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SECOND ORDER ABSTRACT DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE SET IN $\mathbb{R}_+$

Abstract. In this paper we give some new results on complete abstract second order differential equations of elliptic type set in $\mathbb{R}_+$. In the framework of UMD spaces, we use the celebrated Dore–Venni Theorem to prove existence and uniqueness for the strict solution. We will use also the Da Prato–Grisvard Sum Theory to furnish results when the space is not supposed to be a UMD space.

1. Introduction and hypotheses

Let us consider, in the complex Banach space X, the abstract differential equation of the second order

$$u''(x) + 2Bu'(x) + Au(x) = f(x), \qquad x \in (0, R), \tag{1}$$

together with the boundary conditions

$$\begin{cases} u(0) = u_0, \\ u(R) = u_R. \end{cases} \tag{2}$$

Here, A, B are two closed linear operators in X with domains $D(A)$ and $D(B)$ respectively, $0 < R \leqslant +\infty$, $f \in L^p(0, R; X)$, $1 < p < +\infty$ and u_0, u_R are given elements in X, with $u_R = 0$ in the case $R = +\infty$.

Several authors have studied (1)–(2) when $R < +\infty$ and $f \in L^p(0, R; X)$, $1 < p < +\infty$: see for example A. Favini, R. Labbas, S. Maingot, H. Tanabe and A. Yagi [13] and [14]. The case when $R < +\infty$ and $f \in C^\theta([0, R]; X)$, $0 < \theta < 1$, has been also treated: see for example A. El Haial and R. Labbas [8], A. Favini, R. Labbas, H. Tanabe and A. Yagi [12], A. Favini, R. Labbas, S. Maingot, H. Tanabe and A. Yagi, [10], [11], [15].

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The case when $R = +\infty$ and $B = 0$ has been considered by J. Prüss [18], Theorem 3.3, p. 316.

In all this work, we will suppose that $R = +\infty$ and that $f \in L^p(0, +\infty; X)$, $1 < p < +\infty$. We will search a strict solution u to (1)–(2), i.e. a function u such that

$$u \in W^{2,p}(0, +\infty; X), \;\; Bu', \;\; Au \in L^p(0, +\infty; X),$$

and which satisfies (1)–(2).

We must also mention that many authors have studied the same equation (1) with Cauchy data

$$u(0) = u_0, \;\; u'(0) = v_0,$$

see for instance, A. Favini [9], J. Liang and T. Xiao [16] and recently R. Chill and S. Srivastava [4], C. J. K. Batty, R. Chill and S. Srivastava [1]. Their assumptions on operators, the techniques used and the results are completely different from ours. For example in [9], [16] and [4], the authors deal with the parabolicity of the operator pencil defined by

$$P(\lambda) : D(A) \cap D(B) \longrightarrow X; \; x \longmapsto P(\lambda)x = (\lambda^2 I + 2\lambda B + A)x,$$

here we will consider in fact the elliptic case expressed by

$$(3) \quad \begin{cases} B^2 - A \text{ is a linear closed operator in } X, \quad \mathbb{R}_- \subset \rho(B^2 - A) \text{ and} \\ \sup\limits_{\lambda \geqslant 0} \|\lambda(\lambda + B^2 - A)^{-1}\|_{\mathcal{L}(X)} < +\infty, \end{cases}$$

see section 6, and then we build an explicit representation formula of the solution.

If P, Q are two linear operators in X we write $P \subset Q$ if

$$\begin{cases} D(P) \subset D(Q) \;\; \text{and} \\ Px = Qx, \quad x \in D(P). \end{cases}$$

Our assumptions on the operators A and B are the following: there exist L_1, L_2 operators in X such that

$$(4) \quad \begin{cases} L_1 - L_2 \subset 2B, \\ L_1 L_2 \subset -A, \end{cases}$$

$$(5) \quad \begin{cases} D(L_1) = D(L_2), \\ L_1 L_2 = L_2 L_1, \end{cases}$$

$$(6) \quad 0 \in \rho(L_1) \cap \rho(L_2),$$

$$(7) \quad 0 \in \rho(L_1 + L_2),$$

(8) L_1 and L_2 generate a bounded analytic semigroup on X.

Under these hypotheses, we will study (1)–(2) in the two following cases

1. First case:
In order to find a strict solution u to (1)–(2) with no more regularity on f than
$$f \in L^p(0,+\infty;X), \quad 1 < p < +\infty,$$
we will assume here that
$$X \text{ is a UMD space.} \tag{9}$$
Moreover $L_1,\ L_2$ will satisfy
$$\exists \theta \in]0, \tfrac{\pi}{2}[: -L_1, -L_2 \in \mathrm{BIP}(\theta, X). \tag{10}$$
We recall that
   - X is a UMD space if and only if for some $p > 1$ (and thus for all p) the Hilbert transform is continuous from $L^p(\mathbb{R};X)$ into itself (see J. Bourgain [2], D. L. Burkholder [3]).
   - Let $\alpha \in [0,\pi[$. A closed linear densely defined operator U belongs to the class $\mathrm{BIP}(\alpha, X)$ if
$$\begin{cases}]-\infty, 0[\subset \rho(U),\ N(U) = \{0\}, \overline{R(U)} = X \\ \text{and } \exists c \geqslant 1 : \forall \lambda > 0, \left\| (U+\lambda I)^{-1} \right\| \leqslant c/\lambda, \end{cases}$$
and
$$\begin{cases} \text{for all } s \in \mathbb{R}, U^{is} \in \mathcal{L}(X) \text{ and} \\ \exists c \geqslant 1 : \forall s \in \mathbb{R}, \left\| U^{is} \right\| \leqslant c e^{\alpha |s|}, \end{cases}$$
where $N(U)$ is the kernel of U and $R(U)$ the range of U (see J. Prüss and H. Sohr [19], p. 430).
2. Second case:
Here, we avoid assumptions (9) and (10), but we need more regularity on f that is
$$f \in W^{\theta,p}(0,+\infty;X), \quad 0 < \theta < \tfrac{1}{p} \text{ and } 1 < p < +\infty.$$
We recall that $f \in W^{\theta,p}(0,R;X)$ if $f \in L^p(0,R;X)$ and satisfies
$$[f]_{W^{\theta,p}(0,R;X)} = \int_0^R \int_0^R \frac{\|f(x) - f(y)\|_X^p}{|x-y|^{1+\theta p}} dx dy < +\infty,$$
see G. Da Prato and P. Grisvard [5], p. 331.
Then $(W^{\theta,p}(0,R;X), \|.\|_{W^{\theta,p}(0,R;X)})$ is a Banach space, where
$$\|f\|_{W^{\theta,p}(0,R;X)}^p = \|f\|_{L^p(0,R;X)}^p + [f]_{W^{\theta,p}(0,R;X)}.$$

