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Özcan Gelişgen, Rüstem Kaya

THE ALPHA-VERSION OF THE STEWART'S THEOREM

Abstract. G. Chen [1] developed Chinese checker metric for the plane on the question "how to develop a metric which would be similar to the movement made by playing Chinese checker" by E. F. Krause [13]. Tian [17] developed α-metric which is defined by

$$d_\alpha(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + (\sec\alpha - \tan\alpha)\min\{|x_1 - x_2|, |y_1 - y_2|\}$$

where $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are two points in analytical plane, and $\alpha \in [0, \pi/4]$. Stewart's theorem yields a relation between lengths of the sides of a triangle and the length of a cevian of the triangle. A taxicab and Chinese checkers analogues of Stewart's theorem are given in [12] and [9], respectively. In this work, we give an α-analog of the theorem of Stewart by using the *base line* concept and we give a α-analog of formulae for the medians which is the application of Stewart's theorem.

1. Introduction

As stated in [16], Minkowski geometry is a non-Euclidean geometry in a finite number of dimensions that is different from elliptic and hyperbolic geometry (and from the Minkowskian geometry of space-time). Here the linear structure is the same as the Euclidean one but distance is not *uniform* in all directions. Instead of the usual sphere in Euclidean space, the unit ball is a general symmetric convex set. Therefore, although the parallel axiom is valid, Pythagoras' theorem is not.

The taxicab metric was given in a family of metrics of the real plane by Minkowski. Later, Chen [1] developed a Chinese checker metric, and Tian [17] gave a family of metrics, α-metric for $\alpha \in [0, \pi/4]$, which include the taxicab and Chinese checker metrics as special cases. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in analytical plane $\mathbb{R}^2$. α-distance function is defined by

$$d_\alpha(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + (\sec\alpha - \tan\alpha)\min\{|x_1 - x_2|, |y_1 - y_2|\}$$

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where $\alpha \in [0, \pi/4]$. According to the definition of d_α-metric, the shortest path between the points P_1 and P_2 is the line segment which is parallel to a coordinate axis and a line segment making the α angle with the other coordinate axis as shown in Figure 1. Thus, the shortest distance d_α between P_1 and P_2 is the sum of the Euclidean lengths of such two line segments.

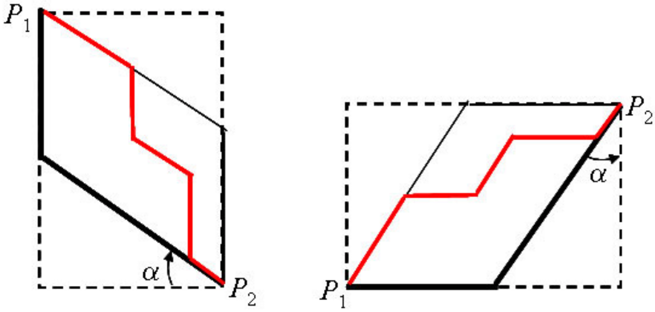

Fig. 1

A metric geometry consists of a set $\mathcal{P}$, whose elements are called points, together with a collection $\mathcal{L}$ of non-empty subsets of $\mathcal{P}$, called lines, and a distance function d, such that

(1) Every two distinct points in $\mathcal{P}$ lie on a unique line.
(2) There exist three points in $\mathcal{P}$, which do not lie all on one line.
(3) There exists a bijective function $f : l \to \mathbb{R}$, for all lines in $\mathcal{L}$ such that $|f(P) - f(Q)| = d(P, Q)$, for each pair of points P and Q on l.

A metric geometry defined above is denoted by $\{\mathcal{P}, \mathcal{L}, d\}$. Let $\mathcal{P}_E$ and $\mathcal{L}_E$ denote sets of all points and lines in the Euclidean geometry, respectively. α-plane geometry consisting of $\mathcal{P}_E$, $\mathcal{L}_E$ and d_α is a metric geometry.

Furthermore, if a metric geometry satisfies the plane separation axiom below, and it has an angle measure function m, then it is called protractor geometry and denoted by $\{\mathcal{P}, \mathcal{L}, d, m\}$.

(4) For every l in $\mathcal{L}$, there are two subsets H_1 and H_2 of $\mathcal{P}$ (called half planes determined by l) such that

   (i) $H_1 \cup H_2 \neq \mathcal{P} - l$ ($\mathcal{P}$ with l removed),
   (ii) H_1 and H_2 are disjoint and each is convex,
   (iii) if $A \in H_1$ and $B \in H_2$, then $[AB] \cap l = \emptyset$.

If $\mathcal{L}_E$ is the set of all lines in the Cartesian coordinate plane, and m_E is the standard angle measure function in the Euclidean plane, then $\{\mathbb{R}^2, \mathcal{L}_E, d_\alpha, m_E\}$, called α-plane, is a model of protractor geometry. (This can be shown easily: the proof is similar to that of taxicab plane; refer to [15] or [4] to see that the taxicab plane is a model of protractor geometry.) α-plane is also in the class of non-Euclidean geometries since it fails to satisfy the side-angle-side axiom. However, α-plane is almost the same as Euclidean plane $\{\mathbb{R}^2, \mathcal{L}_E, d_E, m_E\}$ since the points are the same, the lines are the same, and the angles are measured in the same way. Since the α-plane $\left(\mathbb{R}^2_\alpha\right)$ geometry has a different distance function it seems interesting to study the α-analog of the topics that include the concepts of distance in the Euclidean geometry. α-analogues of some of the topics that include the concept of α-distance have been studied by some authors [17], [7], [8]. The group of isometries of the α-plane has been given in [11]. Finally, two different α-analogues of the Pythagoras' theorem have been introduced in [3]. In this work, we give α-versions of Stewart's theorem and the median property. Also, the taxicab and CC-analogues of Stewart's theorem are given in [12] and [9], respectively.

2. Preliminaries

The following propositions and corollaries give some results of $\mathbb{R}^2_\alpha$ by summarizing from [5].

Proposition 2.1. *Every Euclidean translation is an isometry of* $\mathbb{R}^2_\alpha$.

