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ON THE GENERALIZED ORDER AND GENERALIZED TYPE OF ENTIRE MONOGENIC FUNCTIONS

Abstract. In the present paper we study the generalized growth of entire monogenic functions. The generalized order, generalized lower order and generalized type of entire monogenic functions have been obtained in terms of its Taylor's series coefficients.

1. Introduction

Firstly, following Constales, Almeida and Krausshar (see [2] and [3]), we give some definitions and associated properties. Let $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$ be the n-dimensional multi-index and $\mathbf{x} \in \mathbb{R}^n$, then we define

$$\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \ldots x_n^{m_n}, \quad \mathbf{m!} = m_1! \ldots m_n!, \quad |\mathbf{m}| = m_1 + \cdots + m_n.$$

By $\{e_1, e_2, \ldots, e_n\}$ we denote the canonical basis of the Euclidean vector space $\mathbb{R}^n$. The associated real Clifford algebra Cl_n is the free algebra generated by $\mathbb{R}^n$ modulo $\mathbf{x}^2 = -\|\mathbf{x}\|^2 e_0$, where e_0 is the neutral element with respect to multiplication of the Clifford algebra Cl_n. In the Clifford algebra Cl_n following multiplication rule holds:

$$e_i e_j + e_j e_i = -2\delta_{ij}\, e_0, \quad i, j = 1, 2 \ldots\ldots, n,$$

where δ_{ij} is Kronecker symbol. A basis for Clifford algebra Cl_n is given by the set $\{e_A : A \subseteq \{1, 2, \ldots, n\}\}$ with $e_A = e_{l_1} e_{l_2} \ldots e_{l_r}$, where $1 \leq l_1 < l_2 < \cdots < l_r \leq n$, $e_\phi = e_0 = 1$. Each $a \in Cl_n$ can be written in the form $a = \sum_{A \subseteq (1,2,\ldots,n)} a_A e_A$ with $a_A \in \mathbb{R}$. The conjugation in Clifford algebra Cl_n is defined by $\bar{a} = \sum_{A \subseteq (1,2,\ldots,n)} a_A\, \bar{e}_A$, where $\bar{e}_A = \bar{e}_{l_r} \bar{e}_{l_{r-1}} \ldots\ldots \bar{e}_{l_1}$ and $\bar{e}_j = -e_j$, for $j = 1, 2, \ldots n$, $\bar{e}_0 = e_0 = 1$. The linear subspace $\mathrm{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} = \mathbb{R} \oplus \mathbb{R}^n \subset Cl_n$ is the so called space of paravectors $z = x_0 + x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$ which we simply identify with $\mathbb{R}^{n+1}$. Here

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$x_0 = \mathrm{Sc}(z)$ is the scalar part and $\mathbf{x} = x_1e_1 + x_2e_2 + \cdots + x_ne_n = \mathrm{Vec}(z)$ is the vector part of a paravector z. The Clifford norm of an arbitrary $a = \sum_{A\subseteq(1,2,\ldots,n)} a_A\, e_A$ is given by

$$\|a\| = \Big(\sum_{A\subseteq(1,2,\ldots,n)} |a_A|^2\Big)^{1/2}.$$

The generalized Cauchy–Riemann operator in $\mathbb{R}^{n+1}$ is given by

$$D = \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}.$$

If $U \subseteq \mathbb{R}^{n+1}$ is an open set, then a function $g : U \to Cl_n$ is called left (right) monogenic at a point $z \in U$ if $D\, g(z) = 0$ $(g\, D(z) = 0)$. The functions which are left (right) monogenic in the whole space are called left (right) entire monogenic functions.

Let A_{n+1} be the n-dimensional surface area of the $n+1$-dimensional unit ball and $q_{\mathbf{0}}(z) = \frac{\overline{z}}{\|z\|^{n+1}}$ be the Cauchy kernel function, then every function g which is left monogenic in a neighborhood of closure $\overline{G}$ of domain G satisfies the following equation (see [2], p. 766)

$$g(z) = \frac{1}{A_{n+1}} \int_{\partial G} q_{\mathbf{0}}(z-\zeta)\; d\tau(\zeta)\, g(\zeta), \text{ for all } \; z \in G,$$

where

$$d\tau(\zeta) = \sum_{j=0}^{n} (-1)^j e_j \widehat{d\zeta_j}$$

with

$$\widehat{d\zeta_j} = d\zeta_0 \wedge \ldots \wedge d\zeta_{j-1} \wedge d\zeta_{j+1} \wedge \ldots \wedge d\zeta_n,$$

is the oriented outer normal surface measure. If g is a left monogenic function in a ball $\|z\| < R$, then for all $\|z\| < r$ with $0 < r < R$,

$$g(z) = \sum_{|\mathbf{m}|=0}^{\infty} V_{\mathbf{m}}(z)\, a_{\mathbf{m}}. \tag{1.1}$$

In (1.1) $V_{\mathbf{m}}(z)$ are called Fueter polynomials and are given as

$$V_{\mathbf{m}}(z) = \frac{\mathbf{m}!}{|\mathbf{m}|!} \sum_{\pi\in perm(\mathbf{m})} z_{\pi(m_1)} \cdots z_{\pi(m_n)},$$

where $perm(\mathbf{m})$ is the set of all permutations of the sequence $(m_1, m_2 \ldots, m_n)$ and $z_i = x_i - x_0e_i$, for $i = 1, \ldots, n$ and $V_{\mathbf{0}}(z) = 1$. Also in