Remark 1.

1. By our methods, we will solve
$$u''(x) + (L_1 - L_2)\, u'(x) - L_1 L_2 u(x) = f(x), \quad x \in (0, +\infty),$$
so a function u such that
$$u \in W^{2,p}(0, +\infty; X), \ (L_1 - L_2)u', \ L_1 L_2 u \in L^p(0, +\infty; X),$$
and which satisfies (1)–(2) will be called a (L_1, L_2)-strict solution of Problem (1)–(2). Of course such a solution will be in particular a strict solution of Problem (1)–(2) in the sense defined previously.
2. It is well known that assumption (10) implies (8) (see J. Prüss and H. Sohr [19], Theorem 2, p. 437).

Our main results in this paper are:

Theorem 2. *Assume* (4)$\sim$(10) *and let*
$$f \in L^p(0, +\infty; X), \quad 1 < p < +\infty.$$
Then the two following assertions are equivalent.

1. $u_0 \in (D(L_1 L_2), X)_{\frac{1}{2p}, p}$.
2. *Problem* (1)–(2) *as a unique* (L_1, L_2)*-strict solution.*

Theorem 3. *Assume* (4)$\sim$(8)*, let* X *be a complex Banach space and*
$$f \in W^{\theta,p}(0, +\infty; X), \quad 0 < \theta < \tfrac{1}{p} \text{ and } 1 < p < +\infty.$$
Then the two following assertions are equivalent.

1. $u_0 \in (D(L_1 L_2), X)_{\frac{1}{2p}, p}$.
2. *Problem* (1)–(2) *as a unique* (L_1, L_2)*-strict solution.*

These results will be completed by Theorem 10 and 11, in which L_1 and L_2 are precised.

The plan of the paper is as follows.

In Section 2, we prove some technical lemmas.

Section 3 is devoted to the construction of a representation formula for the solution u of (1)–(2). The uniqueness of the solution is also proved.

Sections 4 and 5 contain the proof of our main results, obtained by the study of the regularity of the previous representation formula.

In Section 6, we give sufficient conditions on operators A, B which allow us to build operators
$$L_1 = B - (B^2 - A)^{\frac{1}{2}} \text{ and } L_2 = -B - (B^2 - A)^{\frac{1}{2}},$$
satisfying our assumptions.

Finally in Section 7, we give some examples of application to partial differential equations.

2. Technical Lemmas

Lemma 4. *Let L, M be two linear operators in X whose domains $D(L)$, $D(M)$, satisfy*

$$D(L) = D(M) \quad \text{and} \quad D(LM) = D(ML).$$

Then

1. *For $l \in \{0,1,2\}$, $n \in \mathbb{N}$ and $P, Q \in \{L, M\}$, we have*
$$\mathcal{P}_{l,n} : \quad D(P^l Q^n) = D(Q^{l+n}).$$
2. *For $l, n \in \{0,1,2\}$ and $P, Q \in \{L, M\}$, we have*
$$D(P^l Q^n) = D(Q^l P^n) = D(Q^{l+n}) = D(P^{l+n}).$$

Proof. It is enough to show statement 1, from which statement 2 is easily deduced. We have the following steps.

Step 1: $\mathcal{P}_{0,n}$ is true for $n \in \mathbb{N}$.

Step 2: $\mathcal{P}_{1,n}$ is true for $n \in \mathbb{N}$. Indeed, $\mathcal{P}_{1,0}$ is true and if $\mathcal{P}_{1,k}$ is true for some $k \in \mathbb{N}$ then

$$\begin{aligned} x \in D(PQ^{k+1}) &\iff x \in D(Q) \ \text{ and } \ Qx \in D(PQ^k) \\ &\iff x \in D(Q) \ \text{ and } \ Qx \in D(Q^{k+1}) \\ &\iff x \in D(Q^{k+2}), \end{aligned}$$

i.e. $\mathcal{P}_{1,k+1}$ is true.

Step 3: $\mathcal{P}_{2,n}$ is true for $n \in \mathbb{N}$. Indeed $\mathcal{P}_{2,0}$ is true since

$$\begin{aligned} x \in D(P^2) &\iff x \in D(P) \ \text{ and } \ Px \in D(P) \\ &\iff x \in D(P) \ \text{ and } \ Px \in D(Q) \\ &\iff x \in D(QP) = D(PQ) \\ &\iff x \in D(Q) \ \text{ and } \ Qx \in D(P) = D(Q) \\ &\iff x \in D(Q^2). \end{aligned}$$

