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SOME NEW GENERALIZATIONS OF MARONI INEQUALITY ON TIME SCALES

Abstract. The aim of present paper is to establish some new integral inequalities on time scales involving several functions and their derivatives which in the special cases yield the well known Maroni inequality and some of its generalizations.

1. Introduction

Opial type inequality have many applications in the theory of differential and difference equation, for instance some inequalities are used to prove existence of solutions (see [1]). For some classical results on Opial inequality, the readers are referred to the articles (see [2]-[9], [15]).

Now, we give some of earliest version of the Opial inequlality as follows:

Theorem 1.1. *Let $f(x) \in C^1[0,a]$ with $f(0) = f(a) = 0$ and $f(x) > 0$, $(0 < x < a)$. Then*

$$\int_0^a \left|f(x)f'(x)\right| dx \leqslant \frac{a}{4} \int_0^a \left|f'(x)\right|^2 dx, \tag{1.1}$$

where the constant factor $\frac{a}{4}$ is the best possible. Equality holds true in (1.1) *if and only if*

$$f(x) = \begin{cases} cx, & 0 \leqslant x \leqslant \frac{a}{2}, \\ c(a-x), & \frac{a}{2} \leqslant x \leqslant a, \end{cases} \tag{1.2}$$

where c is a positive constant.

Beesack in [16] generalized Opial's inequality as follows

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Theorem 1.2. *Let the function $p(x)$ be nonnegative and continuous on $[a,\tau]$. If the function $f(x)$ is absolutely continuous on $[a,\tau]$ with $f(a)=0$. The following inequality holds true:*

$$\int_a^\tau \left|f(x)f'(x)\right| dx \leqslant \frac{1}{2}\Big(\int_a^\tau (p(x))^{-1}dx\Big)\Big(\int_a^\tau p(x)\left|f'(x)\right|^2 dx\Big). \tag{1.3}$$

Subsequently, Maroni in [10] further generalized Beesack's result in a form given by Theorem 1.3.

Theorem 1.3. *Let the function $p(x)$ be nonnegative and continuous on $[a,\tau]$ with $\int_a^\tau (p(x))^{1-u}dx < \infty, (u \geq 1)$. Also, let the function $f(x)$ be absolutely continuous on $[a,\tau]$ and suppose that $f(a) = f(\tau) = 0$. The following inequality holds true:*

$$\int_a^\tau \left|f(x)f'(x)\right| dx \leqslant \frac{1}{2}\Big(\int_a^\tau (p(x))^{1-u}dx\Big)^{\frac{2}{u}}\Big(\int_a^\tau p(x)\left|f'(x)\right|^v dx\Big)^{\frac{2}{v}} \tag{1.4}$$

where $\frac{1}{u}+\frac{1}{v}=1$.

Therefore, some very interesting generalizations are given by B. G. Pachpatte who works with several functions in Opial type inequalities. We give the following case (see [8], [9], [11]).

Theorem 1.4. *For $f(x), g(x) \in C^1[0,a]$ with $f(a)=g(a)=0$, we have*

$$\int_a^b (\left|f(x)g'(x)\right| + \left|f'(x)g(x)\right|)dx \leqslant \frac{b-a}{2}\int_a^b (\left|f'(x)\right|^2 + \left|g'(x)\right|^2)dx \tag{1.5}$$

with equality $f(x)=g(x)=c(x-a)$ for $x \in [a,b]$ where c is constant.

Next, R. P. Agarwal extended Theorem A to hold true on time scales in [12]. H. M. Srivastava etc. obtained some generalizations of Maroni inequality on time scales in [13]. For more results on time scales see [14].

In this paper, we obtain some new Maroni type inequality on time scales involving several functions. Our results are not only new for arbitrary time scales, but also new for the continuous and the discrete cases. Our results generalized some of the works done by B. G. Pachpathe.

2. Time scales essential

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$
$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where the supremum of the empty set is defined to be the infimum of $\mathbb{T}$.

