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ON THE CATEGORY OF PSEUDO-BCI-ALGEBRAS

Abstract. The category **psBCI** of pseudo-BCI-algebras and homomorphisms between them is investigated. It is also shown that the category $\mathbf{psBCI_p}$ of p-semisimple pseudo-BCI-algebras and homomorphisms between them is a reflective subcategory of **psBCI**.

1. Introduction

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are (pseudo-)MV-algebras, (pseudo-)BL-algebras, (pseudo-)BCK-algebras, (pseudo-)BCI-algebras and others. They are strongly connected with logic. For example, BCI-algebras introduced by K. Iséki in 1966 ([7]) have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming.

The notion of pseudo-BCI-algebras has been introduced by W. A. Dudek and Y. B. Jun in [3] as an extension of BCI-algebras and it was investigated by several authors in [4], [5], [8] and [9]. Pseudo-BCI-algebras are algebraic models of some extension of a non-commutative version of the BCI-logic. These algebras have also connections with other algebras of logic such as pseudo-BCK-algebras, pseudo-BL-algebras and pseudo-MV-algebras.

In this paper, the category **psBCI** of pseudo-BCI-algebras and homomorphisms between them is considered. We prove that it has equalizers, coequalizers, products, pullbacks, limits, kernel pairs and it is complete. Moreover, we show that in **psBCI** surjective morphisms and coequalizers coincide. Finally, the category $\mathbf{psBCI_p}$ of p-semisimple pseudo-BCI-algebras and homomorphisms between them is studied. We show that it is a reflective subcategory of **psBCI** and it is isomorphic with the category **Grp** of groups and group homomorphisms.

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2. Preliminaries

We include some necessary material concerning pseudo-BCI-algebras, needed in the sequel.

A *pseudo*-BCI-*algebra* is a structure $(X, \leq, \to, \rightsquigarrow, 1)$, where $\leq$ is a binary relation on a set X, $\to$ and $\rightsquigarrow$ are binary operations on X and 1 is an element of X such that, for all $x, y, z \in X$, we have

(a1) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$, $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$,
(a2) $x \leq (x \to y) \rightsquigarrow y$, $x \leq (x \rightsquigarrow y) \to y$,
(a3) $x \leq x$,
(a4) if $x \leq y$ and $y \leq x$, then $x = y$,
(a5) $x \leq y$ iff $x \to y = 1$ iff $x \rightsquigarrow y = 1$.

It is obvious that any pseudo-BCI-algebra $(X, \leq, \to, \rightsquigarrow, 1)$ can be regarded as a universal algebra $(X, \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$. Note that every pseudo-BCI-algebra satisfying $x \to y = x \rightsquigarrow y$, for all $x, y \in X$ is a BCI-algebra.

Every pseudo-BCI-algebra satisfying $x \leq 1$, for all $x \in X$ is a pseudo-BCK-algebra. A pseudo-BCI-algebra which is not a pseudo-BCK-algebra will be called *proper*.

Later in the paper, we will usually use the symbol X in place of $(X, \to, \rightsquigarrow, 1)$.

Any pseudo-BCI-algebra $(X, \to, \rightsquigarrow, 1)$ satisfies the following, for all $x, y, z \in X$,

(b1) if $1 \leq x$, then $x = 1$,
(b2) if $x \leq y$, then $y \to z \leq x \to z$ and $y \rightsquigarrow z \leq x \rightsquigarrow z$,
(b3) if $x \leq y$ and $y \leq z$, then $x \leq z$,
(b4) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,
(b5) $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$,
(b6) $x \to y \leq (z \to x) \to (z \to y)$, $x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$,
(b7) if $x \leq y$, then $z \to x \leq z \to y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$,
(b8) $1 \to x = 1 \rightsquigarrow x = x$,
(b9) $((x \to y) \rightsquigarrow y) \to y = x \to y$, $((x \rightsquigarrow y) \to y) \rightsquigarrow y = x \rightsquigarrow y$,
(b10) $x \to y \leq (y \to x) \rightsquigarrow 1$,
(b11) $x \rightsquigarrow y \leq (y \rightsquigarrow x) \to 1$,
(b12) $(x \to y) \to 1 = (x \to 1) \rightsquigarrow (y \rightsquigarrow 1)$,
(b13) $(x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \to (y \to 1)$,
(b14) $x \to 1 = x \rightsquigarrow 1$.

If $(X, \leq, \to, \rightsquigarrow, 1)$ is a pseudo-BCI-algebra, then by (a3), (a4), (b3) and (b1), $(X, \leq)$ is a poset with 1 as a maximal element.

For any pseudo-BCI-algebra $(X, \to, \rightsquigarrow, 1)$ the set

$$K(X) = \{x \in X : x \leq 1\}$$

is a subalgebra of X (called pseudo-BCK-part of X, see [3]).