Moreover, if $\mathcal{P}_{2,k}$ is true for some $k \in \mathbb{N}$ then

$$\begin{aligned} x \in D(P^2Q^{k+1}) &\iff x \in D(Q) \ \text{ and } \ Qx \in D(P^2Q^k) \\ &\iff x \in D(Q) \ \text{ and } \ Qx \in D(Q^{k+2}) \\ &\iff x \in D(Q^{k+3}), \end{aligned}$$

i.e. $\mathcal{P}_{2,k+1}$ is satisfied. ∎

Remark 5. If L_1, L_2 are operators in X, satisfying (5), then due to Lemma 4, we have

$$D(L_iL_j) = D(L_i^2) = D(L_j^2), \quad i,j \in \{1,2\}.$$

Lemma 6. *Under assumption* (9)*, consider L, a closed linear operator in X, satisfying*

$$0 \in \rho(L) \text{ and } -L \in BIP(\theta_L, X), \quad \theta_L \in]0, \tfrac{\pi}{2}[.$$

Then, for $R \in \,]0,+\infty[$, $1 < p < +\infty$, $\Psi \in L^p(0,R;X)$ and $\Phi \in L^p(0,+\infty;X)$, we get

1. $\mathcal{L}(\psi) : x \longmapsto L\int_0^x e^{(x-y)L}\Psi(y)dy \in L^p(0,R;X)$.

   Moreover, there exists $C_R > 0$ such that

$$\|\mathcal{L}(\psi)\|_{L^p(0,R;X)} \leqslant C_R\|\psi\|_{L^p(0,R;X)}, \quad \psi \in L^p(0,R;X). \tag{11}$$

2. $x \longmapsto L\int_x^R e^{(y-x)L}\Psi(y)dy \in L^p(0,R;X)$.
3. $x \longmapsto L\int_0^x e^{(x-y)L}\Phi(y)dy \in L^p(0,+\infty;X)$.
4. $x \longmapsto L\int_x^{+\infty} e^{(y-x)L}\Phi(y)dy \in L^p(0,+\infty;X)$.
5. $x \longmapsto Le^{xL}\int_0^{+\infty} e^{yL}\Phi(y)dy \in L^p(0,+\infty;X)$.

Proof.

1. See G. Dore and A. Venni [7].
2. It is an easy consequence of statement 1, since for a. e. $x \in (0,R)$

$$L\int_x^R e^{(y-x)L}\Psi(y)dy = L\int_0^{R-x} e^{((R-x)-s)L}\Psi(R-s)ds.$$

3. It is a result of G. Dore [6] extending 1.
4. We proceed as in G. Dore [6], pp. 28–29.

   We have to prove that $F_1+F_2 \in L^p(0,+\infty;X)$ where for a.e. $x \in (0,+\infty)$

$$\begin{cases} F_1(x) = L\displaystyle\int_x^{x+1} e^{(y-x)L}\Phi(y)dy, \\ F_2(x) = L\displaystyle\int_{x+1}^{+\infty} e^{(y-x)L}\Phi(y)dy. \end{cases}$$

We first show that $F_2 \in L^p(0,+\infty;X)$. In fact, since L generate a bounded analytic semigroup and $0 \in \rho(L)$ then there exist $M \geqslant 1$ and $\omega > 0$ such that for any $y > 0$

$$\|e^{yL}\| \leq Me^{-\omega y} \text{ and } \|Le^{yL}\| \leq My^{-1}e^{-\omega y}, \tag{12}$$

(see A. Pazy [17], Theorem 6.13, p. 74). So

$$\begin{aligned}
\int_0^{+\infty} \|F_2(x)\|^p dx &= \int_0^{+\infty} \Big\| \int_{x+1}^{+\infty} Le^{(y-x)L}\Phi(y)dy\Big\|^p dx \\
&\leqslant M^p \int_0^{+\infty} \Big(\int_{x+1}^{+\infty} \frac{1}{y-x} e^{-(y-x)\omega} \|\Phi(y)\| dy\Big)^p dx \\
&\leqslant M^p \int_0^{+\infty} \Big(\int_{x+1}^{+\infty} e^{-(y-x)\omega} \|\Phi(y)\| dy\Big)^p dx \\
&\leqslant M^p \int_{-\infty}^{+\infty} |(g*h)(x)|^p dx,
\end{aligned}$$

where g and h are defined by

$$g(x) = \begin{cases} 0 & \text{if } x \geqslant -1, \\ e^{x\omega} & \text{if } x < -1, \end{cases} \quad \text{and} \quad h(x) = \begin{cases} \|\Phi(x)\| & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases}$$

But $\Phi \in L^p(0,+\infty;X)$ so $h \in L^p(\mathbb{R})$, moreover $g \in L^1(\mathbb{R})$ since

$$\int_{\mathbb{R}} |g(x)|\, dx = \int_{-\infty}^{-1} e^{x\omega} dx < +\infty,$$

and then $g*h \in L^p(\mathbb{R})$, which gives

$$\int_0^{+\infty} \|F_2(x)\|^p dx < +\infty.$$

It remains to prove that $F_1 \in L^p(0,+\infty;X)$. Let $j \in \mathbb{N}$, we set

$$\phi_j \ : \ x \longmapsto \ \chi_{[j,j+1[}(x)\phi(x),$$

($\chi_{[j,j+1[}$ denotes the characteristic function of $[j,j+1[$) and

$$\begin{cases} I_j = \displaystyle\int_j^{j+1} \Big\| L \int_x^{j+1} e^{(y-x)L}\Phi_j(y)dy\Big\|^p dx, \\ J_j = \displaystyle\int_j^{j+1} \Big\| L \int_{j+1}^{x+1} e^{(y-x)L}\Phi_{j+1}(y)dy\Big\|^p dx. \end{cases}$$