A point $t \in \mathbb{T}$ is said to be right-scattered if $\sigma(t) > t$ and right-dense if $\sigma(t) = t$, and $t \in \mathbb{T}$ with $t > \inf \mathbb{T}$ is said to be left-scattered if $\rho(t) < t$ and left-dense if $\rho(t) = t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided g is continuous at right-dense points and has finite left-sided limits at left-dense points in $\mathbb{T}$. The graininess function μ for a time scales $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for every function $f : \mathbb{T} \to \mathbb{R}$ the notation f^σ means the composition $f \circ \sigma$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the (delta) derivative $f^\Delta(t)$ at $t \in \mathbb{T}$ is defined to be the number (if it exists) such that for all $\varepsilon > 0$, there is a neighborhood U of t with

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| < \varepsilon|\sigma(t) - s|, \tag{2.1}$$

for all $s \in U$. If the (delta) derivative $f^\Delta(t)$ exits for all $t \in \mathbb{T}$, then we say that f is (delta) differentiable on $\mathbb{T}$. We will make use of the following product and rules for the derivatives of the product fg and the quotient f/g (where $gg^\sigma \neq 0$) of two (delta) differentiable functions f and g,

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \tag{2.2}$$

$$\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \tag{2.3}$$

A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable with D, provided $D \subset \mathbb{T}^\kappa$, $\mathbb{T}^\kappa \backslash D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$. Let f be rd-continuous. Then there exists a function F which is pre-differentiable with region of differentiation D, such that $F^\Delta(x) = f(t)$ holds for all $t \in D$. We define the Cauchy integral by

$$\int_b^c f(t)\Delta t = F(c) - F(b), \tag{2.4}$$

where F is a pre-antiderivative of f and $b, c \in \mathbb{T}$. The existence theorem [12, p. 27, Theorem 1.74] reads as follows: Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$ then F defined by $F(t) = \int_{t_0}^t f(\tau)\Delta\tau$ is an antiderivative of f. An integration by parts formula reads

$$\int_b^c f(t)g^\Delta(t)\Delta t = f(t)g(t)|_b^c - \int_b^c f^\Delta(t)g(\sigma(t))\Delta t. \tag{2.5}$$

Note that in the case $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = \rho(t) = t, \mu(t) = 0$, $f^\Delta(t) = f'(t)$,

$$\int_b^c f^\Delta(t)\Delta t = \int_b^c f'(t)dt \tag{2.6}$$

and in the case $\mathbb{T} = \mathbb{Z}$, we have $\sigma(t) = t + 1, \rho(t) = t - 1, \mu(t) \equiv 1$,

$$f^{\Delta}(t) = \Delta f(t) := f(t+1) - f(t) \tag{2.7}$$

and (if $b < c$)

$$\int_b^c f(t)\Delta t = \sum_{t=b}^{c-1} f(t). \tag{2.8}$$

Lemma 2.1. [12, p. 259, Theorem 6.13] *Let $a, b \in \mathbb{T}$. For $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$, we have*

$$\int_a^b |f(x)g(x)|\,\Delta x \leqslant \Big(\int_a^b |f(x)|^p\,\Delta x\Big)^{\frac{1}{p}} \Big(\int_a^b |g(x)|^q\,\Delta x\Big)^{\frac{1}{q}}, \tag{2.9}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

3. Main results

Theorem 3.1. *For $\tau \in [a, b] \cap \mathbb{T}$, let $p(x) \in C_{rd}([a, \tau], \mathbb{R})$ be nonnegative with $\int_a^\tau (p(x))^{1-u}\Delta x < \infty$ $(u > 1)$. Let the function $q(x)$ be positive, bounded and non-increasing on $[a, \tau] \cap \mathbb{T}$. Suppose that the functions $f(x)$ and $g(x)$ are delta differentiable on $[a, \tau] \cap \mathbb{T}$ and $f(a) = g(a) = 0$. Then, the following inequality holds true:*

$$\begin{aligned} &\int_a^\tau q^\sigma(x)(\left|f^\Delta(x)g(x)\right| + \left|f(x)g^\Delta(x)\right|)\Delta x \\ &\leqslant \frac{1}{2}\Big(\int_a^\tau (p(x))^{1-u}\Delta x\Big)^{\frac{2}{u}} \Big(\int_a^\tau p(x)(q^\sigma(x))^{\frac{v}{2}}(\left|f^\Delta(x)\right|^v + \left|g^\Delta(x)\right|^v)\Delta x\Big)^{\frac{2}{v}} \end{aligned} \tag{3.1}$$

where $\frac{1}{u} + \frac{1}{v} = 1$ and $1 < v \leq 2$.