Let $(X, \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Then X is p-*semisimple* if it satisfies for all $x \in X$,

$$\text{if } x \leq 1, \text{ then } x = 1.$$

Note that if X is a p-semisimple pseudo-BCI-algebra, then $K(X) = \{1\}$. Hence, if X is a p-semisimple pseudo-BCK-algebra, then $X = \{1\}$. It is proved in [5] that $(X, \to, \rightsquigarrow, 1)$ is p-semisimple if and only if for all $x, y \in X$, $(x \to 1) \rightsquigarrow y = (y \rightsquigarrow 1) \to x$.

Let $(X, \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. We say that a subset D of X is a *deductive system* of X if it satisfies: (1) $1 \in D$, (2) for all $x, y \in X$, if $x \in D$ and $x \to y \in D$, then $y \in D$. Under this definition, $\{1\}$ and X are the simplest examples of deductive systems. Note that the condition (2) can be replaced by (2') for all $x, y \in X$, if $x \in D$ and $x \rightsquigarrow y \in D$, then $y \in D$. It can be easily proved that for any $x, y \in X$, if $x \in D$ and $x \leq y$, then $y \in D$.

A deductive system D of a pseudo-BCI-algebra $(X, \to, \rightsquigarrow, 1)$ is called *closed* if D is closed under operations $\to$ and $\rightsquigarrow$, that is, if D is a subalgebra of X. It is not difficult to show (see [4]) that a deductive system D of a pseudo-BCI-algebra $(X, \to, \rightsquigarrow, 1)$ is closed if and only if for any $x \in D$, $x \to 1 = x \rightsquigarrow 1 \in D$. Obviously, the pseudo-BCK-part $K(X)$ is a closed deductive system of X.

A deductive system D of a pseudo-BCI-algebra $(X, \to, \rightsquigarrow, 1)$ is said to be *compatible* if for all $x, y \in X$,

$$x \to y \in D \text{ iff } x \rightsquigarrow y \in D.$$

Further, if D is a compatible deductive system of X, then the relation Θ_D defined by

$$(1) \qquad (x, y) \in \Theta_D \text{ iff } x \to y \in D \text{ and } y \to x \in D$$

is a congruence. We say that $\Theta \in \mathrm{Con}(X)$ is a *relative congruence* of $(X, \to, \rightsquigarrow, 1)$ if the quotient algebra $(X/\Theta, \to, \rightsquigarrow, [1]_\Theta)$ is a pseudo-BCI-algebra. It is proved in [4] that relative congruences of X correspond one-to-one to closed compatible deductive systems of X, that is, every relative congruence of X is given by (1) for some closed compatible deductive system D. For every relative congruence Θ_D, the quotient algebra $(X/\Theta_D, \to, \rightsquigarrow, [1]_{\Theta_D})$ will be usually denoted by $(X/D, \to, \rightsquigarrow, 1/D)$ and then we will write x/D instead of $[x]_{\Theta_D}$.

We know that pseudo-BCK-part $K(X)$ of a pseudo-BCI-algebra $(X, \to, \rightsquigarrow, 1)$ is a closed deductive system of X. It is proved in [4] that it is also compatible and we have that $(X/K(X), \to, \rightsquigarrow, 1/K(X))$ is a p-semisimple pseudo-BCI-algebra.

Moreover we will need the following fact.

Lemma 2.1. *Let $f : X \to Y$ be a homomorphism of pseudo-*BCI*-algebras X, Y. Then $Ker(f) = \{x \in X : f(x) = 1\}$ is a closed compatible deductive system of X.*

Proof. Routine. ∎

3. The category psBCI

All notions from the category theory occuring in this section the reader can find in [1] or [11].

If we consider the class of all pseudo-BCI-algebras as the class of objects and the class of all homomorphisms between pseudo-BCI-algebras as the class of morphisms, then we obtain the category of pseudo-BCI-algebras. We denote it by **psBCI**. In the section, we investigate this category.

First, remark that the class of objects in **psBCI** is not a set. Therefore, **psBCI** is not a small category. Moreover, we can define a forgetful functor F : **psBCI** $\to$ **Set** which is faithful. Hence, the category **psBCI** is concrete and embedded in the category **Set** of sets and functions.

Observe yet that in **psBCI**, $\{1\}$ is a zero object because it is an initial object as well as a terminal object. Indeed, there is an unique morphism $f : \{1\} \to X$ for any object X, so $\{1\}$ is an initial object. Similarly, there exists an unique morphism $g : X \to \{1\}$ for any object X, so $\{1\}$ is also a terminal object. Further, note that $0_{\{1\}} : X \to \{1\}$ is a zero morphism in **psBCI**, since it is in the same time a constant morphism and coconstant morphism.

Theorem 3.1. *For any morphism $f : X \to Y$ in* **psBCI** *the following are equivalent:*

(i) *f is injective,*
(ii) *for all morphisms g, h, if $f \circ g = f \circ h$, then $g = h$,*
(iii) *$Ker(f) = \{1\}$.*

Proof. (i)⇒(ii): Assume that f is an injective morphism between objects X, Y. Let Z be another object, and let $g, h : Z \to X$ be morphisms such that $f \circ g = f \circ h$. Then for all $z \in Z$, $f(g(z)) = f(h(z))$. Hence since f is injective, we get $g(z) = h(z)$. Thus $g = h$.