Then by the changes of variable $\tau = 1-x+j$ and $\sigma = 1-y+j$, we obtain

$$\begin{aligned} I_j &= \int_0^1 \left\| L \int_{1-\tau+j}^{j+1} e^{(y-1+\tau-j)L} \Phi_j(y) dy \right\|^p d\tau \\ &= \int_0^1 \left\| L \int_0^{\tau} e^{(\tau-\sigma)L} \Phi_j(1-\sigma+j) d\sigma \right\|^p d\tau \\ &= \left\| \mathcal{L}(\Phi_j(1-\cdot+j)) \right\|^p_{L^p(0,1;X)}, \end{aligned}$$

so, due to (11), we get

$$I_j \leqslant (C_1)^p \left\| \Phi_j(1-\cdot+j)) \right\|^p_{L^p(0,1;X)} \leqslant (C_1)^p \left\| \Phi_j \right\|_{L^p(j,j+1;X)}. \tag{13}$$

Now, taking into account (12), we have

$$\begin{aligned} J_j &= \int_0^1 \left\| L \int_{j+1}^{2-\tau+j} e^{(y-x)L} \Phi_{j+1}(y) dy \right\|^p d\tau \\ &= \int_0^1 \left\| L \int_0^{1-\tau} e^{(s+\tau)L} \Phi_{j+1}(s+j+1) ds \right\|^p d\tau \\ &\leqslant \int_0^1 \left(\int_0^1 \left\| L e^{(s+\tau)L} \Phi_{j+1}(s+j+1) ds \right\| \right)^p d\tau \\ &\leqslant M^p \int_0^1 \left(\int_0^1 \frac{1}{s+\tau} \left\| \Phi_{j+1}(s+j+1) \right\| ds \right)^p d\tau, \end{aligned}$$

and since the kernel $\dfrac{1}{s+\tau}$ defines a bounded operator on $L^p(0,1;\mathbb{R})$, there exists $C > 0$ such that

$$\begin{aligned} J_j &\leqslant C M^p \left\| \Phi_{j+1}(\cdot+j+1)) \right\|^p_{L^p(0,1;X)} \\ &\leqslant C M^p \left\| \Phi_{j+1} \right\|^p_{L^p(j+1,j+2;X)}. \end{aligned} \tag{14}$$

Finally, we write

$$\int_0^{+\infty} \|F_1(x)\|^p dx = \sum_{j=0}^{\infty} \int_j^{j+1} \|F_1(x)\|^p dx,$$

then using (13), (14) and

$$\sum_{j=0}^{\infty} \left\| \Phi_j \right\|^p_{L^p(j,j+1;X)} = \left\| \Phi \right\|^p_{L^p(0,+\infty;X)},$$

we obtain

$$\begin{aligned}\int_0^{+\infty} \|F_1(x)\|^p dx &\leqslant 2^{p-1} \sum_{j=0}^{\infty} (I_j + J_j) \\ &\leqslant 2^{p-1} (C_1)^p \sum_{j=0}^{\infty} \|\Phi_j\|^p_{L^p(j,j+1;X)} \\ &\quad + 2^{p-1} C M^p \sum_{j=0}^{\infty} \|\Phi_{j+1}\|^p_{L^p(j+1,j+2;X)} \\ &\leqslant 2^{p-1} \max\left((C_1)^p, CM^p\right) \|\Phi\|_{L^p(0,+\infty;X)} \\ &< +\infty.\end{aligned}$$

5. This last point is deduced from statements 3 and 4, indeed

$$\begin{aligned}Le^{xL} \int_0^{+\infty} e^{yL}\Phi(y)dy &= Le^{xL} \int_0^{x} e^{yL}\Phi(y)dy + Le^{xL} \int_x^{+\infty} e^{yL}\Phi(y)dy \\ &= L\int_0^{x} e^{(x-y)L} e^{2yL}\Phi(y)dy + e^{2xL} L \int_x^{+\infty} e^{(y-x)L}\Phi(y)dy,\end{aligned}$$

and we take into account the fact that, due to (12)

$$y \longmapsto e^{2yL}\Phi(y) \in L^p(0,+\infty;X). \blacksquare$$

Lemma 7. *Let X be a Banach space and L be the infinitesimal generator of a bounded analytic semigroup on X, satisfying moreover $0 \in \rho(L)$.*

Then, for $R \in]0,+\infty]$, $1 < p < +\infty$, $0 < \theta < \frac{1}{p}$, $\psi \in W^{\theta,p}(0,R;X)$, we get

1. $x \longmapsto L\int_0^{x} e^{(x-y)L}\psi(y)dy \in W^{\theta,p}(0,R;X) \subset L^p(0,R;X)$.
2. *If $R < +\infty$*
$$x \longmapsto L\int_x^{R} e^{(y-x)L}\psi(y)dy \in W^{\theta,p}(0,R;X) \subset L^p(0,R;X).$$
3. *If $R = +\infty$*
$$x \longmapsto L\int_x^{+\infty} e^{(y-x)L}\psi(y)dy \in L^p(0,+\infty;X).$$

Proof.

1. It is a result of G. Da Prato and P. Grisvard (See [5], Theorem 4.7, p. 334).
2. This point is deduced from statement 1 (as in Lemma 6, statement 2).
3. We proceed as in Lemma 6, statement 4. $\blacksquare$

Lemma 8. *Let L, be the infinitesimal generator of a bounded analytic semigroup $\left(e^{xL}\right)_{x\geqslant 0}$ in X and assume that $0\in\rho(L)$.*

Then, for $f\in L^p(0,+\infty;X)$, we get

1. $\lim\limits_{x\to+\infty}\int_0^x e^{(x-y)L}f(y)dy=0.$
2. $\lim\limits_{x\to+\infty}\int_x^{+\infty} e^{(y-x)L}f(y)dy=0.$

Proof.