(1.1) a_{m} are Clifford numbers which are defined by

$$a_{\mathbf{m}} = \frac{1}{\mathbf{m}!\, A_{n+1}} \int\limits_{\|\zeta\|<r} q_{\mathbf{m}}(\zeta)\ d\tau(\zeta)\, g(\zeta)$$

and satisfy the inequality

$$\|a_{\mathbf{m}}\| \leq c(n, \mathbf{m}) \frac{M(r,g)}{r^{|\mathbf{m}|}}.$$

Here $M(r,g) = \max\limits_{\|z\|=r} \{\|g(z)\|\}$ denotes the maximum modulus of the function g in the closed ball of radius r and

$$q_{\mathbf{m}}(z) = \frac{\partial^{m_0+m_1+\cdots+m_n}}{\partial x_0^{m_0} \partial x_1^{m_1} \ldots \partial x_n^{m_n}} q_{\mathbf{0}}(z), \quad c(n, \mathbf{m}) = \frac{n(n+1)\ldots(n+|\mathbf{m}|-1)}{\mathbf{m}!}.$$

We now introduce two classes of functions as defined by Seremeta [4]. Hence, we denote by L^0 the class of functions h satisfying the following conditions:

(i) $h(x)$ is defined on $[a, \infty)$, $a \in \mathbb{R}$, and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$,

(ii) $\lim\limits_{x\to\infty} \frac{h[\{1+1/\psi(x)\}x]}{h(x)} = 1$, for every function $\psi(x)$ such that $\psi(x) \to \infty$ as $x \to \infty$.

Let χ denote the class of functions h satisfying conditions (i) and

(iii) $\lim\limits_{x\to\infty} \frac{h(cx)}{h(x)} = 1$, for every $c > 0$, that is $h(x)$ is slowly increasing.

Using the growth functions of the above classes L^0 and χ, following Seremeta [4] and Shah [5], we define generalized order and generalized type of entire monogenic functions. For an entire monogenic function $g(z)$ and functions $\alpha(x) \in \chi$, $\beta(x) \in L^0$, we define the generalized order ρ and generalized lower order λ of $g(z)$ as

$$\begin{matrix} \rho(\alpha,\beta,g) \\ \lambda(\alpha,\beta,g) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha\ [\log M(r,g)]}{\beta\ (\log r)}. \tag{1.2}$$

Further, for $\alpha(x)$, $\beta(x)$ and $\gamma(x) \in L^0$, we define the generalized type σ of an entire monogenic function $g(z)$ as

$$\sigma(\alpha,\beta,\rho,g) = \lim_{r\to\infty} \sup \frac{\alpha\ [\log M(r,g)]}{\beta\ [\{\gamma(r)\}^{\rho}]}, \tag{1.3}$$

where $0 < \rho < \infty$ is a fixed number.

Now following Almeida and Kraußhar [1], we define the maximum term and central index of entire monogenic functions. Hence, let $g : \mathbb{R}^{n+1} \to Cl_n$ be an entire monogenic function whose Taylor's series representation is given by $g(z) = \sum_{|\mathbf{m}|=0}^{\infty} V_{\mathbf{m}}(z)\, a_{\mathbf{m}}$. Then for $r > 0$, the maximum term of this entire monogenic function is given by $\mu(r,g) = \max_{|\mathbf{m}|\geq 0}\{\|a_{\mathbf{m}}\| r^{|\mathbf{m}|}\}$. Also the index $\mathbf{m}$ with maximal length $|\mathbf{m}|$, for which maximum term is achieved, is called the central index and is denoted by $\nu(r,g) = \mathbf{m}$.

For an entire monogenic function $g(z)$ and functions $\alpha(x) \in \chi$, $\beta(x) \in L^0$, we define the generalized order and generalized lower order of $g(z)$ in terms of maximum term and central index as

$$\begin{matrix}\rho_1(\alpha,\beta,g) \\ \lambda_1(\alpha,\beta,g)\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha\{\log \mu(r,g)\}}{\beta(\log r)} \tag{1.4}$$

and

$$\begin{matrix}\rho_2(\alpha,\beta,g) \\ \lambda_2(\alpha,\beta,g)\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha\{|\nu(r,g)|\}}{\beta(\log r)}. \tag{1.5}$$

On the pattern of classical definitions of growth parameters Constales, Almeida and Krausshar ([2] and [3]) defined the order and type of an entire monogenic function and obtained their coefficient characterizations (see [2], Th. 1 and [3], Th. 1). They also obtained a lower estimate for the lower order of the entire function (see [2], Th. 2). Almeida and Krausshar obtained a lower estimate for the order and lower order of entire monogenic function in terms of maximum term and central indices (see [1], Prop. 5.3). In this paper, we extend these results for generalized order, generalized lower order and generalized type. We have also obtained a coefficient characterization for the generalized lower order. The results obtained here are valid for both left monogenic or right monogenic entire functions.

2. Main results

We now prove

Theorem 2.1. *Let $g : \mathbb{R}^{n+1} \to Cl_n$ be an entire monogenic function whose Taylor's series representation is given by $g(z) = \sum_{|\boldsymbol{m}|=0}^{\infty} V_{\boldsymbol{m}}(z)\, a_{\boldsymbol{m}}$. If $\alpha(x) \in \chi$ and $\beta(x) \in L^0$ then the generalized order ρ of $g(z)$ as defined in* (1.2) *is given as*

$$\rho = \rho(\alpha,\beta,g) = \lim_{|\boldsymbol{m}|\to\infty} \sup \frac{\alpha(|\boldsymbol{m}|)}{\beta\left\{\log \|a_{\boldsymbol{m}}/c(n,\boldsymbol{m})\|^{-1/|\boldsymbol{m}|}\right\}}. \tag{2.1}$$