(ii)⇒(iii): Suppose that $Ker(f) \neq \{1\}$. Then there exists $x \in Ker(f)$ and $x \neq 1$. Let us consider morphisms $i : Ker(f) \to X$ and $j : Ker(f) \to X$ such that $i(x) = x$ and $j(x) = 1$, for all $x \in Ker(f)$. Then $f \circ i = f \circ j$. Now, by (ii), $i = j$. Thus we get a contradiction. Therefore $Ker(f) = \{1\}$.

(iii)⇒(i): Let $Ker(f) = \{1\}$ and $x_1, x_2 \in X$ be such that $f(x_1) = f(x_2)$. Then $f(x_1 \to x_2) = f(x_1) \to f(x_2) = 1$ and $f(x_2 \to x_1) = f(x_2) \to f(x_1) = 1$. Hence $x_1 \to x_2, x_2 \to x_1 \in Ker(f) = \{1\}$. Thus $x_1 \to x_2 = x_2 \to x_1 = 1$, so, $x_1 \leq x_2$ and $x_2 \leq x_1$. Now it is clear that $x_1 = x_2$ and f is injective. ∎

Corollary 3.2. *In the category* **psBCI** *injective morphisms and monomorphisms coincide.*

Proposition 3.3. *Let $f : X \to Y$ be a morphism in* **psBCI**. *If f is surjective, then for all morphisms g, h, if $g \circ f = h \circ f$, then $g = h$.*

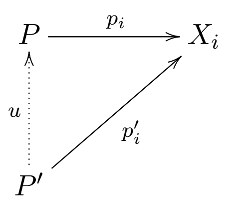
Proof. Let $f : X \to Y$ be a surjective morphism, Z be an object and $g, h : Y \to Z$ be morphisms such that $g \circ f = h \circ f$. Since f is surjective, for any $y \in Y$ there exists $x \in X$ such that $y = f(x)$. Then $g(y) = g(f(x)) = h(f(x)) = h(y)$, for all $y \in Y$. Therefore $g = h$. ∎

Corollary 3.4. *A morphism in the category* **psBCI** *is an epimorphism if it is surjective.*

Remark. It is well-known that any Hilbert algebra is a pseudo-BCI-algebra (precisely, a BCK-algebra). In [2] there is given an example of an epimorphism between Hilbert algebras (so, pseudo-BCI-algebras) which is not surjective. Thus, in the category **psBCI** isomorphisms and bimorphisms are not the same.

Corollary 3.5. *The category* **psBCI** *is not balanced.*

Let **C** be a category and $(X_i)_{i \in I}$ a family of objects in **C**. A *direct product* of a family $(X_i)_{i \in I}$ is a pair $(P, (p_i))_{i \in I}$, where P is an object in **C** and $(p_i)_{i \in I}$ is a family of morphisms in **C**, $p_i : P \to X_i$, such that for any other pair $(P', (p'_i))_{i \in I}$ composed by an object P' and a family of morphisms $(p'_i)_{i \in I}$, $p'_i : P' \to X_i$, there is an unique morphism $u : P' \to P$ such that $p_i \circ u = p'_i$ for every $i \in I$, so that for every $i \in I$ the following diagram is commutative:



We say that a category **C** *has products* if there exists a direct product of any family of objects from **C**.

Theorem 3.6. *The category* **psBCI** *has products.*

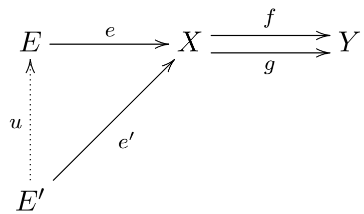
Proof. Let $(X_i)_{i \in I}$ be a familly of objects. Consider the set $P = \prod_{i \in I} X_i$ of all functions $f : I \to \bigcup_{i \in I} X_i$ such that $f(i) \in X_i$ for all $i \in I$. A function $1 : I \to \bigcup_{i\in I} X_i$ such that $1(i) = 1$ for all $i \in I$, is a special element of P. Define binary operations $\to$ and $\rightsquigarrow$ on P as follows: for $f, g \in P$, $(f \to g)(i) = f(i) \to g(i)$ and $(f \rightsquigarrow g)(i) = f(i) \rightsquigarrow g(i)$ for all $i \in I$. We can verify that the structure $(P, \to, \rightsquigarrow, 1)$ forms a pseudo-BCI-algebra, that is P is an object in **psBCI**.