1. For $x\in(0,+\infty)$, we set

$$\begin{aligned}\alpha(x)&=\int_0^x e^{(x-y)L}f(y)dy=\int_0^{\frac{x}{2}} e^{(x-y)L}f(y)dy+\int_{\frac{x}{2}}^x e^{(x-y)L}f(y)dy\\&=\alpha_1(x)+\alpha_2(x).\end{aligned}$$

As in (12), there exist $M\geqslant 1$ and $\omega>0$ such that for any $y>0$

$$\|e^{yL}\|\leq Me^{-\omega y},$$

so, for $x\geqslant 0$

$$\|\alpha_2(x)\|\leqslant\int_{\frac{x}{2}}^x\|e^{(x-y)L}\|\|f(y)\|dy\leqslant M\int_{\frac{x}{2}}^x e^{-\omega(x-y)}\|f(y)\|dy,$$

and, setting $q=\dfrac{p}{p-1}$, we obtain by Hölder inequality

$$\|\alpha_2(x)\|\leqslant M\Big(\int_{\frac{x}{2}}^x e^{-\omega q(x-y)}dy\Big)^{\frac{1}{q}}\Big(\int_{\frac{x}{2}}^x\|f(y)\|^pdy\Big)^{\frac{1}{p}},$$

so

$$\begin{aligned}\|\alpha_2(x)\|&\leqslant\frac{M}{(\omega q)^{\frac{1}{q}}}\Big(1-e^{-\omega q\frac{x}{2}}\Big)^{\frac{1}{q}}\Big(\int_{\frac{x}{2}}^x\|f(y)\|^pdy\Big)^{\frac{1}{p}}\\&\leqslant\frac{M}{(\omega q)^{\frac{1}{q}}}\Big(\int_{\frac{x}{2}}^{+\infty}\|f(y)\|^pdy\Big)^{\frac{1}{p}}.\end{aligned}$$

But $f\in L^p(0,+\infty;X)$ thus

$$\lim_{x\to+\infty}\Big(\int_{\frac{x}{2}}^{+\infty}\|f(y)\|^pdy\Big)^{\frac{1}{p}}=0,$$

so

$$\lim_{x\to+\infty}\|\alpha_2(x)\|=0.$$

For α_1, we write

$$\begin{aligned}\|\alpha_1(x)\| &\leqslant \int_0^{\frac{x}{2}} \|e^{(x-y)L}\| \|f(y)\| dy \\ &\leqslant M\Big(\int_0^{\frac{x}{2}} e^{-\omega q(x-y)} dy\Big)^{\frac{1}{q}} \Big(\int_0^{\frac{x}{2}} \|f(y)\|^p dy\Big)^{\frac{1}{p}} \\ &\leqslant \frac{M}{(\omega q)^{\frac{1}{q}}} \|f\|_{L^p(0,+\infty;X)} \left(e^{-\omega q \frac{x}{2}} - e^{-\omega q x}\right),\end{aligned}$$

which gives

$$\lim_{x\to+\infty} \|\alpha_1(x)\| = 0.$$

2. As in statement 1, we write

$$\begin{aligned}\Big\| \int_x^{+\infty} e^{(y-x)L} f(y) dy \Big\| &\leqslant M\Big(\int_x^{+\infty} e^{-\omega q(y-x)} dy\Big)^{\frac{1}{q}} \Big(\int_x^{+\infty} \|f(y)\|^p dy\Big)^{\frac{1}{p}} \\ &\leqslant \frac{M}{(\omega q)^{\frac{1}{q}}} \Big(\int_x^{+\infty} \|f(y)\|^p dy\Big)^{\frac{1}{p}},\end{aligned}$$

which gives

$$\lim_{x\to+\infty} \Big(\Big\| \int_x^{+\infty} e^{(y-x)L} f(y) dy \Big\|\Big) = 0.$$

3. Construction of a representation formula for the solution u of (1)–(2)

We build a representation formula of the solution in the abstract case, as in the scalar case. We obtain

$$(15) \qquad \begin{cases} u(x) = e^{xL_2} u_0 - e^{xL_2} v_0 + \xi_1(x) + \xi_2(x), \text{ where} \\ v_0 = (L_1 + L_2)^{-1} \int_0^{+\infty} e^{yL_1} f(y) dy, \\ \xi_1(x) = (L_1 + L_2)^{-1} \int_0^x e^{(x-y)L_2} f(y) dy, \\ \xi_2(x) = (L_1 + L_2)^{-1} \int_x^{+\infty} e^{(y-x)L_1} f(y) dy. \end{cases}$$

In order to prove Theorem 2, we must first show a uniqueness result, then we will show that u given by (15) is the solution of problem (1)–(2) with the desired regularity.

Proposition 9. *Assume* (9)∼(10) *and let*

$$f \in L^p(0,+\infty;X), \quad 1 < p < +\infty.$$

If u is a (L_1, L_2)-strict solution of (1)–(2)*, then u is uniquely determined.*

Proof. Let u_1, u_2 be (L_1, L_2)-strict solutions of (1)–(2) and fix $x_0 \geqslant 0$. We have to show that

$$u_1(x_0) = u_2(x_0).$$

Consider some $R \geqslant x_0$, and set $u_R = u_1(R) - u_2(R)$.

Then $u = u_1 - u_2$ is a (L_1, L_2)-strict solution on $(0, R)$ of problem

$$\begin{cases} u''(x) + (L_1 - L_2)u'(x) - L_1 L_2 u(x) = 0, & x \in (0,R), \\ u(0) = 0, \\ u(R) = u_R. \end{cases} \tag{16}$$

Using Krein's method, we get that problem (16) as a unique (L_1, L_2)-strict solution thus u is uniquely determined on $(0, R)$ by

$$u(x) = e^{xL_2}\zeta_0 + e^{(R-x)L_1}\zeta_R, \tag{17}$$

with

$$\begin{cases} u(0) = \zeta_0 + e^{RL_1}\zeta_R = 0, \\ u(R) = e^{RL_2}\zeta_0 + \zeta_R = u_R. \end{cases}$$