Proposition 2.2. *Let l be a line through the points P_1 and P_2 in the analytical plane. If l has slope m, then*

$$d_\alpha\left(P_1, P_2\right) = \frac{M}{\sqrt{1+m^2}} d_E\left(P_1, P_2\right),$$

where

$$M = \begin{cases} 1 + (\sec\alpha - \tan\alpha)\left|m\right|, & \text{if } \left|m\right| \le 1, \\ (\sec\alpha - \tan\alpha) + \left|m\right|, & \text{if } \left|m\right| \ge 1, \end{cases}$$

and d_E denote the Euclidean distance function.

Proof. If l is parallel to the x-axis or y-axis, then $m = 0$ and $M/\sqrt{m^2+1} = 1$ or $m \to \infty$ and $\lim_{m\to\infty}(M/\sqrt{m^2+1}) = 1$. Then, $d_\alpha(A,B) = d_E(A,B)$ in both of the cases above. If l is not parallel to the x-axis and y-axis, then $x_1 \neq x_2$ and $y_1 \neq y_2$, $m = (y_1 - y_2)/(x_1 - x_2)$, where m is the slope of l, and

$$\begin{aligned} d_\alpha(P_1, P_2) &= \max\{|x_1 - x_2|, |y_1 - y_2|\} \\ &\quad + (\sec\alpha - \tan\alpha)\min\{|x_1 - x_2|, |y_1 - y_2|\} \\ &= \begin{cases} |x_1 - x_2|\left(1 + (\sec\alpha - \tan\alpha)\left|m\right|\right), & \text{if } \left|m\right| \le 1, \\ |x_1 - x_2|\left(\left|m\right| + \sec\alpha - \tan\alpha\right), & \text{if } \left|m\right| \ge 1. \end{cases} \end{aligned}$$

Similarly,

$$d_E\left(P_1, P_2\right) = \left|x_1 - x_2\right|\sqrt{1+m^2}, \quad \text{for all } m \in \mathbb{R},$$

and consequently the given equality is valid. ∎

The above proposition says that d_α-distance along any line is some positive constant multiple of Euclidean distance along the same line.

Corollary 2.3. *Let P_1, P_2 and X are any three collinear points in $\mathbb{R}^2$. Then $d_E(P_1, X) = d_E(P_2, X)$ if and only if $d_\alpha(P_1, X) = d_\alpha(P_2, X)$.*

Corollary 2.4. *If P_1, P_2 and X are any three distinct collinear points in $\mathbb{R}^2$, then $d_\alpha(P_1, X)/d_\alpha(P_2, X) = d_E(P_1, X)/d_E(P_2, X)$.*

That is, the ratios of the Euclidean and d_α-distances along a line are the same. Notice that the latter corollary gives us the validity of the theorem of Menelaus and Ceva in $\mathbb{R}^2_\alpha$.

We need the following definitions given in [12] and [14]:

Let ABC be any triangle in the $\mathbb{R}^2_\alpha$. Clearly, there exists a pair of lines passing through every vertex of the triangle, each of which is parallel to a coordinate axis. A line l is called a *base line* of ABC iff

(1) l passes through a vertex,
(2) l is parallel to a coordinate axis,
(3) l intersects the opposite side (as a line segment) to the vertex in condition 1. Clearly, at least one of the vertices of a triangle always has one or two base lines. Such a vertex of a triangle is called a *basic vertex.* A *base segment* is a line segment on a base line, which is bounded by a basic vertex and its opposite side.

Finally, we consider the following separation of $\mathbb{R}^2_\alpha$ to eight regions S_i $(i = 0, 1, \ldots, 7)$ such that

$$S_0 = \{(x, y) | x \geq y \geq 0\}$$
$$S_1 = \{(x, y) | y \geq x \geq 0\}$$
$$S_2 = \{(x, y) | y \geq |x| \geq 0,\ x < 0\}$$
$$S_3 = \{(x, y) |\ |x| \geq y \geq 0,\ x < 0\}$$
$$S_4 = \{(x, y) | x \leq y \leq 0\}$$
$$S_5 = \{(x, y) | y \leq x \leq 0\}$$
$$S_6 = \{(x, y) |\ |y| \geq x \geq 0,\ y < 0\}$$
$$S_7 = \{(x, y) | x \geq |y| \geq 0,\ y < 0\}$$

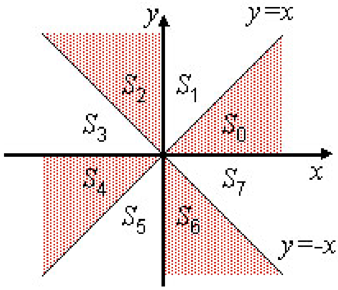

Fig. 2

as shown in Figure 2. In what follows S_{i+1}, S_{i+2}, S_{i+3} and S_{i+4} stand for $S_{i+1(\text{mod } 8)}$, $S_{i+2(\text{mod } 8)}$, $S_{i+3(\text{mod } 8)}$ and $S_{i+4(\text{mod } 8)}$, respectively.

3. The alpha analog of the Stewart's theorem

In geometry, Stewart's theorem yields a relation between lengths of the sides of the triangle and the length of a cevian of the triangle. It is named in honor of the Scottish mathematician Matthew Stewart who published the theorem in 1746. Although, it was probably discovered by Archimedes about 300 B.C., the first known proof is due to R. Simon in 1751 (see [2]). It is known for any triangle ABC in the Euclidean plane that if $X \in BC$ and $a = d(B,C)$, $b = d(A,C)$, $c = d(A,B)$, $p = d(B,X)$, $q = d(C,X)$, $x = d(A,X)$, then

$$x^2 = \frac{b^2p + c^2q}{p+q} - pq$$

which is known as Stewart's theorem.

As applications of the Stewart's theorem, we find the formulae for the medians and angle bisectors of a triangle.

The next theorem gives an alpha version of the Stewart's theorem. We consider the formulae for the length of a cevian of a triangle, consisting of two parts. The first part is nearly like Euclidean one, but the second part is different. In the second part, there is a parameter Δ changing according to position of the triangle in the analytical plane. The table in the Theorem 3.1 explains value of Δ for every position of the triangle.