Proof. Let

$$F(x) = \int_a^x \sqrt{q^\sigma(s)}\,\left|f^\Delta(s)\right|\Delta s, \quad G(x) = \int_a^x \sqrt{q^\sigma(s)}\,\left|g^\Delta(s)\right|\Delta s. \tag{3.2}$$

Then

$$F^\Delta(x) = \sqrt{q^\sigma(x)}\,\left|f^\Delta(x)\right| \geqslant 0, \quad G^\Delta(x) = \sqrt{q^\sigma(x)}\,\left|g^\Delta(x)\right| \geqslant 0 \tag{3.3}$$

and

$$\begin{aligned} F(x) &\geqslant \sqrt{q^\sigma(x)}\int_a^x \left|f^\Delta(s)\right|\Delta s = \sqrt{q^\sigma(x)}\,|f(x)|, \\ G(x) &\geqslant \sqrt{q^\sigma(x)}\,|g(x)|\,. \end{aligned} \tag{3.4}$$

Therefore, we have

$$\int_a^\tau q^\sigma(x)(\left|f^\Delta(x)g(x)\right| + \left|f(x)g^\Delta(x)\right|)\Delta x \tag{3.5}$$
$$\leqslant \int_a^\tau (F^\Delta(x)G(x) + F(x)G^\Delta(x))\Delta x$$
$$\leqslant \int_a^\tau (F^\Delta(x)G(x) + F^\sigma(x)G^\Delta(x))\Delta x = F(x)G(x)\,|_a^\tau$$
$$= \int_a^\tau \sqrt{q^\sigma(x)}\left|f^\Delta(x)\right|\Delta x \int_a^\tau \sqrt{q^\sigma(x)}\left|g^\Delta(x)\right|\Delta x.$$

In the case when $u > 1$, by using AGM inequality and Hölder's inequality, we have

$$\int_a^\tau \sqrt{q^\sigma(x)}\left|f^\Delta(x)\right|\Delta x \int_a^\tau \sqrt{q^\sigma(x)}\left|g^\Delta(x)\right|\Delta x \tag{3.6}$$
$$\leqslant \frac{1}{2}\Big[\Big(\int_a^\tau \sqrt{q^\sigma(x)}\left|f^\Delta(x)\right|\Delta x\Big)^2 + \Big(\int_a^\tau \sqrt{q^\sigma(x)}\left|g^\Delta(x)\right|\Delta x\Big)^2\Big]$$
$$= \frac{1}{2}\Big[\Big(\int_a^\tau (p(x))^{-\frac{1}{v}}(p(x))^{\frac{1}{v}}\sqrt{q^\sigma(x)}\left|f^\Delta(x)\right|\Delta x\Big)^2$$
$$+ \Big(\int_a^\tau (p(x))^{-\frac{1}{v}}(p(x))^{\frac{1}{v}}\sqrt{q^\sigma(x)}\left|g^\Delta(x)\right|\Delta x\Big)^2\Big]$$
$$\leqslant \frac{1}{2}\Big(\int_a^\tau (p(x))^{1-u}dx\Big)^{\frac{2}{u}}\Big[\Big(\int_a^\tau p(x)(q^\sigma(x))^{\frac{v}{2}}\left|f^\Delta(x)\right|^v\Delta x\Big)^{\frac{2}{v}}$$
$$+ \Big(\int_a^\tau p(x)(q^\sigma(x))^{\frac{v}{2}}\left|g^\Delta(x)\right|^v\Delta x\Big)^{\frac{2}{v}}\Big]$$
$$\leqslant \frac{1}{2}\Big(\int_a^\tau (p(x))^{1-u}\Delta x\Big)^{\frac{2}{u}}\Big(\int_a^\tau p(x)(q^\sigma(x))^{\frac{v}{2}}(\left|f^\Delta(x)\right|^v + \left|g^\Delta(x)\right|^v)\Delta x\Big)^{\frac{2}{v}}$$

where we use the elementary inequality

$$a^m + b^m \leqslant (a+b)^m, \qquad (m \geqslant 1,\ a, b \geqslant 0). \tag{3.7}$$

Thus, by means of (3.5) and (3.6), we complete the proof of Theorem 3.1. ■

Remark 3.1. Taking $\mathbb{T} = \mathbb{R}$, $p(x) = q(x) = 1$, $\tau = b$ and $u = v = 2$, Theorem 3.1 reduces to Theorem 1.4 of B. G. Pachpatte.