So

$$\begin{cases} e^{RL_2}\zeta_0 + e^{RL_2}e^{RL_1}\zeta_R = 0, \\ e^{RL_2}\zeta_0 + \zeta_R = u_R, \end{cases}$$

thus

$$\begin{cases} (I - e^{RL_2}e^{RL_1})\zeta_R = u_R, \\ \zeta_0 + e^{RL_1}\zeta_R = 0. \end{cases}$$

Due to (12) applied to L_1 and L_2, there exists $R_0 \geqslant x_0$ such that for any $R \geqslant R_0$

$$\|e^{RL_2}e^{RL_1}\| \leqslant \frac{1}{2}.$$

Now, we consider $R \geqslant R_0$ and then $I - e^{RL_2}e^{RL_1}$ is boundedly invertible, moreover setting $z_R = (I - e^{RL_2}e^{RL_1})^{-1}$, we have

$$\|z_R\| \leqslant 2, \tag{18}$$

and

$$\begin{cases} \zeta_R = z_R u_R, \\ \zeta_0 = -e^{RL_1} z_R u_R. \end{cases} \tag{19}$$

Hence, due to (17) and (19), we get for any $R \geqslant R_0$

$$u(x_0) = -e^{x_0 L_2} e^{R L_1} z_R u_R + e^{(R-x_0)L_1} z_R u_R = \alpha_R.$$

But

$$\lim_{R\to\infty} u_R = \lim_{R\to\infty} (u_1(R) - u_2(R)) = 0,$$

so due to (12) and (18), we obtain

$$u(x_0) = \lim_{R\to\infty} \alpha_R = 0,$$

that is $u_1(x_0) = u_2(x_0)$. ■

4. Proof of Theorem 2

Assume that

$$u_0 \in (D(L_1 L_2); X)_{\frac{1}{2p},p},$$

and consider u given by (15).

We first study the regularity of u. Due to (15), (5) and Remark 5, we have, for a.e. $x \in (0, +\infty)$

$$\begin{cases} L_1 L_2 \xi_1(x) = L_1 (L_1 + L_2)^{-1} L_2 \int_0^x e^{(x-y)L_2} f(y) dy \\ L_1 L_2 \xi_2(x) = L_2 (L_1 + L_2)^{-1} L_1 \int_x^{+\infty} e^{(y-x)L_1} f(y) dy, \end{cases}$$

and since

$$L_1(L_1 + L_2)^{-1} \text{ and } L_2(L_1 + L_2)^{-1} \in \mathcal{L}(X),$$

we deduce from Lemma 6, statements 3 and 4, that

$$L_1 L_2 \xi_1, \quad L_1 L_2 \xi_2 \in L^p(0, +\infty; X), \quad 1 < p < +\infty.$$

Moreover

$$L_1 \xi_1' = L_1 L_2 \xi_1 + L_1 (L_1 + L_2)^{-1} f \in L^p(0, +\infty; X), \quad 1 < p < +\infty,$$

similarly, for $i, j \in \{1, 2\}$

$$L_i \xi_j' \in L^p(0, +\infty; X), \quad 1 < p < +\infty,$$

and also, since $\xi_1' + \xi_2' = L_2 \xi_1 + L_1 \xi_2$, we obtain

$$(\xi_1 + \xi_2)'' \in L^p(0, +\infty; X), \quad 1 < p < +\infty.$$

Thus, setting $\xi = \xi_1 + \xi_2$, we deduce that

$$\xi \in W^{2,p}(0, +\infty; X), \; (L_1 - L_2)\xi', \; L_1 L_2 \xi \in L^p(0, +\infty; X). \tag{20}$$

Now, from (15), we have $u = w - v + \xi$ where

$$v(x) = e^{x L_2} v_0 \text{ and } w(x) = e^{x L_2} u_0, \; \text{a.e. } x \in (0, +\infty),$$

but, from Lemma 6, statement 5, we obtain that

$$v \in W^{2,p}(0,+\infty;X),\ (L_1 - L_2)v',\ L_1L_2v \in L^p(0,+\infty;X). \tag{21}$$

Due to Remark 5, we get that

$$u_0 \in (D(L_1L_2);X)_{\frac{1}{2p},p} = (D(L_2^2);X)_{\frac{1}{2p},p},$$

so

$$x \longmapsto L_2^2 e^{xL_2} u_0 \in L^p(0,+\infty;X),$$

(see H. Triebel [21] p. 96), which gives, for $i = 1, 2$

$$x \longmapsto L_i L_2 e^{xL_2} u_0 \in L^p(0,+\infty;X),$$

(we have used the fact that $L_iL_2 = L_iL_2^{-1}L_2^2$ with $L_iL_2^{-1} \in \mathcal{L}(X)$).

Finally

$$w \in W^{2,p}(0,+\infty;X),\ (L_1 - L_2)w',\ L_1L_2w \in L^p(0,+\infty;X), \tag{22}$$

and, in virtue of (20), (21) and (22), $u = w - v + \xi$ verifies

$$u \in W^{2,p}(0,+\infty;X),\ (L_1 - L_2)u',\ L_1L_2u \in L^p(0,+\infty;X).$$

To conclude, it is enough to show that u satisfies (1)–(2). In fact, it is clear that u satisfies (1) and $u(0) = u_0$. Moreover, due to Lemma 8 and (12), we get $u(+\infty) = 0$. In fact, since we have shown in particular that $u \in W^{1,p}(0,+\infty;X)$ then the condition $u(+\infty) = 0$ is necessarily satisfied.

Conversely assume that Problem (1)–(2) has a (L_1, L_2)-strict solution then, using Remark 1, statement 6 in [13], we deduce that

$$u_0 \in (D(L_1L_2);X)_{\frac{1}{2p},p}. \ \blacksquare$$

5. Proof of Theorem 3

For Theorem 3, we proceed as in proof of Theorem 2; we have just to replace Lemma 6 by Lemma 7.