Proof. Write

$$\theta = \lim_{|\mathbf{m}|\to\infty} \sup \frac{\alpha(|\mathbf{m}|)}{\beta\left\{\log \|a_{\mathbf{m}}/c(n,\mathbf{m})\|^{-1/|\mathbf{m}|}\right\}}.$$

Now first we prove that $\rho \geq \theta$. The coefficients of a monogenic Taylor's series satisfy Cauchy's inequality, that is

$$\|a_{\mathbf{m}}\| \leq M(r,g)\, c(n,\mathbf{m})\, r^{-|\mathbf{m}|}. \tag{2.2}$$

Also from (1.2), for arbitrary $\varepsilon > 0$ and all $r > r_0(\varepsilon)$, we have

$$M(r,g) \le \exp\left[\alpha^{-1}\left\{\overline{\rho}\,\beta(\log r)\right\}\right], \ \ \overline{\rho} = \rho + \varepsilon$$

or

$$\|a_{\mathbf{m}}\| \le c(n,\mathbf{m})\, r^{-|\mathbf{m}|} \exp\left[\alpha^{-1}\left\{\overline{\rho}\,\beta(\log r)\right\}\right].$$

Putting $r = \exp\left[\beta^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\rho}\right\}\right]$ in the above inequality, we get for all large values of $|\mathbf{m}|$,

$$\|a_{\mathbf{m}}\| \le c(n,\mathbf{m}) \exp\left[|\mathbf{m}| - |\mathbf{m}|\beta^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\rho}\right\}\right]$$

or

$$\beta^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\rho}\right\} \le 1 - \frac{1}{|\mathbf{m}|}\left\{\log\|a_{\mathbf{m}}/c(n,\mathbf{m})\|\right\}$$

or

$$\frac{\alpha(|\mathbf{m}|)}{\beta\left\{1 + \log\|a_{\mathbf{m}}/c(n,\mathbf{m})\|^{-1/|\mathbf{m}|}\right\}} \le \overline{\rho}.$$

Since $\beta(x) \in L^0$, $\beta(1+x) \simeq \beta(x)$. Hence proceeding to limits as $|\mathbf{m}| \to \infty$, we get

$$\theta = \limsup_{|\mathbf{m}|\to\infty} \frac{\alpha(|\mathbf{m}|)}{\beta\left\{\log\|a_{\mathbf{m}}/c(n,\mathbf{m})\|^{-1/|\mathbf{m}|}\right\}} \le \overline{\rho}.$$

Since $\varepsilon > 0$ is arbitrarily small, we finally get

$$\theta \le \rho. \tag{2.3}$$

Now we will prove that $\theta \ge \rho$. If $\theta = \infty$, then there is nothing to prove. So let us assume that $0 \le \theta < \infty$. Therefore, for a given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all multi-indices $\mathbf{m}$ with $|\mathbf{m}| > n_0$, we have

$$0 \le \frac{\alpha(|\mathbf{m}|)}{\beta\left\{\log\|a_{\mathbf{m}}/c(n,\mathbf{m})\|^{-1/|\mathbf{m}|}\right\}} < \theta + \varepsilon = \overline{\theta}$$

or

$$\|a_{\mathbf{m}}/c(n,\mathbf{m})\| \le \exp\left[-|\mathbf{m}|\beta^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\theta}\right\}\right].$$

Now from the property of maximum modulus, we have

$$M(r,g) \le \sum_{|\mathbf{m}|=0}^{\infty} \|a_{\mathbf{m}}\| r^{|\mathbf{m}|}$$

or

$$M(r,g) \le \sum_{|\mathbf{m}|=0}^{n_0} \|a_{\mathbf{m}}\| r^{|\mathbf{m}|} + \sum_{|\mathbf{m}|=n_0+1}^{\infty} c(n,\mathbf{m})\, r^{|\mathbf{m}|} \exp\left[-|\mathbf{m}|\beta^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\theta}\right\}\right].$$

Now for $r > 1$, we have

$$M(r,g) \le A_1 r^{n_0} + \sum_{|\mathbf{m}|=n_0+1}^{\infty} c(n,\mathbf{m})\, r^{|\mathbf{m}|} \exp\left[-|\mathbf{m}|\beta^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\theta}\right\}\right], \tag{2.4}$$

where A_1 is a positive real constant. We take

$$N(r) = \left[\alpha^{-1}\left\{\overline{\theta}\beta[\log\{(n+1)r\}]\right\}\right],$$

where $[x]$ denotes the integer part of $x \geq 0$. Since $\alpha(x) \in \chi$ and $\beta(x) \in L^0$, the integer $N(r)$ is well defined. Now we choose

$$r > \max\left\{1, \frac{1}{n+1}\exp\left(\beta^{-1}\left(\frac{\alpha(n_0+1)}{\bar{\theta}}\right)\right)\right\}.$$

Then from (2.4), we have

$$M(r,g) \leq A_1 r^{n_0} + r^{N(r)} \sum_{n_0+1 \leq |\mathbf{m}| \leq N(r)} c(n,\mathbf{m}) \exp\left[-|\mathbf{m}|\beta^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\theta}\right\}\right]$$
$$+ \sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m})\, r^{|\mathbf{m}|} \exp\left[-|\mathbf{m}|\beta^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\theta}\right\}\right]$$

or

$$M(r,g) \leq A_1 r^{n_0} + r^{N(r)} \sum_{|\mathbf{m}|=1}^{\infty} c(n,\mathbf{m}) \exp\left[-|\mathbf{m}|\beta^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\theta}\right\}\right]$$
$$+ \sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m})\, r^{|\mathbf{m}|} \exp\left[-|\mathbf{m}|\beta^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\theta}\right\}\right]. \tag{2.5}$$

