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APPLICATIONS OF CONVOLUTION PROPERTIES

Abstract. K. I. Noor (2007 Appl. Math. Comput. 188, 814–823) has defined the classes $\mathcal{Q}_k(a,b,\lambda,\gamma)$ and $\mathcal{T}_k(a,b,\lambda,\gamma)$ of analytic functions by means of linear operator connected with incomplete beta function. In this paper, we have extended some of the results and have given other properties concerning these classes.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f analytic in the open unit disc $U = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Denote by $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)(0 \le \alpha < 1)$ the subfamilies consisting of functions in $\mathcal{A}$ that are starlike of order α and convex of order α respectively. For $0 \le \gamma < 1$ and $k \ge 2$ let $\mathcal{P}_k(\gamma)$ denote the class of functions p analytic in U satisfying the conditions $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{p(z)-\gamma}{1-\gamma} \right| d\theta \le k\pi \tag{1}$$

where $z = re^{i\theta}$. The class $\mathcal{P}_k(\gamma)$ has been introduced by Padmanabhan and Parvatham (see [16]). For special choices of parameters, we obtain the known classes of functions. For example, for $k = 2$ we have the class $\mathcal{P}(\gamma)$ of functions with real part greater than γ and consequently, for $k = 2$ and $\gamma = 0$ we obtain the class of functions with positive real part. For $\gamma = 0$ we have the class $\mathcal{P}_k$ defined by Pinchuk [19]. From (1), we conclude that

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1-2\gamma)ze^{-it}}{1 - ze^{-it}} d\mu(t)$$

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where $\mu(t)$ is a function with bounded variation on $[0,2\pi]$ such that

$$\int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \le k.$$

It follows from (1) that $p \in \mathcal{P}_k(\gamma)$ can be expressed in the form

$$(2) \quad p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z), \ p_i \in \mathcal{P}(\gamma), \ i=1,2, \ z \in U.$$

For the functions f and g with the series expansions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$, the Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

This product is associative, commutative and distributive over addition and the function $\frac{1}{1-z}$ is an identity for it.

For $a>0$, $b>0$, a linear operator $\mathcal{I}_{a,b} : \mathcal{A} \longrightarrow \mathcal{A}$ is defined in [2] by

$$\mathcal{I}_{a,b} f(z) = f_{a,b}(z) * f(z)$$

where

$$(3) \qquad \frac{z}{(1-z)^a} * f_{a,b}(z) = \frac{z}{(1-z)^b}.$$

A simple computation leads to the relation

$$(4) \qquad f_{a,b}(z) = \sum_{k=0}^{\infty} \frac{(b)_k}{(a)_k} z^{k+1} = \phi(a,b;z)$$

where $(x)_k$ denotes the Pochhammer symbol defined by

$$(x)_k = \begin{cases} 1, & \text{for} \quad k=0, \quad x \in C \setminus \{0\}, \\ x(x+1)\dots(x+k-1), & \text{for } k \in N = \{1,2,3,\dots\}, \quad x \in C, \end{cases}$$

and $\phi(a,b;z)$ is the incomplete beta function connected with the hypergeometric function by the identity

$$\phi(a,b;z) = z {}_2F_1(1,b;a,z).$$

Therefore, we have immediately that $\mathcal{I}_{a,b} f = \mathcal{L}(b,a) f$ where $\mathcal{L}(b,a)$ is the well known Carlson–Shaffer operator (see [1]). As a special case, we note that for $a=1$ and $b=n+1$, we obtain

$$\mathcal{I}_{1,n+1} f(z) = \mathcal{D}^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!},$$

that is the Ruscheweyh derivative of order n. We recall here the fact that Dziok and Srivastava [5] have introduced and considered more general the Dziok–Srivastava operator

$$\mathcal{H} : \mathcal{A}_{p,k} \to \mathcal{A}_{p,k}$$

such that

$$\begin{aligned}\mathcal{H}f(z) &= \mathcal{H}(a_1, \ldots, a_q; c_1, \ldots, c_s)f(z)\\ &= [z^p \cdot {}_qF_s(a_1, \ldots, a_q; c_1, \ldots, c_s; z)] * f(z),\end{aligned}$$

where ${}_qF_s$ is given by

$${}_qF_s(a_1, \ldots, a_q; c_1, \ldots, c_s; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdot \ldots \cdot (a_q)_n}{(c_1)_n \cdot \ldots \cdot (c_s)_n} \frac{z^n}{n!} \quad (z \in \mathcal{U}),$$

and $\mathcal{A}_{p,k}$ denotes the class of functions with the series expansion

$$f(z) = z^p + \sum_{n=k}^{\infty} a_n z^n \quad (p < k; p, k \in N = \{1, 2, \ldots\}).$$