Theorem 3.1. *Let the sides of a triangle ABC in the $\mathbb{R}^2_\alpha$ have lengths $\mathbf{a} = d_\alpha(B,C)$, $\mathbf{b} = d_\alpha(A,C)$ and $\mathbf{c} = d_\alpha(A,B)$. If $X \in BC$ and $\mathbf{p} = d_\alpha(B,X)$, $\mathbf{q} = d_\alpha(C,X)$ and $\mathbf{x} = d_\alpha(A,X)$, then*

$$\mathbf{x} = \frac{\mathbf{bp} + \mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{\boldsymbol{\Delta}}{\mathbf{p}+\mathbf{q}},$$

where $\boldsymbol{\Delta}$ is as in the Table 1, and $B = (b_1, b_2)$, $C = (c_1, c_2)$, $|b_1| = \theta$, $|b_2| = \beta$, $|c_1| = \gamma$, $|c_2| = \delta$ and $w = \sec\alpha - \tan\alpha$.

Proof. Without loss of generality, the vertex A of the triangle ABC in $\mathbb{R}^2_\alpha$ can be chosen at origin by Proposition 2.1. Let $B = (b_1, b_2)$, $C = (c_1, c_2)$, $X = (x_1, x_2)$, $|b_1| = \theta$, $|b_2| = \beta$, $|c_1| = \gamma$ and $|c_2| = \delta$. Thus

$$\begin{aligned}
\mathbf{b} &= \max\{\gamma, \delta\} + (\sec\alpha - \tan\alpha)\min\{\gamma, \delta\},\\
\mathbf{c} &= \max\{\theta, \beta\} + (\sec\alpha - \tan\alpha)\min\{\theta, \beta\},\\
\mathbf{x} &= \max\{|x_1|, |x_2|\} + (\sec\alpha - \tan\alpha)\min\{|x_1|, |x_2|\}.
\end{aligned}$$

Three main cases are possible for the base line through the vertex A:

Case I: Let ABC be a triangle which has no base line through the vertex A. Since vertex A is at origin, AB and AC are in the same quadrant. So one can easily obtain

$$\mathbf{q}(\gamma - \theta) = (\mathbf{p}+\mathbf{q})(\gamma - x_1), \qquad \mathbf{p}(\delta - \beta) = (\mathbf{p}+\mathbf{q})(x_2 - \beta),$$

by Corollary 2.3 and Corollary 2.4. Thus $x_1 = \frac{\mathbf{p}\gamma + \mathbf{q}\theta}{\mathbf{p}+\mathbf{q}}$ and $x_2 = \frac{\mathbf{p}\delta + \mathbf{q}\beta}{\mathbf{p}+\mathbf{q}}$. Depending on the positions of AB, AC and AX, one can obtain $d_\alpha(A,X) = \mathbf{x}$ as follows:

Table 1

Δ	Number of base line	AB lies in	AX lies in	AC lies in	
0	0	S_i	S_i	S_i	$\forall i$
$(1-w)\lvert\gamma-\delta\rvert\mathbf{p}$	0	S_i	S_i	S_{i+1}	$\forall i$
$(1-w)\lvert\theta-\beta\rvert\mathbf{q}$	0	S_i	S_{i+1}	S_{i+1}	$\forall i$
$2w\mathbf{q}\min\{\theta,\beta\}$	1	S_i	S_{i+1}	S_{i+1}	$\forall i$
$2w\mathbf{p}\min\{\gamma,\delta\}$	1	S_i	S_i	S_{i+1}	$\forall i$
$(1-w)\lvert\gamma-\delta\rvert\mathbf{p}+2w\mathbf{q}\min\{\theta,\beta\}$	1	S_i	S_{i+1}	S_{i+2}	$i\in\{1,3,5,7\}$
$(1+w)\,\mathbf{p}\max\{\gamma+\frac{w-1}{w+1}\delta,\ \delta+\frac{w-1}{w+1}\gamma\}$	1	S_i	S_i	S_{i+2}	$i\in\{1,3,5,7\}$
$(1+w)\,\mathbf{q}\min\{\theta-\frac{w-1}{w+1}\beta,\ \beta-\frac{w-1}{w+1}\theta\}$	1	S_i	S_{i+2}	S_{i+2}	$i\in\{1,3,5,7\}$
$2w\mathbf{p}\min\{\gamma,\delta\}+(1-w)\lvert\theta-\beta\rvert\mathbf{q}$	1	S_i	S_{i+1}	S_{i+2}	$i\in\{0,2,4,6\}$
$(1+w)\,\mathbf{q}\max\{\theta+\frac{w-1}{w+1}\beta,\ \beta+\frac{w-1}{w+1}\theta\}$	1	S_i	S_{i+2}	S_{i+2}	$i\in\{0,2,4,6\}$
$(1+w)\,\mathbf{p}\min\{\gamma-\frac{w-1}{w+1}\delta,\ \delta-\frac{w-1}{w+1}\gamma\}$	1	S_i	S_i	S_{i+2}	$i\in\{0,2,4,6\}$
$(1-w)\lvert\gamma-\delta\rvert\mathbf{p}+(1+w)\,\mathbf{q}\max\{\theta+\frac{w-1}{w+1}\beta,\ \beta+\frac{w-1}{w+1}\theta\}$	1	S_i	S_{i+2}	S_{i+3}	$\forall i$
$(1+w)\,\mathbf{p}\max\{\gamma+\frac{w-1}{w+1}\delta,\ \delta+\frac{w-1}{w+1}\gamma\}+(1-w)\mathbf{q}\lvert\theta-\beta\rvert$	1	S_i	S_{i+1}	S_{i+3}	$\forall i$
$2\mathbf{q}\max\{\theta,\beta\}$	1	S_i	S_{i+3}	S_{i+3}	$\forall i$
$2\mathbf{p}\max\{\gamma,\delta\}$	1	S_i	S_i	S_{i+3}	$\forall i$
$(1+w)\,\mathbf{p}\min\{\gamma-\frac{w-1}{w+1}\delta,\ \delta-\frac{w-1}{w+1}\gamma\}+2w\mathbf{q}\min\{\theta,\beta\}$	2	S_i	S_{i+1}	S_{i+3}	$\forall i$
$(1+w)(\gamma+\delta)\mathbf{p}$	2	S_i	S_i	S_{i+3}	$\forall i$
$2w\mathbf{p}\min\{\gamma,\delta\}+(1+w)\,\mathbf{q}\min\{\theta-\frac{w-1}{w+1}\beta,\ \beta-\frac{w-1}{w+1}\theta\}$	2	S_i	S_{i+2}	S_{i+3}	$\forall i$
$(1+w)(\theta+\beta)\mathbf{q}$	2	S_i	S_{i+3}	S_{i+3}	$\forall i$
$2\mathbf{p}\max\{\gamma,\delta\}+2w\mathbf{q}\min\{\theta,\beta\}$	2	S_i	S_{i+1}	S_{i+4}	$i\in\{1,3,5,7\}$
$(1+w)\,\mathbf{p}\max\{\gamma+\frac{w-1}{w+1}\delta,\ \delta+\frac{w-1}{w+1}\gamma\}+(1+w)\,\mathbf{q}\min\{\theta-\frac{w-1}{w+1}\beta,\ \beta-\frac{w-1}{w+1}\theta\}$	2	S_i	S_{i+2}	S_{i+4}	$i\in\{1,3,5,7\}$
$(1-w)\lvert\gamma-\delta\rvert\mathbf{p}+(1+w)(\theta+\beta)\mathbf{q}$	2	S_i	S_{i+3}	S_{i+4}	$i\in\{1,3,5,7\}$
$2w\mathbf{p}\min\{\gamma,\delta\}+2\mathbf{q}\max\{\theta,\beta\}$	2	S_i	S_{i+3}	S_{i+4}	$i\in\{0,2,4,6\}$
$(1+w)\,\mathbf{p}\min\{\gamma-\frac{w-1}{w+1}\delta,\ \delta-\frac{w-1}{w+1}\gamma\}+(1+w)\,\mathbf{q}\max\{\theta+\frac{w-1}{w+1}\beta,\ \beta+\frac{w-1}{w+1}\theta\}$	2	S_i	S_{i+2}	S_{i+4}	$i\in\{0,2,4,6\}$
$(1+w)(\gamma+\delta)\mathbf{p}+(1-w)\lvert\theta-\beta\rvert\mathbf{q}$	2	S_i	S_{i+1}	S_{i+4}	$i\in\{0,2,4,6\}$
$2\mathbf{pb}$	2	S_i	S_i	S_{i+4}	$\forall i$
$2\mathbf{qc}$	2	S_i	S_{i+4}	S_{i+4}	$\forall i$