Now the first series in (2.5) can be rewritten as

$$\sum_{p=1}^{\infty}\Big(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m})\Big) \exp\left[-p\beta^{-1}\left\{\alpha(p)/\overline{\theta}\right\}\right]. \tag{2.6}$$

Now from ([2], Lemma 1), we have

$$\limsup_{p\to\infty}\Big(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m})\Big)^{1/p} = n.$$

Hence, we have

$$\limsup_{p\to\infty}\Big[\Big(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m})\Big) \exp\left[-p\beta^{-1}\left\{\alpha(p)/\overline{\theta}\right\}\right]\Big]^{1/p}$$
$$= n \limsup_{p\to\infty} \exp\left[-\beta^{-1}\left\{\alpha(p)/\overline{\theta}\right\}\right] = 0.$$

Hence the series (2.6) converges to a positive real constant A_2. So from (2.5), we get

$$M(r,g) \leq A_1 r^{n_0} + A_2 r^{N(r)}$$
$$+ \sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m})\, r^{|\mathbf{m}|} \exp\left[-|\mathbf{m}|\beta^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\theta}\right\}\right]$$

or

$$M(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m})\, r^{|\mathbf{m}|} \exp\left[-|\mathbf{m}| \log\{(n+1)r\}\right]$$

or

$$M(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m}) \left(\frac{1}{n+1}\right)^{|\mathbf{m}|}$$

or

$$M(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{|\mathbf{m}|=1}^{\infty} c(n,\mathbf{m}) \left(\frac{1}{n+1}\right)^{|\mathbf{m}|}. \tag{2.7}$$

The series in (2.7) can be rewritten as

$$\sum_{p=1}^{\infty} \Big(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m}) \Big) \left(\frac{1}{n+1}\right)^{p}. \tag{2.8}$$

So we have

$$\limsup_{p\to\infty} \left[\Big(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m}) \Big) \left\{\frac{1}{n+1}\right\}^{p} \right]^{1/p} = \frac{n}{n+1} < 1.$$

Hence the series (2.8) converges to a positive real constant A_3. Therefore from (2.7), we get

$$M(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + A_3.$$

Since $N(r) \to \infty$ as $r \to \infty$, we can write the above inequality as

$$\log M(r,g) \le \; [1+o(1)]\; N(r) \log r$$

or

$$\log M(r,g) \le \; [1+o(1)] \; \left[\alpha^{-1}\left\{\overline{\theta}\beta[\log\{(n+1)r\}]\right\}\right] \; \log r$$

or

$$\log M(r,g) \le \left[\alpha^{-1}\left\{\left(\overline{\theta}+\delta_1\right)\beta[\log\{(n+1)r\}]\right\}\right][1+o(1)],$$

where $\delta_1 > 0$ is suitably small. Using the properties of $\alpha(x)$ and $\beta(x)$, we get

$$\frac{\alpha[\log M(r,g)]}{\beta[\log r]} \le (\overline{\theta}+\delta_1)\,[1+o(1)].$$

Since δ_1 and ε are arbitrary, proceeding to limits as $r \to \infty$, we get $\rho \le \theta$. Combining this with the inequality (2.3), we get (2.1). Hence Theorem 2.1 is proved. ■

Next we prove

Theorem 2.2. *Let $g : \mathbb{R}^{n+1} \to Cl_n$ be an entire monogenic function whose Taylor's series representation is given by $g(z) = \sum_{|\boldsymbol{m}|=0}^{\infty} V_{\boldsymbol{m}}(z)\, a_{\boldsymbol{m}}$. Also let $\alpha(x), \beta(x), \gamma(x) \in L^0$ and $0 < \rho < \infty$, then the generalized type σ of $g(z)$ as defined in* (1.3) *is given as*

$$\sigma = \sigma(\alpha,\beta,\rho,g) = \lim_{|\boldsymbol{m}|\to\infty} \sup \frac{\alpha(|\boldsymbol{m}|/\rho)}{\beta\left[\left\{\gamma\left(e^{1/\rho}\|a_{\boldsymbol{m}}/c(n,\boldsymbol{m})\|^{-1/|\boldsymbol{m}|}\right)\right\}^{\rho}\right]}. \tag{2.9}$$

Proof. Write

$$\eta = \lim_{|\mathbf{m}|\to\infty} \sup \frac{\alpha(|\mathbf{m}|/\rho)}{\beta\left[\left\{\gamma\left(e^{1/\rho}\|a_{\mathbf{m}}/c(n,\mathbf{m})\|^{-1/|\mathbf{m}|}\right)\right\}^{\rho}\right]}.$$

First we prove that $\sigma \geq \eta$. From (1.3), for arbitrary $\varepsilon > 0$ and all $r > r_0(\varepsilon)$, we have

$$M(r,g) \leq \exp\left[\alpha^{-1}\left\{\overline{\sigma}\,\beta[\{\gamma(r)\}^{\rho}]\right\}\right],$$

where $\overline{\sigma} = \sigma + \varepsilon$. Now using (2.2), we get

$$\|a_{\mathbf{m}}\| \leq c(n,\mathbf{m})\, r^{-|\mathbf{m}|} \exp\left[\alpha^{-1}\left\{\overline{\sigma}\,\beta[\{\gamma(r)\}^{\rho}]\right\}\right].$$