6. Construction of L_1 and L_2

Let us assume that operators A and B satisfy

$$\begin{cases} B^2 - A \text{ is closed, } \mathbb{R}_- \subset \rho(B^2 - A) \text{ and} \\ \sup\limits_{\lambda \geqslant 0} \|\lambda(\lambda + B^2 - A)^{-1}\|_{\mathcal{L}(X)} < +\infty, \end{cases} \tag{23}$$

(then it is well known that $-(B^2 - A)^{\frac{1}{2}}$ is infinitesimal generator of an analytic semigroup)

$$D((B^2 - A)^{\frac{1}{2}}) \subseteq D(B), \tag{24}$$

$$\forall y \in D(B), B(B^2 - A)^{-\frac{1}{2}} y = (B^2 - A)^{-\frac{1}{2}} By, \tag{25}$$

$$\pm B - (B^2 - A)^{\frac{1}{2}} \text{ are boundedly invertible.} \tag{26}$$

Moreover, we suppose that

$$\exists \theta \in]0, \tfrac{\pi}{2}[: \ \pm B + (B^2 + A)^{\frac{1}{2}} \in \mathrm{BIP}(\theta, X), \tag{27}$$

or

$$\pm B - (B^2 - A)^{\frac{1}{2}} \text{ generates a bounded analytic semigroup on } X, \tag{28}$$

(recall that (27) implies (28)).

Then, if we set

$$L_1 = B - (B^2 - A)^{\frac{1}{2}} \text{ and } L_2 = -B - (B^2 - A)^{\frac{1}{2}}, \tag{29}$$

we have the following lemma (see Lemma 7, p.178 in [14]).

Lemma 10. *Assume* (23)∼(26). *Then* L_1 *and* L_2 *defined by* (29)

$$\begin{cases} D(L_1) = D(L_2) = D((B^2 - A)^{\frac{1}{2}}), \\ D(L_1 L_2) = D(L_2 L_1) = D(B^2 - A) \subset D(-A), \\ L_1 L_2 = L_2 L_1 \subset -A, \end{cases}$$

and $(L_1 + L_2)^{-1} = -\dfrac{1}{2}(B^2 - A)^{-\frac{1}{2}} \in \mathcal{L}(X)$. *Note that* $L_1 L_2 = L_2 L_1 = -A$ *if and only if* $D(A) \subset D(B^2)$.

Finally Theorem 2 and the previous lemma lead us to the following result.

Theorem 11. *Assume* (9), (23)∼(27) *and*

$$f \in L^p(0, +\infty; X), \quad 1 < p < +\infty.$$

Then the following assertions are equivalent

1. $u_0 \in (D(B^2 - A), X)_{\frac{1}{2p}, p}$.
2. *Problem* (1)–(2) *as a unique strict solution satisfying moreover*

$$u \in L^p(0, +\infty; D(B^2 - A)) \text{ and } u' \in L^p(0, +\infty; D((B^2 - A)^{\frac{1}{2}})).$$

Similarly, Theorem 3 and Lemma 10 give:

Theorem 12. *Assume* (23)∼(26) *and* (28), *let* X *be a Banach space and*

$$f \in W^{\theta,p}(0, +\infty; X), \quad 0 < \theta < \tfrac{1}{p} \text{ and } 1 < p < +\infty.$$

Then the following assertions are equivalent

1. $u_0 \in (D(B^2 - A), X)_{\frac{1}{2p}, p}$.
2. *Problem* (1)–(2) *as a unique strict solution satisfwing moreover*

$$u \in L^p(0, +\infty; D(B^2 - A)) \text{ and } u' \in L^p(0, +\infty; D((B^2 - A)^{\frac{1}{2}})).$$

7. Examples

Example 1. Here, we describe a general model to which our previous theory applies. Let X be a UMD space, $\alpha \in]-\infty, 0[$, $\beta \in]0, +\infty[$, $m \in \mathbb{N}^*$ and C be a linear operator such that

$$C \in \mathrm{BIP}(\theta_C, X) \text{ and } 0 \in \rho(C), \text{ with } 0 < \theta_C < \tfrac{\pi}{2m},$$

and consider A, B defined by

$$A = \alpha\beta C^{2m}, \ B = \frac{\alpha+\beta}{2} C^m.$$

Since $C \in \mathrm{BIP}(\theta_C, X)$ then for any $\mu > 0$, $\mu C^m \in \mathrm{BIP}(m\theta_C, X)$ (see J. Prüss and H. Sohr [19], Corollary 3, p. 444 and Corollary 1, p. 435).

Now, by taking $L_1 = \alpha C^m$ and $L_2 = -\beta C^m$, we verify that all the assumptions are fulfilled on L_1 and L_2 and so, we can apply our previous results as well.

As a simple example, we will consider $m = 1$, Ω a bounded domain in $\mathbb{R}^n$ with C^2-boundary $\partial\Omega$, $X = L^q(\Omega)$ with $1 < q < +\infty$ and C such that

$$\begin{cases} D(C) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), \\ Cu = -\Delta u, \end{cases}$$

see Example 3 in [13] p. 210. Then $0 \in \rho(C)$ and $C \in BIP(\theta, X)$ for $\theta \in]0, \frac{\pi}{2}[$ see Theorem C, p. 166–167, in [20].

Note that here

$$D(A) = D(B^2) = \{u \in W^{4,q}(\Omega) : u_{|\partial\Omega} = \Delta u_{|\partial\Omega} = 0\}.$$

Then by Theorem 2, we have

Proposition 13. *Let $p \in]1; +\infty[$, $f \in L^p(0, +\infty; L^q(\Omega))$.*

Then the two following assertions are equivalent

1. $u_0 \in (D(A), L^q(\Omega))_{\frac{1}{2p}, p}$.
2. *The problem*

$$(30) \quad \begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) - (\alpha+\beta)\Delta_y \dfrac{\partial u}{\partial x}(x,y) + \alpha\beta\Delta_y^2 u(x,y) = f(x,y), \\ \qquad\qquad\qquad\qquad (x,y) \in (0,+\infty)\times\Omega, \\ u(0,y) = u_0(y), \quad y \in \Omega, \\ u(+\infty, y) = 0, \quad y \in \Omega, \\ u(x,\xi) = \Delta_y u(x,\xi) = 0, \quad (x,\xi) \in (0,+\infty) \times \partial\Omega, \end{cases}$$

has a unique strict solution u, that is

$u \in W^{2,p}(0,+\infty; L^q(\Omega)) \cap L^p(0,+\infty; L^q(\Omega))$ *and* $u' \in L^p(0,+\infty; L^q(\Omega))$,

and satisfies (30).