For each $i \in I$, there is a natural projection $p_i : P \to X_i$ defined by $p_i(f) = f(i)$ for all $f \in P$. Further, for all objects P' and morphisms $p'_i : P' \to X_i$ for $i \in I$ the map $u : P' \to P$ defined by

$$(u(x))(i) = p'_i(x) \text{ for all } x \in P' \text{ and } i \in I$$

is the unique morphism such that $p_i \circ u = p'_i$. Thus the category **psBCI** has products. ■

By a couple of morphisms (f, g) in a category **C** we understand two morphisms $f, g : X \to Y$, where X, Y are objects in **C**. A pair (E, e) with E an object in **C** and $e : E \to X$ a morphism in **C**, will be called an *equalizer* of a couple (f, g) if $f \circ e = g \circ e$ and for every other pair (E', e') with E' an object and $e' : E' \to X$ a morphism such that $f \circ e' = g \circ e'$, there exists an unique morphism $u : E' \to E$ such that $e' = e \circ u$:



We say that a category **C** *has equalizers* if there exists an equalizer for any couple of morphisms in **C**.

Theorem 3.7. *The category* **psBCI** *has equalizers.*

Proof. Let (f, g) be a couple of morphisms, $f, g : X \to Y$. Then nonempty set $E = \{x \in X : f(x) = g(x)\}$ is a subalgebra of X and if we consider the empedding $e : E \to X$, then $f \circ e = g \circ e$.

Further, let E' be other object and let $e' : E' \to X$ be a morphism such that $f \circ e' = g \circ e'$. We define $u : E' \to E$, $u(x) = e'(x)$ for all $x \in E'$. Then u is well defined, since from $f \circ e' = g \circ e'$ we have $e'(x) \in E$ for every $x \in E'$. It is clear that u is a morphism and $e \circ u = e'$.

The uniqueness of u follows from the fact that e is a monomorphism. ■

Corollary 3.8. *The category* **psBCI** *has pullbacks, limits and it is complete.*

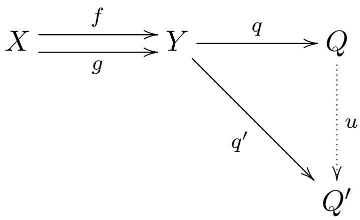
Let $f : X \to Y$ be a morphism in $\mathbf{C}$. We say that f is an *equalizer* if there exists a couple of morphisms (α, β) such that $\alpha, \beta : Y \to Z$ and (X, f) is an equalizer of (α, β). Obviously, every equalizer in $\mathbf{C}$ is a monomorphism.

Thus by Corollary 3.2, we have the following theorem.

Theorem 3.9. *In the category* **psBCI** *every equalizer is injective.*

Remark. The converse of Theorem 3.9 is not true. In [6], there is given an example of an injective morphism between Hilbert algebras (so, pseudo-BCI-algebras) which can not be an equalizer for any couple of morphisms.

Let $f, g : X \to Y$, where X, Y are objects in a category $\mathbf{C}$. A pair (Q, q) with Q an object in $\mathbf{C}$ and $q : Y \to Q$ a morphism in $\mathbf{C}$, will be called a *coequalizer* of a couple (f, g) if $q \circ f = q \circ g$ and for every other pair (Q', q') with Q' an object and $q' : Y \to Q'$ a morphism such that $q' \circ f = q' \circ g$, there exists an unique morphism $u : Q \to Q'$ such that $q' = u \circ q$:



We say that a category $\mathbf{C}$ *has coequalizers* if there exists a coequalizer for any couple of morphisms in $\mathbf{C}$.

Theorem 3.10. *The category* **psBCI** *has coequalizers.*

Proof. Let (f, g) be a couple of morphisms, $f, g : X \to Y$. Put

$$R = \{(f(x), g(x)) \in Y \times Y : x \in X\}.$$

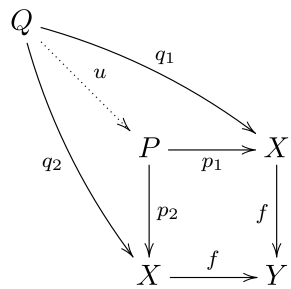
Let Θ be the intersection of all relative congruences on Y (that is, congruences determined by closed compatible deductive systems of Y) which contain R. Then $Q = Y/\Theta$ is an object in **psBCI**. Let $q : Y \to Q$ be the canonical surjection. We show that (Q, q) is a coequalizer of (f, g). Since $(f(x), g(x)) \in \Theta$ for all $x \in X$, we have $(q \circ f)(x) = q(f(x)) = [f(x)]_\Theta = [g(x)]_\Theta = q(g(x)) = (q \circ g)(x)$ for all $x \in X$. Thus $q \circ f = q \circ g$.

Let Q' be another object and let $q' : Y \to Q'$ be a morphism such that $q' \circ f = q' \circ g$. Let $\Theta' = \{(y_1, y_2) \in Y \times Y : q'(y_1) = q'(y_2)\} = \{(y_1, y_2) \in Y \times Y : y_1 \to y_2, y_2 \to y_1 \in Ker(q')\}$. Then Θ' is a relative congruence determined by a closed compatible deductive system $Ker(q')$. Since for every $x \in X$ we have $q'(f(x)) = q'(g(x))$, we obtain $(f(x), g(x)) \in \Theta'$ for every $x \in X$. Hence $R \subset \Theta'$. Thus $\Theta \subset \Theta'$. We can define now a morphism $u : Q \to Q'$ such that $u([y]_\Theta) = q'(y)$. Then u is well defined because for

$[y_1]_\Theta = [y_2]_\Theta$ we have $(y_1, y_2) \in \Theta \subset \Theta'$ whence $q'(y_1) = q'(y_2)$. Clearly, $u \circ q = q'$.