Putting $r = \gamma^{-1}\left[\left\{\beta^{-1}\left(\frac{1}{\overline{\sigma}}\alpha(|\mathbf{m}|/\rho)\right)\right\}^{1/\rho}\right]$, we get

$$\|a_{\mathbf{m}}\| \leq c(n,\mathbf{m})\left(\gamma^{-1}\left[\left\{\beta^{-1}\left(\frac{1}{\overline{\sigma}}\alpha(|\mathbf{m}|/\rho)\right)\right\}^{1/\rho}\right]\right)^{-|\mathbf{m}|}\exp(|\mathbf{m}|/\rho)$$

or

$$\|a_{\mathbf{m}}/c(n,\mathbf{m})\|^{-1/|\mathbf{m}|} \geq \left(\gamma^{-1}\left[\left\{\beta^{-1}\left(\frac{1}{\overline{\sigma}}\alpha(|\mathbf{m}|/\rho)\right)\right\}^{1/\rho}\right]\right)\exp(-1/\rho)$$

or

$$\frac{\alpha(|\mathbf{m}|/\rho)}{\beta\left[\left\{\gamma\left(e^{1/\rho}\|a_{\mathbf{m}}/c(n,\mathbf{m})\|^{-1/|\mathbf{m}|}\right)\right\}^{\rho}\right]} \leq \overline{\sigma}.$$

Now proceeding to limits as $|\mathrm{m}| \to \infty$, we get

$$\eta \leq \overline{\sigma}.$$

Since $\varepsilon > 0$ is arbitrarily small, so finally we get

$$\eta \leq \sigma.$$

Now we will prove that $\eta \geq \sigma$. If $\eta = \infty$, then there is nothing to prove. So let us assume that $0 \leq \eta < \infty$. Therefore, for a given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all multi-indices $\mathbf{m}$ with $|\mathbf{m}| > n_0$, we have

$$0 \leq \frac{\alpha(|\mathbf{m}|/\rho)}{\beta\left[\left\{\gamma\left(e^{1/\rho}\|a_{\mathbf{m}}/c(n,\mathbf{m})\|^{-1/|\mathbf{m}|}\right)\right\}^{\rho}\right]} \leq \eta + \varepsilon = \overline{\eta}$$

or

$$\|a_{\mathbf{m}}/c(n,\mathbf{m})\| \leq \left(\gamma^{-1}\left[\left\{\beta^{-1}\left(\frac{1}{\overline{\eta}}\alpha(|\mathrm{m}|/\rho)\right)\right\}^{1/\rho}\right]\right)^{-|\mathbf{m}|} e^{|\mathbf{m}|/\rho}.$$

Now from the property of maximum modulus, we have

$$M(r,g) \leq \sum_{|\mathbf{m}|=0}^{n_0} \|a_{\mathbf{m}}\| r^{|\mathbf{m}|} + \sum_{|\mathbf{m}|=n_0+1}^{\infty} c(n,\mathbf{m}) \times \left(\gamma^{-1}\left[\left\{\beta^{-1}\left(\frac{1}{\overline{\eta}}\alpha(|\mathbf{m}|/\rho)\right)\right\}^{1/\rho}\right]\right)^{-|\mathrm{m}|} r^{|\mathbf{m}|} e^{|\mathbf{m}|/\rho}.$$

Now for $r > 1$, we have

$$M(r,g) \leq B_1 r^{n_0} + \sum_{|\mathbf{m}|=n_0+1}^{\infty} c(n,\mathbf{m}) \times \left(\gamma^{-1}\left[\left\{\beta^{-1}\left(\frac{1}{\overline{\eta}}\alpha(|\mathbf{m}|/\rho)\right)\right\}^{1/\rho}\right]\right)^{-|\mathbf{m}|} r^{|\mathbf{m}|} e^{|\mathbf{m}|/\rho}, \tag{2.10}$$

where B_1 is a positive real constant. We take

$$N(r) = \left[\rho\,\alpha^{-1}\left\{\overline{\eta}\,\beta\left(\left[\gamma\{(n+1)re^{1/\rho}\}\right]^{\rho}\right)\right\}\right],$$

where $[x]$ denotes the integer part of $x \geq 0$. Since $\alpha(x), \beta(x)$ and $\gamma(x) \in L^0$, the integer $N(r)$ is well defined. Now we choose

$$r > \max\left[1, \frac{e^{-1/\rho}}{n+1}\left\{\gamma^{-1}\left(\beta^{-1}\left(\frac{1}{\overline{\eta}}\,\alpha\left(\frac{n_0+1}{\rho}\right)\right)\right)^{1/\rho}\right\}\right].$$

Then from (2.10), we have

$$M(r,g) \leq B_1 r^{n_0} + r^{N(r)} \sum_{n_0+1\leq|\mathbf{m}|\leq N(r)} c(n,\mathbf{m}) \times \left(\gamma^{-1}\left[\left\{\beta^{-1}\left(\frac{1}{\overline{\eta}}\alpha(|\mathbf{m}|/\rho)\right)\right\}^{1/\rho}\right]\right)^{-|\mathbf{m}|} e^{|\mathbf{m}|/\rho}$$
$$+ \sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m}) \left(\gamma^{-1}\left[\left\{\beta^{-1}\left(\frac{1}{\overline{\eta}}\alpha(|\mathbf{m}|/\rho)\right)\right\}^{1/\rho}\right]\right)^{-|\mathbf{m}|} r^{|\mathbf{m}|} e^{|\mathbf{m}|/\rho}$$

or

$$M(r,g) \leq B_1 r^{n_0} + r^{N(r)} \sum_{|\mathbf{m}|=1}^{\infty} c(n,\mathbf{m}) \times \left(\gamma^{-1}\left[\left\{\beta^{-1}\left(\frac{1}{\eta}\alpha(|\mathbf{m}|/\rho)\right)\right\}^{1/\rho}\right]\right)^{-|\mathbf{m}|} e^{|\mathbf{m}|/\rho} + \sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m})\left(\gamma^{-1}\left[\left\{\beta^{-1}\left(\frac{1}{\eta}\alpha(|\mathbf{m}|/\rho)\right)\right\}^{1/\rho}\right]\right)^{-|\mathbf{m}|} r^{|\mathbf{m}|}\, e^{|\mathbf{m}|/\rho}. \tag{2.11}$$