If AB, AC and AX are in S_i, as in Figure 3, then $\mathbf{x}=\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}$.
If AB and AX are in S_i and AC is in S_{i+1}, as in Figure 3, then

$$\mathbf{x}=\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{(1-(\sec\alpha-\tan\alpha))\,\lvert\gamma-\delta\rvert\,\mathbf{p}}{\mathbf{p}+\mathbf{q}}.$$

If AB is in S_i and AX and AC are in S_{i+1}, as in Figure 3, then

$$\mathbf{x} = \frac{\mathbf{bp} + \mathbf{qc}}{\mathbf{p} + \mathbf{q}} - \frac{(1 - (\sec\alpha - \tan\alpha))\,|\theta - \beta|\,\mathbf{q}}{\mathbf{p} + \mathbf{q}}.$$

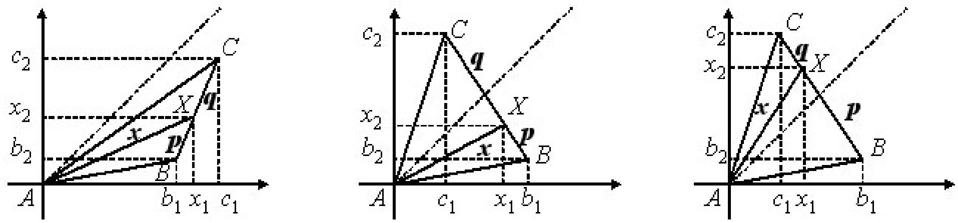

Fig. 3

Case II: Let ABC be a triangle which has only one base line through the vertex A. Since vertex A is at origin, AB and AC are in a neighbor quadrant. That is, $b_1c_1 < 0$ or $b_2c_2 < 0$. Depending on the positions of AB, AC and AX, one can obtain

$$x_1 = \frac{\mathbf{p}\gamma + \mathbf{q}\theta}{\mathbf{p} + \mathbf{q}} \quad \text{and} \quad x_2 = \frac{\mathbf{p}\delta - \mathbf{q}\beta}{\mathbf{p} + \mathbf{q}} \quad \text{or} \quad x_2 = \frac{-\mathbf{p}\delta + \mathbf{q}\beta}{\mathbf{p} + \mathbf{q}}$$

or

$$x_1 = \frac{\mathbf{p}\gamma - \mathbf{q}\theta}{\mathbf{p} + \mathbf{q}} \quad \text{or} \quad x_1 = \frac{-\mathbf{p}\gamma + \mathbf{q}\theta}{\mathbf{p} + \mathbf{q}} \quad \text{and} \quad x_2 = \frac{\mathbf{p}\delta + \mathbf{q}\beta}{\mathbf{p} + \mathbf{q}}.$$

Now using these values, $d_\alpha(A, X) = \mathbf{x}$ is obtained as follows:

If AB is in S_i and AX and AC are in S_{i+1}, then

$$\begin{aligned}\mathbf{x} &= \frac{\mathbf{bp} + \mathbf{q}(\mathbf{c} - 2(\sec\alpha - \tan\alpha)\min\{\theta, \beta\})}{\mathbf{p} + \mathbf{q}} \\ &= \frac{\mathbf{bp} + \mathbf{qc}}{\mathbf{p} + \mathbf{q}} - \frac{2(\sec\alpha - \tan\alpha)\min\{\theta, \beta\}\,\mathbf{q}}{\mathbf{p} + \mathbf{q}}.\end{aligned}$$

If AB and AX are in S_i and AC is in S_{i+1}, as in Figure 4, then

$$\begin{aligned}\mathbf{x} &= \frac{(\mathbf{b} - 2(\sec\alpha - \tan\alpha)\min\{\gamma, \delta\}\,)\mathbf{p} + \mathbf{qc}}{\mathbf{p} + \mathbf{q}} \\ &= \frac{\mathbf{bp} + \mathbf{qc}}{\mathbf{p} + \mathbf{q}} - \frac{2(\sec\alpha - \tan\alpha)\min\{\gamma, \delta\}\,\mathbf{p}}{\mathbf{p} + \mathbf{q}}.\end{aligned}$$