The uniqueness of u follows from the fact that q is an epimorphism. This completes the proof. ∎

Let $\mathbf{C}$ be a category and $f : X \to Y$ a morphism in $\mathbf{C}$. A system $(P; p_1, p_2)$ formed by an object P and two morphisms $p_1, p_2 : P \to X$, is called a *kernel pair* of f if $f \circ p_1 = f \circ p_2$ and for any other system $(Q; q_1, q_2)$ with an object Q and morphisms $q_1, q_2 : Q \to X$ such that $f \circ q_1 = f \circ q_2$, there exists an unique morphism $u : Q \to P$ such that $p_1 \circ u = q_1$ and $p_2 \circ u = q_2$:



We say that a category $\mathbf{C}$ *has kernel pairs* if every morphism in it has a kernel pair.

Theorem 3.11. *The category* **psBCI** *has kernel pairs.*

Proof. Let $f : X \to Y$ be a morphism. Let us put

$$P = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}.$$

Obviously, P is a subalgebra of the product algebra $X \times X$. Let $p_1, p_2 : P \to X$ be the canonical projections, that is, $p_i(x_1, x_2) = x_i$ for $i = 1, 2$ and all $(x_1, x_2) \in P$. We show that $(P; p_1, p_2)$ is a kernel pair of f. Clearly, $f \circ p_1 = f \circ p_2$. Let $(Q; q_1, q_2)$ with an object Q and morphisms $q_1, q_2 : Q \to X$ be another system such that $f \circ q_1 = f \circ q_2$. We take $u : Q \to P$ as

$$u(x) = (q_1(x), q_2(x)) \text{ for all } x \in Q.$$

Now, u is well defined because $f \circ q_1 = f \circ q_2$ implies $f(q_1(x)) = f(q_2(x))$ whence $(q_1(x), q_2(x)) \in P$. Further,

$$\begin{aligned} u(x_1 \to x_2) &= (q_1(x_1 \to x_2), q_2(x_1 \to x_2)) \\ &= (q_1(x_1) \to q_1(x_2), q_2(x_1) \to q_2(x_2)) \\ &= (q_1(x_1), q_2(x_1)) \to (q_1(x_2), q_2(x_2)) \\ &= u(x_1) \to u(x_2) \end{aligned}$$

and

$$\begin{aligned} u(x_1 \rightsquigarrow x_2) &= (q_1(x_1 \rightsquigarrow x_2), q_2(x_1 \rightsquigarrow x_2)) \\ &= (q_1(x_1) \rightsquigarrow q_1(x_2), q_2(x_1) \rightsquigarrow q_2(x_2)) \\ &= (q_1(x_1), q_2(x_1)) \rightsquigarrow (q_1(x_2), q_2(x_2)) \\ &= u(x_1) \rightsquigarrow u(x_2). \end{aligned}$$

Thus u is a morphism in **psBCI**. Moreover, it is easy to see that $p_1 \circ u = q_1$ and $p_2 \circ u = q_2$.

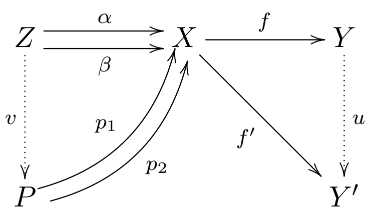
Now, let $u' : Q \to P$ be another morphism such that $p_1 \circ u' = q_1$ and $p_2 \circ u' = q_2$. Let $u'(x) = (x', x'')$. Then $p_1 \circ u' = p_1 \circ u$ gives $p_1(x', x'') = p_1(q_1(x), q_2(x))$ whence $x' = q_1(x)$ and $x'' = q_2(x)$. Thus $u'(x) = (x', x'') = (q_1(x), q_2(x)) = u(x)$ for all $x \in Q$. Hence u is unique. Therefore, the system $(P; p_1, p_2)$ is a kernel pair of f. ■

Let $f : X \to Y$ be a morphism in **C**. We say that f is a *coequalizer* if there exists a couple of morphisms (α, β) such that $\alpha, \beta : Z \to X$ and (Y, f) is a coequalizer of (α, β). Clearly, every coequalizer in **C** is an epimorphism.

Proposition 3.12. *Let $f : X \to Y$ be a coequalizer in* **psBCI**. *Then f is a coequalizer of its kernel pair.*

Proof. Let $\alpha, \beta : Z \to X$ be such that f is a coequalizer of (α, β) and let $(P; p_1, p_2)$ be a kernel pair of f. Since $f \circ p_1 = f \circ p_2$, it is sufficient to prove that for any other morphism $f' : X \to Y'$ such that $f' \circ p_1 = f' \circ p_2$, there exists an unique morphism $u : Y \to Y'$ such that $f' = u \circ f$.

