f_x|) B_\lambda(x)\,(x-a)^{-\beta}\,\mu^{\beta-1}(\lambda)\\ &+ \frac{B_\lambda(x)}{\mu(\lambda)} \int_a^{x-(x-a)/\mu(\lambda)} (x-t)^{-\beta} d_t\left(-\bigvee_t^x \psi(f_x)\right)\\ = &\bigvee_{x-(x-a)/\mu(\lambda)}^{x} \psi(f_x) B_\lambda(x)\,(x-a)^{-\beta}\,\mu^{\beta-1}(\lambda)\\ &+ \frac{B_\lambda(x)}{\mu(\lambda)}\Bigg[-(x-a)^{-\beta}\,\mu^{\beta}(\lambda) \bigvee_{x-(x-a)/\mu(\lambda)}^{x} \psi(|f_x|)\end{aligned}$$

$$+ (x-a)^{-\beta} \bigvee_a^x \psi(|f_x|) + \int_a^{x-(x-a)/\mu(\lambda)} \bigvee_t^x \psi(|f_x|) \frac{\beta}{(x-t)^{\beta+1}} dt \Bigg]$$

$$= \frac{B_\lambda(x)}{\mu(\lambda)} \left[(x-a)^{-\beta} \bigvee_a^x \psi(|f_x|) + \int_a^{x-(x-a)/\mu(\lambda)} \bigvee_t^x \psi(|f_x|) \frac{\beta}{(x-t)^{\beta+1}} dt \right].$$

Changing the variable t by $x-(x-a)/u^{1/\beta}$ in the last integral, we have

$$\int_a^{x-(x-a)/\mu(\lambda)} \bigvee_t^x \psi(|f_x|) \frac{\beta}{(x-t)^{\beta+1}} dt = \frac{1}{(x-a)^\beta} \int_1^{\mu^\beta(\lambda)} \bigvee_{x-(x-a)/u^{1/\beta}}^x \psi(|f_x|)\, du$$

$$\leq \frac{1}{(x-a)^\beta} \sum_{k=1}^{[\mu^\beta(\lambda)]} \bigvee_{x-(x-a)/k^{1/\beta}}^x \psi(|f_x|).$$

Consequently, we obtain

$$|I_1(\lambda,x)| \leq \frac{B_\lambda(x)}{\mu(\lambda)} (x-a)^{-\beta} \left[\bigvee_a^x \psi(|f_x|) + \sum_{k=1}^{[\mu^\beta(\lambda)]} \bigvee_{x-(x-a)/k^{1/\beta}}^x \psi(|f_x|) \right].$$

Using a similar method, we can find

$$|I_3(\lambda,x)| \leq \frac{B_\lambda(x)}{\mu(\lambda)} (b-x)^{-\beta} \left[\bigvee_x^b \psi(|f_x|) + \sum_{k=1}^{[\mu^\beta(\lambda)]} \bigvee_x^{x+(b-x)/k^{1/\beta}} \psi(|f_x|) \right].$$

Collecting the above estimates we get the required result. ■

Acknowledgment. The author would like to thank in advance the referee for his/her critical comments and remarks, which have definitely improved the quality of this study.

References

[1] L. Angeloni, G. Vinti, *Convergence in variation and rate of approximation for nonlinear integral operators of convolution type*, Results Math. 49 (2006), 1–23. Erratum: 57 (2010), 387–391.

[2] L. Angeloni, G. Vinti, *Approximation by means of nonlinear integral operators in the space of functions with bounded φ-variation*, Differential Integral Equations 20 (2007), 339–360. Erratum: 23(7–8) (2010), 795–799.

[3] L. Angeloni, G. Vinti, *Convergence and rate of approximation for linear integral operators in BV_φ-spaces in multidimensional setting*, J. Math. Anal. Appl. 349 (2009), 317–334.

[4] C. Bardaro, J. Musielak, G. Vinti, *Nonlinear Integral Operators and Applications*, de Gruyter Ser. Nonlinear Anal. Appl. Vol. 9, 2003.