If AB, AX and AC are in S_i, S_{i+1}, S_{i+2}, for $i \in \{1, 3, 5, 7\}$, respectively, then

$$\begin{aligned}\mathbf{x} &= \frac{(\mathbf{b} - (1 - w)\,|\delta - \gamma|)\mathbf{p} + \mathbf{q}(\mathbf{c} - 2w\min\{\theta, \beta\})}{\mathbf{p} + \mathbf{q}} \\ &= \frac{\mathbf{bp} + \mathbf{qc}}{\mathbf{p} + \mathbf{q}} - \frac{(1 - w)\,|\delta - \gamma|\,\mathbf{p} + 2w\min\{\theta, \beta\}\,\mathbf{q}}{\mathbf{p} + \mathbf{q}}.\end{aligned}$$

If AB and AX are in S_i and AC is in S_{i+2}, for $i \in \{1,3,5,7\}$, then

$$\mathbf{x} = \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{((w-1)\min\{\gamma,\delta\} + (1+w)\max\{\gamma,\delta\})\,\mathbf{p}}{\mathbf{p}+\mathbf{q}}$$
$$= \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{(1+w)\,\mathbf{p}\max\left\{\gamma + \frac{w-1}{w+1}\delta, \delta + \frac{w-1}{w+1}\gamma\right\}}{\mathbf{p}+\mathbf{q}}.$$

If AB is in S_i and AX and AC are in S_{i+2}, for $i \in \{1,3,5,7\}$, then

$$\mathbf{x} = \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{((1-w)\max\{\theta,\beta\} + (1+w)\min\{\theta,\beta\})\,\mathbf{q}}{\mathbf{p}+\mathbf{q}}$$
$$= \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{(1+w)\,\mathbf{q}\min\left\{\theta - \frac{w-1}{w+1}\beta, \beta - \frac{w-1}{w+1}\theta\right\}}{\mathbf{p}+\mathbf{q}}.$$

If AB, AX and AC are in S_i, S_{i+1}, S_{i+2}, for $i \in \{0,2,4,6\}$, as in Figure 4, respectively, then

$$\mathbf{x} = \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{2w\min\{\gamma,\delta\}\,\mathbf{p} + (1-w)\,|\beta-\theta|\,\mathbf{q}}{\mathbf{p}+\mathbf{q}}.$$

If AB is in S_i and AX and AC are in S_{i+2}, for $i \in \{0,2,4,6\}$, then

$$\mathbf{x} = \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{((w-1)\min\{\theta,\beta\} + (1+w)\max\{\theta,\beta\})\,\mathbf{q}}{\mathbf{p}+\mathbf{q}}$$
$$= \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{(1+w)\,\mathbf{q}\max\left\{\theta + \frac{w-1}{w+1}\beta, \beta + \frac{w-1}{w+1}\theta\right\}}{\mathbf{p}+\mathbf{q}}.$$

If AB and AX are in S_i and AC is in S_{i+2}, for $i \in \{0,2,4,6\}$, then

$$\mathbf{x} = \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{((1-w)\max\{\gamma,\delta\} + (1+w)\min\{\gamma,\delta\})\,\mathbf{p}}{\mathbf{p}+\mathbf{q}}$$
$$= \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{(1+w)\,\mathbf{p}\min\left\{\gamma - \frac{w-1}{w+1}\delta, \delta - \frac{w-1}{w+1}\gamma\right\}}{\mathbf{p}+\mathbf{q}}.$$

If AB, AX and AC are in S_i, S_{i+2}, S_{i+3}, respectively, then

$$\mathbf{x} = \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{(1-w)|\delta-\gamma|\mathbf{p} + \mathbf{q}((w-1)\min\{\theta,\beta\} + (1+w)\max\{\theta,\beta\})}{\mathbf{p}+\mathbf{q}}$$
$$= \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{(1-w)|\delta-\gamma|\mathbf{p} + (1+w)\mathbf{q}\max\{\theta + \frac{w-1}{w+1}\beta, \beta + \frac{w-1}{w+1}\theta\}}{\mathbf{p}+\mathbf{q}}.$$

If AB, AX and AC are in S_i, S_{i+1}, S_{i+3}, as in Figure 4, respectively, then

$$\mathbf{x}=\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{((w-1)\min\{\gamma,\delta\}+(1+w)\max\{\gamma,\delta\})\mathbf{p}+\mathbf{q}(1-w)\,|\beta-\theta|}{\mathbf{p}+\mathbf{q}}$$
$$=\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{(1+w)\,\mathbf{p}\max\left\{\gamma+\frac{w-1}{w+1}\delta,\delta+\frac{w-1}{w+1}\gamma\right\}+\mathbf{q}(1-w)\,|\beta-\theta|}{\mathbf{p}+\mathbf{q}}.$$

If AB is in S_i and AX and AC are in S_{i+3}, then

$$\mathbf{x}=\frac{\mathbf{bp}+\mathbf{q}(\mathbf{c}-2\max\{\theta,\beta\})}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{2\max\{\theta,\beta\}\,\mathbf{q}}{\mathbf{p}+\mathbf{q}}.$$

If AB and AX are in S_i and AC is in S_{i+3}, then

$$\mathbf{x}=\frac{(\mathbf{b}-2\max\{\gamma,\delta\})\mathbf{p}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{2\mathbf{p}\max\{\gamma,\delta\}}{\mathbf{p}+\mathbf{q}}.$$