Now the first series in (2.11) can be rewritten as

$$\sum_{p=1}^{\infty}\Big(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m})\Big)\left(\gamma^{-1}\left[\left\{\beta^{-1}\left(\frac{1}{\eta}\alpha(p/\rho)\right)\right\}^{1/\rho}\right]\right)^{-p} e^{p/\rho}. \tag{2.12}$$

As in the proof of Theorem 2.1, we have

$$\lim_{p\to\infty}\sup\left[\Big(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m})\Big)\left(\gamma^{-1}\left[\left\{\beta^{-1}\left(\frac{1}{\eta}\alpha(p/\rho)\right)\right\}^{1/\rho}\right]\right)^{-p} e^{p/\rho}\right]^{1/p} = n \lim_{p\to\infty}\sup\left(\gamma^{-1}\left[\left\{\beta^{-1}\left(\frac{1}{\eta}\alpha(p/\rho)\right)\right\}^{1/\rho}\right]\right)^{-1} e^{1/\rho} = 0.$$

Hence, the series (2.12) converges to a positive real constant B_2. So from (2.11), we get

$$M(r,g) \leq B_1 r^{n_0} + B_2 r^{N(r)} + \sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m}) \times \left(\gamma^{-1}\left[\left\{\beta^{-1}\left(\frac{1}{\eta}\alpha(|\mathbf{m}|/\rho)\right)\right\}^{1/\rho}\right]\right)^{-|\mathbf{m}|} r^{|\mathbf{m}|}\, e^{|\mathbf{m}|/\rho}$$

or

$$M(r,g) \leq B_1 r^{n_0} + B_2 r^{N(r)} + \sum_{|\mathbf{m}|>N(r)} c(n,\ \mathbf{m})\ [(n+1)re^{1/\rho}]^{-|\mathbf{m}|}\, r^{|\mathbf{m}|}\, e^{|\mathbf{m}|/\rho}$$

or

$$M(r,g) \leq B_1 r^{n_0} + B_2 r^{N(r)} + \sum_{|\mathbf{m}|=1}^{\infty} c(n,\mathbf{m})\left(\frac{1}{n+1}\right)^{|\mathbf{m}|}.$$

Now from Theorem 2.1, we can say that the series

$$\sum_{|\mathbf{m}|=1}^{\infty} c(n,\mathbf{m})\left(\frac{1}{n+1}\right)^{|\mathbf{m}|}$$

converges to a positive real number B_3. Hence

$$M(r,g) \leq B_1 r^{n_0} + B_2 r^{N(r)} + B_3.$$

Proceeding as in proof of Theorem 2.1, we get

$$\log M(r,g) \leq \ [1+o(1)] N(r) \log r$$

or

$$\log M(r,g) \leq \ [1+o(1)] \ \left[\rho\, \alpha^{-1} \left\{\overline{\eta}\, \beta \left(\left[\gamma\{(n+1) r e^{1/\rho}\} \right]^{\rho} \right) \right\} \right] \log r$$

or

$$\log M(r,g) \leq \ [1+o(1)] \alpha^{-1} \left[(\overline{\eta} + \delta_2)\, \beta \left(\left[\gamma\{(n+1) r e^{1/\rho}\} \right]^{\rho} \right) \right],$$

where $\delta_2 > 0$ is suitably small. Hence, using the properties of $\alpha(x)$, we get

$$\alpha[\log M(r,g)] \leq \ (\overline{\eta} + \delta_2)\, \beta \left(\left[\gamma\{(n+1) r e^{1/\rho}\} \right]^{\rho} \right)$$

or

$$\alpha[\log M(r,g)] \leq \ (\overline{\eta} + \delta_2)\, \beta[\{\gamma(r)\}^{\rho} + O(1)]$$

or

$$\alpha[\log M(r,g)] \leq \ (\overline{\eta} + \delta_2)\, \beta[\{\gamma(r)\}^{\rho}\{1+o(1)\}].$$

Using properties of $\beta(x)$, we get

$$\frac{\alpha\ [\log M(r,g)]}{\beta\ [\{\gamma(r)\}^{\rho}]} \leq \ (\overline{\eta} + \delta_2)\ [1+o(1)].$$

Since δ_2 and ε are arbitrary, proceeding to limits as $r \to \infty$, we get $\sigma \leq \eta$. Combining this with the reverse inequality obtained earlier, we get (2.9). Hence Theorem 2.2 is proved. ∎

Next we prove

Theorem 2.3. *Let $g : \mathbb{R}^{n+1} \to Cl_n$ be an entire monogenic function whose Taylor's series representation is given by $g(z) = \sum_{|\boldsymbol{m}|=0}^{\infty} V_{\boldsymbol{m}}(z)\, a_{\boldsymbol{m}}$. Also suppose that $\alpha(x) \in \chi$, $\beta(x) \in L^0$ and λ_2, ρ_2 is defined as in (1.5) with $0 \leq \rho_2 < \infty$. If for arbitrary $\varepsilon > 0$ and $\overline{\rho_2} = \rho_2 + \varepsilon$*

$$\lim_{r\to\infty} \frac{\alpha\left\{n \log\left[\alpha^{-1}\left\{\overline{\rho_2}\beta(\log 2r)\right\}\right]\right\}}{\beta(\log r)} = 0, \tag{2.13}$$

then the generalized order $\rho(\alpha,\beta,g)$ and generalized lower order $\lambda(\alpha,\beta,g)$ of this entire monogenic function $g(z)$ satisfy

$$\rho(\alpha,\beta,g) \leq \ \rho_1(\alpha,\beta,g) = \rho_2(\alpha,\beta,g)$$

and

$$\lambda(\alpha,\beta,g) \leq \ \lambda_1(\alpha,\beta,g) = \lambda_2(\alpha,\beta,g).$$