Theorem 3.2. *For $\tau \in [a,b] \cap \mathbb{T}$, let $p(x) \in C_{rd}([\tau,b],\mathbb{R})$ be nonnegative with $\int_\tau^b (p(x))^{1-u}\Delta x < \infty$ $(u > 1)$. Let the function $q(x)$ be positive, bounded and non-decreasing on $[\tau,b] \cap \mathbb{T}$. Suppose that the functions $f(x)$ and $g(x)$ are delta differentiable on $[\tau,b] \cap \mathbb{T}$ and $f(b) = g(b) = 0$. Then*

$$\int_\tau^b q^\sigma(x)(\left|f^\Delta(x)g(x)\right| + \left|f(x)g^\Delta(x)\right|)\Delta x \leqslant \frac{1}{2}\Big(\int_\tau^b (p(x))^{1-u}\Delta x\Big)^{\frac{2}{u}}\Big(\int_\tau^b p(x)(q^\sigma(x))^{\frac{v}{2}}(\left|f^\Delta(x)\right|^v + \left|g^\Delta(x)\right|^v)\Delta x\Big)^{\frac{2}{v}}. \tag{3.8}$$

Proof. Let

$$F(x) = \int_x^b \sqrt{q^\sigma(s)}\left|f^\Delta(s)\right|\Delta s, G(x) = \int_x^b \sqrt{q^\sigma(s)}\left|g^\Delta(s)\right|\Delta s. \tag{3.9}$$

Then

$$F^\Delta(x) = -\sqrt{q^\sigma(x)}\left|f^\Delta(x)\right| \leqslant 0, G(x) = -\sqrt{q^\sigma(x)}\left|g^\Delta(x)\right| \leqslant 0 \tag{3.10}$$

and

$$F(x) \geqslant \sqrt{q^\sigma(x)}\int_x^b \left|f^\Delta(s)\right|\Delta s = \sqrt{q^\sigma(x)}\,|f(x)|, G(x) \geqslant \sqrt{q^\sigma(x)}\,|g(x)|\,. \tag{3.11}$$

Therefore, we have

$$\begin{aligned}\int_\tau^b q^\sigma(x)(\left|f^\Delta(x)g(x)\right| + \left|f(x)g^\Delta(x)\right|)\Delta x &\leqslant -\int_\tau^b (F^\Delta(x)G(x) + F(x)G^\Delta(x))\Delta x \\ &\leqslant \int_b^\tau (F^\Delta(x)G(x) + F^\sigma(x)G^\Delta(x))\Delta x = F(x)G(x)\,|_b^\tau\,.\end{aligned} \tag{3.12}$$

The rest of the proof of Theorem 3.2 follows by suitable modifications of the proof of Theorem 3.1. We omit the further details. ■

Theorem 3.3. *For $\tau \in [a,b] \cap \mathbb{T}$, let $p(x) \in C_{rd}([a,b],\mathbb{R})$ be nonnegative with $\int_a^\tau (p(x))^{1-u}\Delta x < \infty$ $(u > 1)$ and $\int_\tau^b (p(x))^{1-u}\Delta x < \infty$ $(u > 1)$. Suppose that the function $q(x)$ is positive, bounded and non-increasing on $[a,\tau] \cap \mathbb{T}$ and non-decreasing on $[\tau,b] \cap \mathbb{T}$. Suppose that the functions $f(x)$ and $g(x)$ are delta differentiable on $[a,b] \cap \mathbb{T}$ and $f(a) = f(b) = g(a) = g(b) = 0$. Then*

$$\int_a^b q^\sigma(x)(|f^\Delta(x)g(x)| + |f(x)g^\Delta(x)|)\Delta x \tag{3.13}$$
$$\leqslant \frac{A}{2}\int_a^b p(x)(q^\sigma(x))^{\frac{v}{2}}(|f^\Delta(x)|^v + |g^\Delta(x)|^v)\Delta x)^{\frac{2}{v}}$$

where $\frac{1}{u} + \frac{1}{v} = 1,\quad 1 < v \leqslant 2$ *and* A, τ *are so constrained that*

$$\Big(\int_a^\tau (p(x))^{1-u}\Delta x\Big)^{\frac{2}{u}} = \Big(\int_\tau^b (p(x))^{1-u}\Delta x\Big)^{\frac{2}{u}} = A. \tag{3.14}$$