Example 2. Let $X = L^q(\Omega)$, $1 < q < \infty$, where Ω is either $\mathbb{R}^n$, or the half space $\mathbb{R}^n_+$, or a bounded domain with C^2-boundary, or an exterior domain with C^2-boundary. Take A, the operator in X such that

$$\begin{cases} D(A) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), \\ A = \sum\limits_{j,k=1}^{n} \dfrac{\partial}{\partial y_j}\left(a_{jk}\dfrac{\partial}{\partial y_k}\right) - \delta, \quad \delta \geq 0, \end{cases}$$

where $a = (a_{jk})$ satisfies

(A1) $a(y) = (a_{jk}(y))$ is a real symmetric matrix for all $y \in \bar{\Omega}$ and there exists $a_0 > 0$ such that

$$a_0 \leq a(y)\xi \cdot \xi \leq a_0^{-1}, \text{ for all } y \in \bar{\Omega}, \quad \xi \in \mathbb{R}^n, \quad |\xi| = 1;$$

(A2) $a_{jk} \in C^\alpha(\bar{\Omega})$, for some $\alpha \in (0,1)$; moreover if Ω is unbounded then

$$a_{jk}^\infty = \lim_{|y|\to\infty} a_{jk}(y)$$

exists and there is a constant $C > 0$ such that

$$|a_{jk}(y) - a_{jk}^\infty| \leq C|y|^{-\alpha}, \quad y \in \Omega, \quad |y| \geq 1, \quad j,k = 1,\dots,n;$$

(A3) $\dfrac{\partial a_{jk}}{\partial y_j} \in L^{r_k}(\Omega)$, for some numbers r_k verifying

$$p \leq r_k \leq \infty, \; r_k > n, \; j,k = 1,\dots,n.$$

We assume moreover that $\delta > 0$ or Ω is bounded.

Then, by Theorem C, p. 166–167, in [20], $-A$ has bounded imaginary powers. Therefore, Theorem 2 applies and we get

Proposition 14. *Let $p, q \in]1,\infty[$, $f \in L^p(0,+\infty; L^q(\Omega))$ and*

$$u_0 \in \left(W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), L^q(\Omega)\right)_{\frac{1}{2p},p}.$$

Then problem

$$(31) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + \sum\limits_{j,k=1}^{n} \dfrac{\partial}{\partial y_j}\left(a_{jk}\dfrac{\partial u}{\partial y_k}\right)(x,y) - \delta u(x,y) \\ = f(x,y), \qquad (x,y) \in (0,+\infty)\times\Omega, \\ u(0,y) = u_0(y), \; u(+\infty,y) = 0, \qquad y \in \Omega, \\ u(x,\sigma) = 0, \qquad (x,\sigma) \in (0,+\infty)\times\partial\Omega, \end{cases}$$

has a unique strict solution u, that is

$$u \in W^{2,p}(0,+\infty; L^q(\Omega)) \cap L^p(0,+\infty; W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)),$$

and satisfies (31).

Note that here the interpolation space

$$\left(W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), L^q(\Omega)\right)_{\frac{1}{2p},p},$$

is, for bounded domain, the Besov space

$$\left\{u \in \mathcal{B}_{q,p}^{2-1/p}(\Omega); u_{|\partial\Omega} = 0\right\},$$

see Triebel [21], p. 321 (for the case $\Omega = \mathbb{R}^n$ or $\mathbb{R}_+^n$ one applies Triebel, again, Theorem 5.3.3. p. 373).

Example 3. We can generalize (30) in the following manner. Consider

$$L_1 := 2A - cI, L_2 := A$$

where $c \geq 0$ and A is defined as in Example 2.

Then L_1 and L_2 satisfy (5), (6), (7) and (10). Thus, we can apply Theorem 2 and deal with the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + \sum\limits_{j,k=1}^{n} \dfrac{\partial}{\partial y_j}\left(a_{jk}\dfrac{\partial^2 u}{\partial y_k \partial x}\right)(x,y) - c\dfrac{\partial u}{\partial x}(x,y) \\ + c\sum\limits_{j,k=1}^{n} \dfrac{\partial}{\partial y_j}\left(a_{jk}\dfrac{\partial u}{\partial y_k}\right)(x,y) \\ - 2\sum\limits_{j,j',k,k'=1}^{n} \dfrac{\partial}{\partial y_j}\left(a_{jk}\dfrac{\partial u}{\partial y_k}\right)\dfrac{\partial}{\partial y_{j'}}\left(a_{j'k'}\dfrac{\partial u}{\partial y_{k'}}\right)(x,y) \\ = f(x,y), \quad (x,y) \in (0,+\infty)\times\Omega, \\ \\ u(0,y) = u_0(y),\ u(+\infty,y) = 0, \quad y \in \Omega, \\ u(x,\sigma) = \sum\limits_{j,k=1}^{n} \dfrac{\partial}{\partial y_j}\left(a_{jk}\dfrac{\partial u}{\partial y_k}\right)(x,\sigma) = 0, \quad (x,\sigma) \in (0,+\infty)\times\partial\Omega, \end{cases}$$

here for simplicity we have taken $\delta = 0$ and Ω bounded.

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