Proof. For simplicity we write $\rho(\alpha,\beta,g)=\rho,\ \lambda(\alpha,\beta,g)=\lambda$ and

$$\rho_1=\rho_1(\alpha,\beta,g),\ \rho_2=\rho_2(\alpha,\beta,g),$$
$$\lambda_1=\lambda_1(\alpha,\beta,g),\ \lambda_2=\lambda_2(\alpha,\beta,g).$$

Now following ([1], page 805), for sufficiently large value of r, we have

$$M(r,g))\le \mu(r,g)|\nu(2r,g)|^n[1+o(1)]. \tag{2.14}$$

Also from (1.5), for arbitrary $\varepsilon>0$ and all $r>r_0(\varepsilon)$, we have

$$|v(r,g)|\le \alpha^{-1}\left\{\overline{\rho_2}\beta(\log r)\right\}.$$

Therefore from (2.14), we get

$$M(r,g)\le \mu(r,g)\left(\alpha^{-1}\left\{\overline{\rho_2}\beta(\log 2r)\right\}\right)^n[1+o(1)]$$

or

$$\log M(r,g)\le \log\mu(r,g)+n\log\left[\alpha^{-1}\left\{\overline{\rho_2}\beta(\log 2r)\right\}\right]\ +\ \log[1+o(1)]$$

or

$$\alpha[\log M(r,g)]\le \alpha\left\{\log\mu(r,g)+n\log\left[\alpha^{-1}\left\{\overline{\rho_2}\beta(\log 2r)\right\}\right]\ [1+o(1)]\right\}.$$

Since α is a slowly increasing function, therefore we have

$$\alpha[\log M(r,g)]\le \alpha[\log\mu(r,g)]+\alpha\left\{n\log\left[\alpha^{-1}\left\{\overline{\rho_2}\beta(\log 2r)\right\}\right]\right\}$$

or

$$\frac{\alpha[\log M(r,g)]}{\beta(\log r)}\le\frac{\alpha[\log\mu(r,g)]}{\beta(\log r)}+\frac{\alpha\left\{n\log\left[\alpha^{-1}\left\{\overline{\rho_2}\beta(\log 2r)\right\}\right]\right\}}{\beta(\log r)}.$$

Now proceeding to limits and using (2.13), we get

$$\rho\le\rho_1\quad\text{and}\quad\lambda\le\lambda_1.$$

Now following ([1], page 805), we can say that $\rho_1=\rho_2$ and $\lambda_1=\lambda_2$. Hence Theorem 2.3 is proved. ∎

Lastly, we prove

Theorem 2.4. *Let $g:\mathbb{R}^{n+1}\to Cl_n$ be an entire monogenic function whose Taylor's series representation is given by $g(z)=\sum_{|\boldsymbol{m}|=0}^{\infty}a_{\boldsymbol{m}}V_{\boldsymbol{m}}(z)$. Also if $\alpha(x)\in\chi,\ \beta(x)\in L^0$, then the generalized lower order λ of this entire monogenic function $g(z)$ satisfies*

$$\lambda=\lambda(\alpha,\beta,g)\ge\lim_{|\boldsymbol{m}|\to\infty}\inf\frac{\alpha(|\boldsymbol{m}|)}{\beta\left\{\log\|a_{\boldsymbol{m}}/c(n,\boldsymbol{m})\|^{-1/|\boldsymbol{m}|}\right\}}. \tag{2.15}$$

Further, if

$$\psi(k)\ =\ \max_{|\boldsymbol{m}|=k}\left\{\frac{\|a_{\boldsymbol{m}}\|}{\|a_{\boldsymbol{m}'}\|},\ |\boldsymbol{m}'|=|\boldsymbol{m}|+1\right\}$$

is a non-decreasing function of k then equality holds in (2.15).

Proof. Write

$$\Phi = \lim_{|\mathbf{m}|\to\infty} \inf \frac{\alpha(|\mathbf{m}|)}{\beta\left\{\log \|a_{\mathbf{m}}/c(n,\mathbf{m})\|^{-1/|\mathbf{m}|}\right\}}.$$

First we prove that $\lambda \geq \Phi$. From (1.2), for arbitrary $\varepsilon > 0$ and a sequence $r = r_k \to \infty$ as $k \to \infty$, we have

$$M(r,g) \leq \exp[\alpha^{-1}\{\overline{\lambda}\,\beta(\log r)\}],$$

where $\overline{\lambda} = \lambda + \varepsilon$. Now proceeding as in the proof of Theorem 2.1, we get

$$\|a_{\mathbf{m}}\| \ \leq c(n,\mathbf{m})\, r^{-|\mathbf{m}|} \exp[\alpha^{-1}\{\overline{\lambda}\,\beta(\log r)\}].$$

Putting $r = \exp[\beta^{-1}\{\alpha(|\mathbf{m}|)/\overline{\lambda}\}]$ in the above inequality, we get

$$\|a_{\mathbf{m}}\| \ \leq c(n,\mathbf{m}) \exp[|\mathbf{m}| - |\mathbf{m}|\beta^{-1}\{\alpha(|\mathbf{m}|)/\overline{\lambda}\}]$$

or

$$\beta^{-1}\left[\alpha(|\mathbf{m}|)/\overline{\lambda}\right] \leq 1 - \frac{1}{|\mathbf{m}|}\{\log \|a_{\mathbf{m}}/c(n,\mathbf{m})\|\}$$

or

$$\frac{\alpha(|\mathbf{m}|)}{\beta\left\{1 + \ \log \|a_{\mathbf{m}}/c(n,\mathbf{m})\|^{-1/|\mathrm{m}|}\right\}} \leq \overline{\lambda}.$$