Proof. It is easily observed from the hypothesis of Theorem 3.3 that

$$\int_a^b q^\sigma(x)(|f^\Delta(x)g(x)| + |f(x)g^\Delta(x)|)\Delta x \tag{3.15}$$
$$= \int_a^\tau q^\sigma(x)(|f^\Delta(x)g(x)| + |f(x)g^\Delta(x)|)\Delta x$$
$$+ \int_\tau^b q^\sigma(x)(|f^\Delta(x)g(x)| + |f(x)g^\Delta(x)|)\Delta x$$
$$\leqslant \frac{A}{2}\Big[\Big(\int_a^\tau p(x)(q^\sigma(x))^{\frac{v}{2}}(|f^\Delta(x)|^v + |g^\Delta(x)|^v)\Delta x\Big)^{\frac{2}{v}}$$
$$+ \Big(\int_\tau^b p(x)(q^\sigma(x))^{\frac{v}{2}}(|f^\Delta(x)|^v + |g^\Delta(x)|^v)\Delta x\Big)^{\frac{2}{v}}\Big]$$
$$\leqslant \frac{A}{2}\int_a^b p(x)(q^\sigma(x))^{\frac{v}{2}}(|f^\Delta(x)|^v + |g^\Delta(x)|^v)\Delta x)^{\frac{2}{v}}.$$

This completes the proof. ■

Theorem 3.4. *For* $\tau \in [a,b] \cap \mathbb{T}$, $p(x) \in C_{rd}([a,b],\mathbb{R})$ *nonnegative with*

$$\int_a^\tau (p(x))^{-\frac{1}{2m+1}}\Delta x < \infty \quad \text{and} \quad \int_\tau^b (p(x))^{-\frac{1}{2m+1}}\Delta x < \infty. \tag{3.16}$$

Let the functions $f(x), g(x)$ *and* $h(x)$ *be delta differentiable on* $\tau \in [a,b] \cap \mathbb{T}$ *and* $f(a) = f(b) = g(a) = g(b) = h(a) = h(b) = 0$. *Then*

$$\int_a^b \Xi(f(x), g(x))\Delta x \tag{3.17}$$
$$\leqslant \frac{A}{m+1}\int_a^b p(x)(|f^\Delta(x)|^{2m+2} + |g^\Delta(x)|^{2m+2} + |h^\Delta(x)|^{2m+2})\Delta x$$

where $m \geq 0$,

$$\Xi(f(x),g(x)) = |f(x)g(x)|^m \left(\left|f^{\Delta}(x)g(x)\right| + \left|f(x)g^{\Delta}(x)\right|\right) + |g(x)h(x)|^m \left(\left|g^{\Delta}(x)h(x)\right| + \left|g(x)h^{\Delta}(x)\right|\right) + |h(x)f(x)|^m \left(\left|h^{\Delta}(x)f(x)\right| + \left|h(x)f^{\Delta}(x)\right|\right) \tag{3.18}$$

and A, τ *are so constrained that*

$$\left(\int_a^{\tau} (p(x))^{-\frac{1}{2m+1}} \Delta x\right)^{2m+1} = \left(\int_{\tau}^{b} (p(x))^{-\frac{1}{2m+1}} \Delta x\right)^{2m+1} = A. \tag{3.19}$$

Proof. Let

$$F(x) = \int_a^x \left|f^{\Delta}(s)\right| \Delta s, G(x) = \int_a^x \left|g^{\Delta}(s)\right| \Delta s, H(x) = \int_a^x \left|h^{\Delta}(s)\right| \Delta s. \tag{3.20}$$

Then

$$F^{\Delta}(x) = \left|f^{\Delta}(x)\right| \geqslant 0, G^{\Delta}(x) = \left|g^{\Delta}(x)\right| \geqslant 0, H^{\Delta}(x) = \left|h^{\Delta}(x)\right| \geqslant 0 \tag{3.21}$$

and

$$F(x) \geqslant |f(x)|, G(x) \geqslant |g(x)|, H(x) \geqslant |h(x)|. \tag{3.22}$$