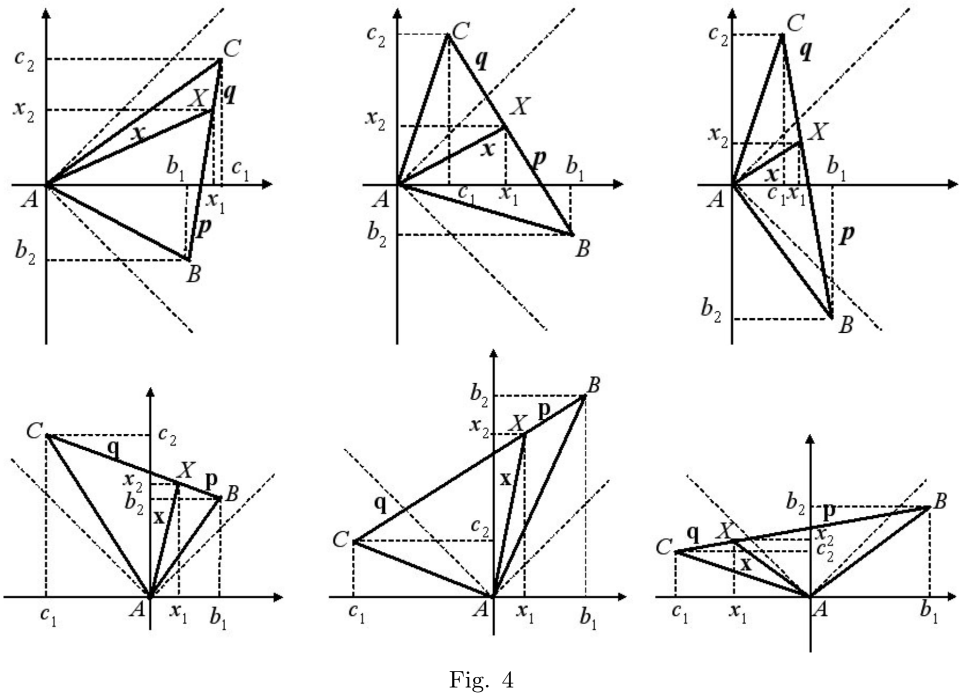

Fig. 4

Case III: Let ABC be a triangle which has two base lines through the vertex A. Since the vertex A is at origin, AB and AC are in the opposite quadrants. Depending on the positions of AB, AC and AX, one can obtain

$$x_1=\frac{-\mathbf{p}\gamma+\mathbf{q}\theta}{\mathbf{p}+\mathbf{q}}\quad\text{or}\quad x_1=\frac{\mathbf{p}\gamma-\mathbf{q}\theta}{\mathbf{p}+\mathbf{q}}$$

and

$$x_2=\frac{-\mathbf{p}\delta+\mathbf{q}\beta}{\mathbf{p}+\mathbf{q}}\quad\text{or}\quad x_2=\frac{\mathbf{p}\delta-\mathbf{q}\beta}{\mathbf{p}+\mathbf{q}}.$$

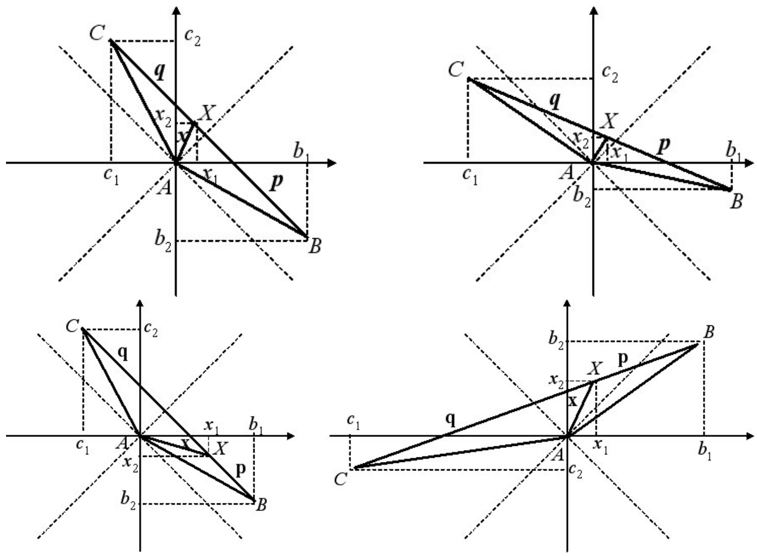

Fig. 5

Using the values of x_1 and x_2, $d_\alpha(A,X)=\mathbf{x}$ is obtained as follows:

If AB, AX and AC are in S_i, S_{i+1}, S_{i+3}, as in Figure 5, respectively, then

$$\begin{aligned}\mathbf{x} &= \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{((1-w)\max\{\gamma,\delta\}+(1+w)\min\{\gamma,\delta\})\mathbf{p}+2w\mathbf{q}\min\{\theta,\beta\})}{\mathbf{p}+\mathbf{q}} \\ &= \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{(1+w)\,\mathbf{p}\min\{\gamma-\frac{w-1}{w+1}\delta,\delta-\frac{w-1}{w+1}\gamma\}+2w\mathbf{q}\min\{\theta,\beta\})}{\mathbf{p}+\mathbf{q}}.\end{aligned}$$

If AB and AX are in S_i and AC is in S_{i+3}, then

$$\mathbf{x} = \frac{(\mathbf{b}-(1+w)\,(\delta+\gamma))\mathbf{p}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} = \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{(1+w)\,(\delta+\gamma)\mathbf{p}}{\mathbf{p}+\mathbf{q}}.$$

If AB, AX and AC are in S_i, S_{i+2}, S_{i+3}, respectively, then

$$\begin{aligned}\mathbf{x} &= \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{2w\min\{\gamma,\delta\}\mathbf{p}+\mathbf{q}((1-w)\max\{\theta,\beta\}+(1+w)\min\{\theta,\beta\})}{\mathbf{p}+\mathbf{q}} \\ &= \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{2w\mathbf{p}\min\{\gamma,\delta\}+(1+w)\mathbf{q}\min\{\theta-\frac{w-1}{w+1}\beta,\beta-\frac{w-1}{w+1}\theta\}}{\mathbf{p}+\mathbf{q}}.\end{aligned}$$

If AB is in S_i and AX and AC are in S_{i+3}, then

$$\mathbf{x} = \frac{\mathbf{bp}+\mathbf{q}(\mathbf{c}-(1+w)\,(\theta+\beta))}{\mathbf{p}+\mathbf{q}} = \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{(1+w)\,(\theta+\beta)\,\mathbf{q}}{\mathbf{p}+\mathbf{q}}.$$

If AB, AX and AC are in S_i, S_{i+1}, S_{i+4}, for $i \in \{1,3,5,7\}$, respectively, then

$$\mathbf{x} = \frac{(\mathbf{b} - 2\max\{\delta,\gamma\})\mathbf{p} + \mathbf{q}(\mathbf{c} - 2(\sec\alpha - \tan\alpha)\min\{\theta,\beta\})}{\mathbf{p}+\mathbf{q}}$$
$$= \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{2\mathbf{p}\max\{\delta,\gamma\} + 2w\mathbf{q}\min\{\theta,\beta\}}{\mathbf{p}+\mathbf{q}}.$$

If AB, AX and AC are in S_i, S_{i+2}, S_{i+4}, for $i \in \{1,3,5,7\}$, as in Figure 5, respectively, then

$$\mathbf{x} = \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{(1+w)\mathbf{p}\max\{\gamma+\frac{w-1}{w+1}\delta, \delta+\frac{w-1}{w+1}\gamma\} + (1+w)\mathbf{q}\min\{\theta-\frac{w-1}{w+1}\beta, \beta-\frac{w-1}{w+1}\theta\}}{\mathbf{p}+\mathbf{q}}.$$