Since $f \circ \alpha = f \circ \beta$ and $(P; p_1, p_2)$ is a kernel pair of f, we get the existence of an unique morphism $v : Z \to P$ such that $\alpha = p_1 \circ v$ and $\beta = p_2 \circ v$:



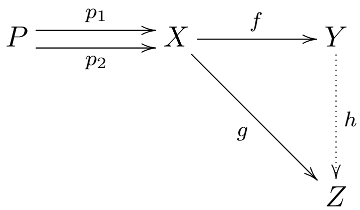
Hence $f' \circ \alpha = (f' \circ p_1) \circ v = (f' \circ p_2) \circ v = f' \circ \beta$. Thus since f is a coequalizer of (α, β), we obtain the existence of an unique $u : Y \to Y'$ such that $f' = u \circ f$. This completes the proof. ■

Theorem 3.13. *Every surjective morphism in* **psBCI** *is a coequalizer.*

Proof. Let $(P; p_1, p_2)$ be a kernel pair of a surjective morphism $f : X \to Y$. Then as we know $P = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ and $p_1, p_2 : P \to X$ are the canonical projections. It is sufficient to prove that (Y, f) is a coequalizer of (p_1, p_2). Clearly, $f \circ p_1 = f \circ p_2$. Let $f' : X \to Y'$ be a morphism such that $f' \circ p_1 = f' \circ p_2$. Since f is surjective, for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$. Now, take $u : Y \to Y'$ as follows: $u(y) = f'(x)$. It is well defined because if $f(x_1) = f(x_2) = y$, then $(x_1, x_2) \in P$ and $u(y) = f'(x_1) = (f' \circ p_1)(x_1, x_2) = (f' \circ p_2)(x_1, x_2) = f'(x_2)$. Next, let $y_1, y_2 \in Y$. Then there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$, and hence $f'(x_1) = u(y_1)$ and $f'(x_2) = u(y_2)$. Further, we have $y_1 \to y_2 = f(x_1) \to f(x_2) = f(x_1 \to x_2)$ and $y_1 \rightsquigarrow y_2 = f(x_1) \rightsquigarrow f(x_2) = f(x_1 \rightsquigarrow x_2)$. Hence $u(y_1 \to y_2) = f'(x_1 \to x_2) = f'(x_1) \to f'(x_2) = u(y_1) \to u(y_2)$ and $u(y_1 \rightsquigarrow y_2) = f'(x_1 \rightsquigarrow x_2) = f'(x_1) \rightsquigarrow f'(x_2) = u(y_1) \rightsquigarrow u(y_2)$. Thus u is a morphism and obviously, $u \circ f = f'$. The uniqueness of u follows from the fact that f is an epimorphism. Therefore f is a coequalizer. ∎

Proposition 3.14. *Let X, Y, Z be objects in* **psBCI** *and $f : X \to Y$ and $g : X \to Z$ be morphisms in* **psBCI** *such that f is surjective and $Ker(f) \subset Ker(g)$. Then there exists an unique morphism $h : Y \to Z$ such that $h \circ f = g$.*

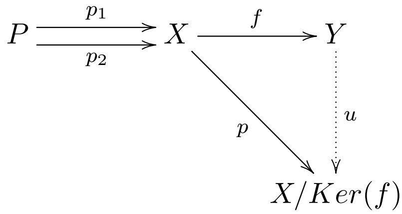
Proof. Let $(P; p_1, p_2)$ be a kernel pair of f, that is, $P = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ and $p_1, p_2 : P \to X$ are the canonical projections. Since f is surjective, by Theorem 3.13, we have that f is a coequalizer, that is, (Y, f) is a coequalizer of (p_1, p_2). Let $(x_1, x_2) \in P$. Then $f(x_1) = f(x_2)$ which gives that $x_1 \to x_2, x_2 \to x_1 \in Ker(f) \subset Ker(g)$, so $g(x_1) = g(x_2)$. Hence, there exists an unique morphism $h : Y \to Z$ such that $h \circ f = g$:



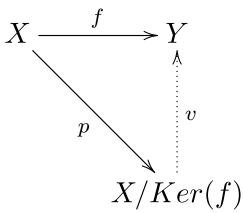
This completes the proof. ∎

Theorem 3.15. *Every coequalizer in* **psBCI** *is surjective.*

Proof. Let $f : X \to Y$ be a coequalizer. By Proposition 3.12, f is a coequalizer of its kernel pair $(P; p_1, p_2)$, where $P = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ and $p_1, p_2 : P \to X$ are the canonical projections. Note that $P = \{(x_1, x_2) \in X \times X : x_1 \to x_2, x_2 \to x_1 \in Ker(f)\}$. Hence P is a relative congruence determined by the closed compatible deductive system $Ker(f)$. Let $X/Ker(f)$ be the corresponding quotient pseudo-BCI-algebra and let $p : X \to X/Ker(f)$ be the canonical surjection. Notice that $p \circ p_1 = p \circ p_2$. Indeed, for every $(x_1, x_2) \in P$ we have $(p \circ p_1)(x_1, x_2) = x_1/Ker(f) = x_2/Ker(f) = (p \circ p_2)(x_1, x_2)$. Since (Y, f) is a coequalizer of (p_1, p_2), there exists an unique morphism $u : Y \to X/Ker(f)$ such that $u \circ f = p$:



Let $x \in Ker(p)$. Then $p(x) = 1/Ker(f)$. Since $p(x) = x/Ker(f)$, we get $(x, 1) \in P$, so $x \in Ker(f)$. This means that $Ker(p) \subset Ker(f)$. Thus by Proposition 3.14, there exists an unique morphism $v : X/Ker(f) \to Y$ such that $v \circ p = f$:



Now,

$$(u \circ v) \circ p = u \circ f = p = 1_{X/Ker(f)} \circ p$$

and

$$(v \circ u) \circ f = v \circ p = f = 1_Y \circ f.$$

Since p is surjective and f is a coequalizer, both are epimorphisms. Hence

$$u \circ v = 1_{X/Ker(f)} \quad \text{and} \quad v \circ u = 1_Y.$$

Thus u and v are isomorphisms, one the inverse of the other. Now, we get that $f = v \circ p$ is surjective, because both v and p are surjective. ∎

Corollary 3.16. *In the category* **psBCI** *surjective morphisms and coequalizers coincide.*

Remark. In the category **psBCI** not every epimorphism is a coequalizer. Indeed, in [6] there is given an example of an epimorphism (not a surjective one) between Hilbert algebras (so, pseudo-BCI-algebras) which is not a coequalizer.

4. The category $\mathbf{psBCI_p}$

The category formed by taking the class of objects as the class of all p-semisimple pseudo-BCI-algebras and the class of morphisms as the class of all homomorphisms between them is called the category of p-semisimple pseudo-BCI-algebras. We denote this category by $\mathbf{psBCI_p}$. We have an inclusion functor $I : \mathbf{psBCI_p} \hookrightarrow \mathbf{psBCI}$, which is faithful and full. Hence

psBCI$_\mathbf{p}$ is a full subcategory of the category **psBCI**. Like **psBCI**, the category **psBCI$_\mathbf{p}$** is not a small category, it is concrete and embedded in the category **Set**; it also has zero objects ($\{1\}$ is so) and zero morphisms ($0_{\{1\}} : X \to \{1\}$ is the one).

For p-semisimple pseudo-BCI-algebras we have the following nice fact from [5] (compare with [10] for p-semisimple BCI-algebras).

Theorem 4.1. *A pseudo-*BCI*-algebra* $(X, \to, \rightsquigarrow, 1)$ *is p-semisimple if and only if* $(X, \cdot, ^{-1}, e)$ *is a group, where, for any* $x, y \in X$, $x \cdot y = (x \to 1) \rightsquigarrow y = (y \rightsquigarrow 1) \to x$, $x^{-1} = x \to 1 = x \rightsquigarrow 1$ *and* $e = 1$. *In this case,* $x \to y = y \cdot x^{-1}$ *and* $x \rightsquigarrow y = x^{-1} \cdot y$ *for any* $x, y \in X$.

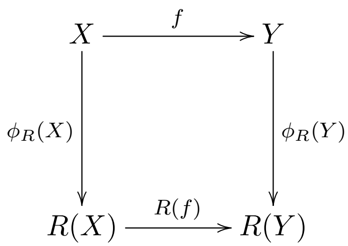
Moreover, it is not difficult to prove that f is a morphism in the category **psBCI$_\mathbf{p}$** if and only if it is a morphism in the category **Grp** of groups and group homomorphisms. Thus we have the following theorem.

Theorem 4.2. *The category* **psBCI$_\mathbf{p}$** *is isomorphic with the category* **Grp**.

Remark. From Theorem 4.2, it follows that the category **psBCI$_\mathbf{p}$** has the same properties as the category **Grp**. For example, it has coproducts and it is balanced and cocomplete.

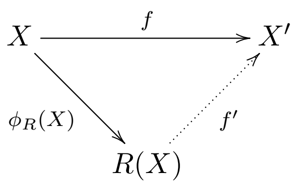
A subcategory **C'** of a category **C** is called *reflective* if there is a covariant functor $R : \mathbf{C} \to \mathbf{C}'$, called *reflector*, such that for every object X from **C** there is a morphism $\phi_R(X) : X \to R(X)$ in **C** with the properties:

(i) if $f : X \to Y$ is a morphism in **C**, then the diagram



is commutative, that is, $\phi_R(Y) \circ f = R(f) \circ \phi_R(X)$,

(ii) if X' is an object in **C'** and $f : X \to X'$ is a morphism in **C**, then there is an unique morphism $f' : R(X) \to X'$ in **C'** such that the diagram



is commutative, that is, $f' \circ \phi_R(X) = f$.