Since $\beta(x) \in L^0$, $\beta(1+x) \simeq \beta(x)$. Hence proceeding to limits as $|\mathbf{m}| = |\mathbf{m}(k)| \to \infty$, we get

$$\Phi \ = \lim_{|\mathbf{m}|\to\infty} \inf \frac{\alpha(|\mathbf{m}|)}{\beta\left\{\log \|a_{\mathbf{m}}/c(n,\mathbf{m})\|^{-1/|\mathbf{m}|}\right\}} \leq \overline{\lambda}.$$

Since $\varepsilon > 0$ is arbitrarily small, so finally we get

$$\Phi \leq \lambda.$$

Now we prove that $\lambda \leq \Phi$. From the assumption on ψ, $\psi(k) \to \infty$ as $k \to \infty$. By the definition given in section 1, if $\|a_{\mathbf{m}}\|\, r^{|\mathbf{m}|}$ is the maximum term for r then for $|\mathbf{m}_1| \leq |\mathbf{m}| < |\mathbf{m}_2|$, $\|a_{\mathbf{m}_1}\|\, r^{|\mathbf{m}_1|} \leq \|a_{\mathbf{m}}\|\, r^{|\mathbf{m}|} > \|a_{\mathbf{m}_2}\|\, r^{|\mathbf{m}_2|}$ and for $|\mathbf{m}| = k$, $\psi(k-1) \leq r < \psi(k)$.

Now suppose that $\|a_{\mathbf{m}^1}\| r^{|\mathbf{m}^1|}$ and $\|a_{\mathbf{m}^2}\| r^{|\mathbf{m}^2|}$ are two consecutive maximum terms. Then $|\mathbf{m}^1| \leq |\mathbf{m}^2| - 1$. Let $|\mathbf{m}^1| \leq |\mathbf{m}| \leq |\mathbf{m}^2|$, then $|\nu(r,g)| = |\mathbf{m}^1|$ for $\psi(|\mathbf{m}^{1^*}|) \leq r < \psi(|\mathbf{m}^1|)$, where $|\mathbf{m}^{1^*}| = |\mathbf{m}^1| - 1$. Hence from (1.5), for arbitrary $\varepsilon > 0$ and all $r > r_0(\varepsilon)$, we have

$$|\mathbf{m}^1| \ = \ |\nu(r,g)| > \alpha^{-1}\left\{\lambda'\beta(\log r)\right\}, \quad \text{where } \lambda' = \lambda - \varepsilon,$$

or

$$|\mathbf{m}^1| \ = \ |\nu(r,g)| \ \geq \ \alpha^{-1}\left\{\lambda'\beta[\log\{\psi(|\mathbf{m}^1|) - d\}]\right\},$$

where d is a constant such that $0 < d < \min\left\{1,\ [\psi(|\mathbf{m}^1|) - \psi(|\mathbf{m}^{1^*}|)]/2\right\}$ or

$$\log \psi(|\mathbf{m}^1|) \leq \ O(1) + \beta^{-1}\{\alpha(|\mathbf{m}^1|)/\lambda'\}.$$

Further we have

$$\psi(|\mathbf{m}^1|) = \psi(|\mathbf{m}^1| + 1) \; = \cdots = \psi(|\mathbf{m}| - 1).$$

Using the definition of $\psi(k)$, we have

$$\frac{\|a_{\mathbf{m}^0}\|}{\|a_{\mathbf{m}}\|} \; \leq \psi(|\mathbf{m}^0|) \; \ldots \; \psi(|\mathbf{m}^*|) \; \leq [\psi(|\mathbf{m}^*|]^{|\mathbf{m}|-|\mathbf{m}^0|},$$

where $|\mathbf{m}^*| = |\mathbf{m}| - 1$ and $|\mathbf{m}| \; \gg \; |\mathbf{m}^0|$,
or

$$c(n, \mathbf{m}) \frac{\|a_{\mathbf{m}^0}\|}{\|a_{\mathbf{m}}\|} \; \leq c(n, \mathbf{m}) \, [\psi(|\mathbf{m}^*|]^{|\mathrm{m}|-|\mathrm{m}^0|}$$

or

$$\log \|a_{\mathbf{m}}/c(n, \mathbf{m})\| \,^{-1} \leq \; |\mathbf{m}| \log \psi(|\mathbf{m}^1|) + O(1)$$

or

$$\log \|a_{\mathbf{m}}/c(n, \mathbf{m})\|^{-1} \leq |\mathbf{m}| \beta^{-1}\{\alpha(|\mathbf{m}^1|)/\lambda'\} + O(1)$$

or

$$-\frac{1}{|\mathbf{m}|} \log \|a_{\mathbf{m}}/c(n, \mathbf{m})\| \leq [\beta^{-1}\{\alpha(|\mathbf{m}^1|)/\lambda'\}][1 + o(1)]$$

or

$$-\frac{1}{|\mathbf{m}|} \log \|a_{\mathbf{m}}/c(n, \mathbf{m})\| \leq [\beta^{-1}\{\alpha(|\mathbf{m}|)/\lambda'\}][1 + o(1)]$$

or

$$\lambda' \leq \; \frac{\alpha(|\mathbf{m}|)}{\beta \left\{ \log \|a_{\mathbf{m}}/c(n, \mathbf{m})\|^{-1/|\mathbf{m}|} \right\}} [1 + o(1)].$$

Now taking limits as $|\mathbf{m}| \to \infty$, we get $\lambda \leq \Phi$. Hence, Theorem 2.4 is proved. ∎

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