If AB, AX and AC are in S_i, S_{i+3}, S_{i+4}, for $i \in \{1,3,5,7\}$, respectively, then

$$\mathbf{x} = \frac{(\mathbf{b} - (1-w)\,|\gamma-\delta|)\mathbf{p} + \mathbf{q}(\mathbf{c} - (1+w)\,(\theta+\beta))}{\mathbf{p}+\mathbf{q}}$$
$$= \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{(1-w)\,|\gamma-\delta|\,\mathbf{p} + (1+w)\,\mathbf{q}(\theta+\beta)}{\mathbf{p}+\mathbf{q}}.$$

If AB and AC are in S_i, S_{i+4} and AX is in S_{i+3}, S_{i+2}, S_{i+1}, for $i \in \{1,3,5,7\}$, then

$$\mathbf{x} = \frac{(\mathbf{b} - 2w\min\{\gamma,\delta\})\mathbf{p} + \mathbf{q}(\mathbf{c} - 2\max\{\theta,\beta\})}{\mathbf{p}+\mathbf{q}}$$
$$= \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{2w\mathbf{p}\min\{\gamma,\delta\} + 2\mathbf{q}\max\{\theta,\beta\}}{\mathbf{p}+\mathbf{q}},$$

$$\mathbf{x} = \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{(1+w)\mathbf{p}\min\{\gamma-\frac{w-1}{w+1}\delta, \delta-\frac{w-1}{w+1}\gamma\} + (1+w)\mathbf{q}\max\{\theta+\frac{w-1}{w+1}\beta, \beta+\frac{w-1}{w+1}\theta\})}{\mathbf{p}+\mathbf{q}},$$

$$\mathbf{x} = \frac{(\mathbf{b} - (1+w)\,(\gamma+\delta))\mathbf{p} + \mathbf{q}(\mathbf{c} - (1-w)\,|\theta-\beta|)}{\mathbf{p}+\mathbf{q}}$$
$$= \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{(1+w)\,\mathbf{p}(\gamma+\delta) + (1-w)\,\mathbf{q}\,|\theta-\beta|}{\mathbf{p}+\mathbf{q}},$$

respectively.

If AB and AC are in S_i, S_{i+4}, respectively, and AX is in S_i or S_{i+4}, then

$$\mathbf{x} = \frac{|\mathbf{bp} - \mathbf{qc}|}{\mathbf{p}+\mathbf{q}} = \begin{cases} \dfrac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \dfrac{2\mathbf{pb}}{\mathbf{p}+\mathbf{q}}, & AX \text{ in } S_i \\ \dfrac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \dfrac{2\mathbf{qc}}{\mathbf{p}+\mathbf{q}}, & AX \text{ in } S_{i+4}. \end{cases} \quad \blacksquare$$

As one of applications of the Stewart's theorem, we can find the formulae for length of the median of a triangle. If X is the midpoint of BC of any triangle ABC in the Euclidean plane with $\mathbf{a} = d(B,C)$, $\mathbf{b} = d(A,C)$, $\mathbf{c} = d(A,B)$ and $V_a = d(A,X)$, then

$$2V_a^2 = \mathbf{b}^2 + \mathbf{c}^2 - \mathbf{a}^2/2$$

which is known as *median property.* The following corollary gives an Alpha-version of this property, for $\mathbf{p} = \mathbf{q}$ in Theorem 3.1. The table in the next corollary gives the doubled length of the median about a side of a triangle according to position of the triangle in the analytical plane.

Corollary 3.2. *Let the sides of a triangle ABC in the $\mathbb{R}^2_\alpha$ have lengths $\mathbf{a} = d_\alpha(B,C)$, $\mathbf{b} = d_\alpha(A,C)$. If X is the midpoint of BC and $V_a = d_\alpha(A,X)$, then $2V_a$ is given as in Table 2, where $B = (b_1,b_2)$, $C = (c_1,c_2)$, $|b_1| = \theta$, $|b_2| = \beta$, $|c_1| = \gamma$,$|c_2| = \delta$ and $w = \sec\alpha - \tan\alpha$.*

When $\alpha = 0$ and $\alpha = \pi/4$ in α-distance function, one can immediately obtain Taxicab and Chinese checker distance function, respectively. Therefore, if $\alpha = 0$ and $\alpha = \pi/4$ is used in corollaries of alpha version of Stewart's theorem and median property, we obtain results for Taxicab and Chinese checkers planes, respectively. (See Kaya and Colakoglu [12] and Gelisgen and Kaya [9].)

Let $A = (0,0)$, $B = (3,2)$ and $C = (-4,1)$ be three points in analytical plane, and let $X = (-2, 9/7)$ lies on the side BC of the triangle ABC. Clearly, $\mathbf{x} = 2 + (\sec\alpha - \tan\alpha)\frac{9}{7}$. The sides AB, AX and AC lie in S_1, S_4 and S_4, respectively, the triangle ABC has only one base line through vertex A (see Figure 4). So $\Delta = 2\mathbf{q}\max\{\theta,\beta\} = 12 + \frac{12}{7}(\sec\alpha - \tan\alpha)$. Thus,

$$\mathbf{x} = \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{\Delta}{\mathbf{p}+\mathbf{q}} = 2 + (\sec\alpha - \tan\alpha)\frac{9}{7}$$

since $\mathbf{b} = 4 + (\sec\alpha - \tan\alpha)$, $\mathbf{c} = 3 + 2(\sec\alpha - \tan\alpha)$, $\mathbf{p} = 5 + \frac{5}{7}(\sec\alpha - \tan\alpha)$, $\mathbf{q} = 2 + \frac{2}{7}(\sec\alpha - \tan\alpha)$, $\theta = 3$ and $\beta = 2$. If X is the midpoint of BC of the triangle ABC, then $X = (-1/2, 3/2)$. So,

$$\begin{aligned} 2V_a &= \mathbf{b} + \mathbf{c} - (1-w)\,|\gamma - \delta| - (1+w)\max\left\{\theta + \frac{w-1}{w+1}\beta, \beta + \frac{w-1}{w+1}\theta\right\} \\ &= 3 + w. \end{aligned}$$

Table 2.