Remark. It is a well known fact that **C'** is a reflective subcategory of a category **C** if and only if there exist a function which assigns to every object X in **C**, an object $R(X)$ in **C'** and a function which assigns to every X in **C**, a morphism $\phi_R(X) : X \to R(X)$ in **C** such that for every object X' in **C'** and every morphism $f : X \to X'$ in **C** there is an unique morphism $f' : R(X) \to X'$ in **C'** such that $f' \circ \phi_R(X) = f$.

Theorem 4.3. *The category* $\mathbf{psBCI_p}$ *is a reflective subcategory of the category* **psBCI**.

Proof. Let X be an object in **psBCI**. Then as we know $X/K(X)$ is an object in $\mathbf{psBCI_p}$. Thus, we put $R(X) = X/K(X)$. We define $\phi_R(X) : X \to R(X)$ as follows

$$(\phi_R(X))(x) = x/K(X), \text{ for all } x \in X,$$

that is, $\phi_R(X)$ is the canonical surjection.

Now, take a morphism $f : X \to Y$, where Y is an object in $\mathbf{psBCI_p}$. First, note that $f(x) = 1$, for all $x \in K(X)$. Indeed, $x \to 1 = 1$ gives $1 = f(1) = f(x \to 1) = f(x) \to f(1) = f(x) \to 1$, that is, $f(x) \in K(Y) = \{1\}$ whence $f(x) = 1$. We define $f' : R(X) \to Y$ as follows

$$f'(x/K(X)) = f(x), \text{ for all } x \in X.$$

First of all, we prove that f' is well defined. Let $x_1/K(X) = x_2/K(X)$. Then $x_1 \to x_2 \in K(X)$ and $x_2 \to x_1 \in K(X)$, which gives $f(x_1 \to x_2) = 1$ and $f(x_2 \to x_1) = 1$, that is, $f(x_1) = f(x_2)$. This proves that f' is well defined. Further, it is easy to show that f' is a morphism in $\mathbf{psBCI_p}$ and $f' \circ \phi_R(X) = f$.

The uniqueness of f' follows from the fact that $\phi_R(X)$ is an epimorphism. This completes the proof. ∎

Remark. The reflector $R : \mathbf{psBCI} \to \mathbf{psBCI_p}$ is defined in the following way. If for X from **psBCI** we put

$$R(X) = X/K(X),$$

then we obtain the definition of R on objects. Now, let $f : X \to Y$ be a morphism in **psBCI**. If we define $R(f) : R(X) \to R(Y)$ by

$$(R(f))(x/K(X)) = f(x)/K(Y), \text{ for all } x \in X,$$

then we obtain the definition of R on morphisms. Obviously, R is a left adjoint for the inclusion functor $I : \mathbf{psBCI_p} \hookrightarrow \mathbf{psBCI}$. Moreover, R is faithfull.

5. Conclusions

In the category **psBCI**, monomorphisms and injective morphisms coincide, but epimorphisms and surjective morphisms not. These imply that

psBCI is not balanced. Since in **psBCI**, not every monomorphism is an equalizer and not every epimorphism is a coequalizer, they are not normal, that is, **psBCI** is not abelian. In the same time, since it has arbitrary limits, it is complete. It is an open problem if it is cocomplete.

The category $\mathbf{psBCI_p}$ is a full and reflective subcategory of **psBCI** and it is isomorphic with the category **Grp**. This means that $\mathbf{psBCI_p}$ is among other things balanced and cocomplete.

References

[1] D. Buşneag, *Categories of Algebraic Logic*, Editura Academiei Romane, Bucharest, 2006.
[2] D. Buşneag, M. Ghiţă, *Some properties of epimorphisms of Hilbert algebras*, Cent. Eur. J. Math. 8 (2010), 41–52.
[3] W. A. Dudek, Y. B. Jun, *Pseudo*-BCI *algebras*, East Asian Math. J. 24 (2008), 187–190.
[4] G. Dymek, *On compatible deductive systems of pseudo*-BCI-*algebras*, J. Mult.-Valued Logic Soft Comput., to appear.
[5] G. Dymek, *p-semisimple pseudo*-BCI-*algebras*, J. Mult.-Valued Logic Soft Comput. 19 (2012), 461–474.
[6] M. Ghiţă, *Some categorical properties of Hilbert algebras*, An. Univ. Craiova Ser. Mat. Inform. 36 (2009), 95–104.
[7] K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan Acad. 42 (1966), 26–29.
[8] Y. B. Jun, H. S. Kim, J. Neggers, *On pseudo*-BCI *ideals of pseudo* BCI-*algebras*, Mat. Vesnik 58 (2006), 39–46.
[9] K. J. Lee, C. H. Park, *Some ideals of pseudo*-BCI *algebras*, J. Appl. Math. Informatics 27 (2009), 217–231.
[10] T. D. Lei, C. C. Xi, *p-radical in* BCI-*algebras*, Math. Japonica 30 (1985), 511–517.
[11] L. Popescu, N. Popescu, *Theory of Categories*, Editura Academiei Romane, Bucharest, 1979.

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