[5] C. Bardaro, P. L. Butzer, R. L. Stens, G. Vinti, *Convergence in variation and rates of approximation for Bernstein-type polynomials and singular convolution integrals*, Analysis (Munich) 23(4) (2003), 299–346.
[6] C. Bardaro, H. Karsli, G. Vinti, *On pointwise convergence of linear integral operators with homogeneous kernels*, Integral Transforms Spec. Funct. 19(6) (2008), 429–439.
[7] C. Bardaro, H. Karsli, G. Vinti, *Nonlinear integral operators with homogeneous kernels: pointwise approximation theorems*, Appl. Anal. 90(3,4) (2011), 463–474.
[8] C. Bardaro, I. Mantellini, *Pointwise convergence theorems for nonlinear Mellin convolution operators*, Int. J. Pure Appl. Math. 27(4) (2006), 431–447.
[9] C. Bardaro, S. Sciamannini, G. Vinti, *Convergence in BV_φ by nonlinear Mellin-type convolution operators*, Funct. Approx. 29 (2001), 17–28.
[10] R. Bojanic, F. Cheng, *Estimates for the rate of approximation of functions of bounded variation by Hermite-Fej*ér *polynomials*, Second Edmonton Conference on Approximation Theory, Edmonton, Alta., 1982, 5–17.
[11] R. Bojanic, M. Vuilleumier, *On the rate of convergence of Fourier Legendre series of functions of bounded variation*, J. Approx. Theory 31 (1981), 67–79.
[12] P. L. Butzer, R. J. Nessel, *Fourier Analysis and Approximation*, Vol. 1, Academic Press, New York, London, 1971.
[13] F. Cheng, *On the rate of convergence of Bernstein polynomials of functions of bounded variation*, J. Approx. Theory 39 (1983), 259–274.
[14] V. Gupta, M. K. Gupta, *Rate of convergence for certain families of summation-integral type operators*, J. Math. Anal. Appl. 296 (2004), 608–618.
[15] Y. H. Hua, S. Y. Shaw, *Rate of approximation for functions of bounded variation by integral operators*, Period. Math. Hungar. 46(1) (2003), 41–60.
[16] H. Karsli, *Convergence and rate of convergence by nonlinear singular integral operators depending on two parameters*, Appl. Anal. 85(6,7) (2006), 781–791.
[17] H. Karsli, *On approximation properties of a class of convolution type nonlinear singular integral operators*, Georgian Math. J. 15(1) (2008), 77–86.
[18] H. Karsli, V. Gupta, *Rate of convergence by nonlinear integral operators for functions of bounded variation*, Calcolo 45(2) (2008), 87–99.
[19] H. Karsli, E. Ibikli, *On convergence of convolution type singular integral operators depending on two parameters*, Fasc. Math. 38 (2007), 25–39.
[20] J. Musielak, *On some approximation problems in modular spaces*, in: *Constructive Function Theory*, 1981, Proc. Int. Conf., Varna, June 1–5, 1981, pp. 455–461, Publ. House Bulgarian Acad. Sci., Sofia, 1983.
[21] S. Y. Shaw, W. C. Liaw, Y. L. Lin, *Rates for approximation of functions in $BV[a,b]$ and $DBV[a,b]$ by positive linear operators*, Chinese J. Math. 21(2) (1993), 171–193.
[22] T. Świderski, E. Wachnicki, *Nonlinear singular integrals depending on two parameters*, Comment. Math. Prace Mat. 15 (2000), 181–189.
[23] X. M. Zeng, W. Chen, *On the rate of convergence of the generalized Durrmeyer type operators for functions of bounded variation*, J. Approx. Theory 102 (2000), 1–12.

ABANT IZZET BAYSAL UNIVERSITY FACULTY OF SCIENCE AND ARTS
DEPARTMENT OF MATHEMATICS
14280 GOLKOY BOLU
E-mail: karsli_h@ibu.edu.tr

Received May 4, 2011; revised version May 21, 2012.