Therefore, we have

$$\begin{aligned} \int_a^{\tau} \Xi(f(x),g(x))\Delta x \leqslant \int_a^{\tau} [&(F(x)G(x))^m(F(x)G^{\Delta}(x) + F^{\Delta}(x)G(x)) \\ &+ (G(x)H(x))^m(G^{\Delta}(x)H(x) + G(x)H^{\Delta}(x)) \\ &+ (H(x)F(x))^m(H^{\Delta}(x)F(x) + H(x)F^{\Delta}(x))]\Delta x \end{aligned} \tag{3.23}$$

$$\begin{aligned} &\leqslant \int_a^{\tau} [(F(x)G(x))^m(F(x)G(x))^{\Delta} + (G(x)H(x))^m(G(x)H(x))^{\Delta} \\ &\quad + (H(x)F(x))^m(F(x)H(x))^{\Delta}]\Delta x \\ &\leqslant \frac{(F(\tau)G(\tau))^{m+1} + (G(\tau)H(\tau))^{m+1} + (H(\tau)F(\tau))^{m+1}}{m+1} \\ &\leqslant \frac{(F(\tau))^{2m+2} + (G(\tau))^{2m+2} + (H(\tau))^{2m+2}}{m+1} \\ &= \frac{1}{m+1}\left[\left(\int_a^{\tau} \left|f^{\Delta}(x)\right| \Delta x\right)^{2m+2} + \left(\int_a^{\tau} \left|g^{\Delta}(x)\right| \Delta x\right)^{2m+2} + \left(\int_a^{\tau} \left|h^{\Delta}(x)\right| \Delta x\right)^{2m+2}\right] \end{aligned}$$

where we apply the inequality

$$cd + de + ec \leqslant c^2 + d^2 + e^2. \tag{3.24}$$

Let $u = \frac{2m+2}{2m+1}$ and $v = 2m + 2$. By using Hölder inequality, we have

(3.25)
$$\begin{aligned}
&\Big(\int_a^\tau \left|f^\Delta(x)\right| \Delta x\Big)^{2m+2} + \Big(\int_a^\tau \left|g^\Delta(x)\right| \Delta x\Big)^{2m+2} + \Big(\int_a^\tau \left|h^\Delta(x)\right| \Delta x\Big)^{2m+2} \\
&\quad = \Big(\int_a^\tau (p(x))^{-\frac{1}{2m+1}} (p(x))^{\frac{1}{2m+1}} \left|f^\Delta(x)\right| \Delta x\Big)^{2m+2} \\
&\qquad + \Big(\int_a^\tau (p(x))^{-\frac{1}{2m+1}} (p(x))^{\frac{1}{2m+1}} \left|g^\Delta(x)\right| \Delta x\Big)^{2m+2} \\
&\qquad + \Big(\int_a^\tau (p(x))^{-\frac{1}{2m+1}} (p(x))^{\frac{1}{2m+1}} \left|h^\Delta(x)\right| \Delta x\Big)^{2m+2} \\
&\quad \leqslant \Big(\int_a^\tau (p(x))^{-\frac{1}{2m+1}} \Delta x\Big)^{2m+1} \\
&\qquad \int_a^\tau p(x)(\left|f^\Delta(x)\right|^{2m+2} + \left|g^\Delta(x)\right|^{2m+2} + \left|h^\Delta(x)\right|^{2m+2})\Delta x.
\end{aligned}$$

Considering (3.23) and (3.25), we have

(3.26)
$$\begin{aligned}
\int_a^\tau \Xi(f(x), g(x))\Delta x \leqslant \frac{1}{m+1}&\Big(\int_a^\tau (p(x))^{-\frac{1}{2m+1}} \Delta x\Big)^{2m+1} \\
&\int_a^\tau p(x)(\left|f^\Delta(x)\right|^{2m+2} + \left|g^\Delta(x)\right|^{2m+2} + \left|h^\Delta(x)\right|^{2m+2})\Delta x.
\end{aligned}$$

By following steps very similar to those of the proof above, we have

(3.27)
$$\begin{aligned}
\int_\tau^b \Xi(f(x), g(x))\Delta x \leqslant \frac{1}{m+1}&\Big(\int_\tau^b (p(x))^{-\frac{1}{2m+1}} \Delta x\Big)^{2m+1} \\
&\int_\tau^b p(x)(\left|f^\Delta(x)\right|^{2m+2} + \left|g^\Delta(x)\right|^{2m+2} + \left|h^\Delta(x)\right|^{2m+2})\Delta x.
\end{aligned}$$

Associating (3.26) with (3.27), we complete the proof. ■

Remark 3.2. Result of Theorem 3.4 is not only new for arbitrary time scales, but also new for the continuous and the discrete cases.

Remark 3.3. If $\mathbb{T} = \mathbb{R}$, $p(x) = 1$, $\tau = \frac{a+b}{2}$ and $m = 0$, Theorem 3.4 is a particular case of Theorem 1 of B. G. Pachpatte [15].

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