$2V_\alpha$	Number of base line	AB lies in	AX lies in	AC lies in
$\mathbf{b+c}$	0	S_i	S_i	S_i $\forall i$
$\mathbf{b+c} -(1-w)\lvert\gamma-\delta\rvert$	0	S_i	S_i	S_{i+1} $\forall i$
$\mathbf{b+c} -(1-w)\lvert\theta-\beta\rvert$	0	S_i	S_{i+1}	S_{i+1} $\forall i$
$\mathbf{b+c} -2w\min\{\theta,\beta\}$	1	S_i	S_{i+1}	S_{i+1} $\forall i$
$\mathbf{b+c} -2w\min\{\gamma,\delta\}$	1	S_i	S_i	S_{i+1} $\forall i$
$\mathbf{b+c} -(1-w)\lvert\gamma-\delta\rvert-2w\min\{\theta,\beta\}$	1	S_i	S_{i+1}	S_{i+2} $i\in\{1,3,5,7\}$
$\mathbf{b+c} -(1+w)\max\{\gamma+\frac{w-1}{w+1}\delta,\ \delta+\frac{w-1}{w+1}\gamma\}$	1	S_i	S_i	S_{i+2} $i\in\{1,3,5,7\}$
$\mathbf{b+c} -(1+w)\min\{\theta-\frac{w-1}{w+1}\beta,\ \beta-\frac{w-1}{w+1}\theta\}$	1	S_i	S_{i+2}	S_{i+2} $i\in\{1,3,5,7\}$
$\mathbf{b+c} -2w\min\{\gamma,\delta\}-(1-w)\lvert\theta-\beta\rvert$	1	S_i	S_{i+1}	S_{i+2} $i\in\{0,2,4,6\}$
$\mathbf{b+c} -(1+w)\max\{\theta+\frac{w-1}{w+1}\beta,\ \beta+\frac{w-1}{w+1}\theta\}$	1	S_i	S_{i+2}	S_{i+2} $i\in\{0,2,4,6\}$
$\mathbf{b+c} -(1+w)\min\{\gamma-\frac{w-1}{w+1}\delta,\ \delta-\frac{w-1}{w+1}\gamma\}$	1	S_i	S_i	S_{i+2} $i\in\{0,2,4,6\}$
$\mathbf{b+c} -(1-w)\lvert\gamma-\delta\rvert-(1+w)\max\{\theta+\frac{w-1}{w+1}\beta,\ \beta+\frac{w-1}{w+1}\theta\}$	1	S_i	S_{i+2}	S_{i+3} $\forall i$
$\mathbf{b+c} -(1+w)\max\{\gamma+\frac{w-1}{w+1}\delta,\ \delta+\frac{w-1}{w+1}\gamma\}-(1-w)\lvert\theta-\beta\rvert$	1	S_i	S_{i+1}	S_{i+3} $\forall i$
$\mathbf{b+c} -2\max\{\theta,\beta\}$	1	S_i	S_{i+3}	S_{i+3} $\forall i$
$\mathbf{b+c} -2\max\{\gamma,\delta\}$	1	S_i	S_i	S_{i+3} $\forall i$
$\mathbf{b+c} -(1+w)\min\{\gamma-\frac{w-1}{w+1}\delta,\ \delta-\frac{w-1}{w+1}\gamma\}-2w\min\{\theta,\beta\}$	2	S_i	S_{i+1}	S_{i+3} $\forall i$
$\mathbf{b+c} -(1+w)(\gamma+\delta)$	2	S_i	S_i	S_{i+3} $\forall i$
$\mathbf{b+c} -2w\min\{\gamma,\delta\}-(1+w)\min\{\theta-\frac{w-1}{w+1}\beta,\ \beta-\frac{w-1}{w+1}\theta\}$	2	S_i	S_{i+2}	S_{i+3} $\forall i$
$\mathbf{b+c} -(1+w)(\theta+\beta)$	2	S_i	S_{i+3}	S_{i+3} $\forall i$
$\mathbf{b+c} -2\max\{\gamma,\delta\}-2w\min\{\theta,\beta\}$	2	S_i	S_{i+1}	S_{i+4} $i\in\{1,3,5,7\}$
$\mathbf{b+c} -(1+w)\max\{\gamma+\frac{w-1}{w+1}\delta,\ \delta+\frac{w-1}{w+1}\gamma\}-(1+w)\min\{\theta-\frac{w-1}{w+1}\beta,\ \beta-\frac{w-1}{w+1}\theta\}$	2	S_i	S_{i+2}	S_{i+4} $i\in\{1,3,5,7\}$
$\mathbf{b+c} -(1-w)\lvert\gamma-\delta\rvert-(1+w)(\theta+\beta)$	2	S_i	S_{i+3}	S_{i+4} $i\in\{1,3,5,7\}$
$\mathbf{b+c} -2w\min\{\gamma,\delta\}-2\max\{\theta,\beta\}$	2	S_i	S_{i+3}	S_{i+4} $i\in\{0,2,4,6\}$
$\mathbf{b+c} -(1+w)\min\{\gamma-\frac{w-1}{w+1}\delta,\ \delta-\frac{w-1}{w+1}\gamma\}-(1+w)\max\{\theta+\frac{w-1}{w+1}\beta,\ \beta+\frac{w-1}{w+1}\theta\}$	2	S_i	S_{i+2}	S_{i+4} $i\in\{0,2,4,6\}$
$\mathbf{b+c} -(1+w)(\gamma+\delta)-(1-w)\lvert\theta-\beta\rvert$	2	S_i	S_{i+1}	S_{i+4} $i\in\{0,2,4,6\}$
$\lvert\mathbf{b}-\mathbf{c}\rvert$	2	S_i	S_i	S_{i+4} $\forall i$
$\lvert\mathbf{b}-\mathbf{c}\rvert$	2	S_i	S_{i+4}	S_{i+4} $\forall i$

As a future work, α-analog of law of cosine can be given by using α-trigonometric functions. Also one can study α-analog for the Stewart's theorem by using this α-analog of law of cosine and an α-analog of formulae for the angle bisectors of a triangle.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES
FACULTY OF SCIENCE AND ARTS
UNIVERSITY OF ESKISEHIR OSMANGAZI
ESKISEHIR, TURKEY
E-mails: gelisgen@ogu.edu.tr, rkaya@ogu.edu.tr

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