It is easy to observe that for $p = s = 1$, $q = 2$ and $a_2 = 1$, the Dziok–Srivastava operator becomes the Carlson–Shaffer operator and consequently $\mathcal{I}_{c_1,a_1}f(z) = \mathcal{H}(a_1, 1; c_1)$. Many interesting subclasses of analytic functions, associated with the Dziok–Srivastava operator, were studied recently by (for example) Srivastava et al. [8], [9], [17], [20], (see also [3], [4], [18]).

The following subclasses have been defined in [13], for $k \geq 2$, $0 \leq \lambda \leq 1$ and $0 \leq \gamma < 1$, by using the operator $\mathcal{I}_{a,b}$:

$$\mathcal{Q}_k(a, b, \lambda, \gamma) = \left\{ f \in \mathcal{A} : \left[\frac{z(\mathcal{I}_{a,b}f)' + \lambda z^2(\mathcal{I}_{a,b}f)''}{(1-\lambda)(\mathcal{I}_{a,b}f) + \lambda z(\mathcal{I}_{a,b}f)'} \right] \in \mathcal{P}_k(\gamma), \quad z \in U \right\},$$

$$\mathcal{T}_k(a, b, \lambda, \gamma) = \left\{ f \in \mathcal{A} : \left[(\mathcal{I}_{a,b}f)' + \lambda z(\mathcal{I}_{a,b}f)'' \right] \in \mathcal{P}_k(\gamma), \quad z \in U \right\}.$$

Note that

1) $\mathcal{Q}_k(a, a, 1, 0) = \mathcal{V}_k$ where $\mathcal{V}_k$ is the class of functions of bounded boundary rotation introduced by Loewner [10] and deeply examined by Paatero [14, 15].

2)$\mathcal{Q}_2(a, a, 0, \gamma) = \mathcal{S}^*(\gamma)$.

3)$\mathcal{Q}_2(a, a, 1, \gamma) = \mathcal{K}(\gamma)$.

In [13], many of interesting results concerning the classes $\mathcal{Q}_k(a, b, \lambda, \gamma)$ and $\mathcal{T}_k(a, b, \lambda, \gamma)$ have been obtained. In particular, inclusion results, covering theorem and radius problems have been studied. In this paper, we continue and extend the investigation of the paper [13].

2. Main results

In proofs of our main results, we will use the following lemmas

Lemma 2.1. ([21], p. 54) *If $f \in \mathcal{K}$, $g \in \mathcal{S}^*$, then for each analytic function h,*

$$\frac{(f * hg)(U)}{(f * g)(U)} \subset \overline{co} h(U),$$

where $\overline{co} h(U)$ denotes the closed convex hull of $h(U)$.

Lemma 2.2. [7] *Let $a > 0$. If $b \geq \max\{2, a\}$ then the function $\phi(a, b; z) \in \mathcal{K}$ for $z \in U$.*

Lemma 2.3. [13] *Let $f \in \mathcal{P}_k(\alpha)$, $g \in \mathcal{P}_k(\beta)$, for $\alpha \leq 1$, $\beta \leq 1$. Then $(f * g) \in \mathcal{P}_k(\delta)$, where $\delta = 1 - 2(1 - \alpha)(1 - \beta)$, for $z \in U$.*

Lemma 2.4. [13] *We have*

$$\mathcal{T}_k(a, b, \lambda, \gamma) \subset \mathcal{T}_k(a, b, 0, \delta)$$

where

$$\delta = \gamma + (1 - \gamma)(2\eta - 1) \quad \text{and} \quad \eta = \int_0^1 (1 + t^\lambda)^{-1} dt. \tag{5}$$

Theorem 2.1. *We have*

(i) *for $b_1 > 0$ and $b_2 \geq \max\{2, b_1\}$,*

$$\mathcal{Q}_2(a, b_2, \lambda, \gamma) \subset \mathcal{Q}_2(a, b_1, \lambda, \gamma),$$

(ii) *for $a_1 > 0$ and $a_2 \geq \max\{2, a_1\}$,*

$$\mathcal{Q}_2(a_1, b, \lambda, \gamma) \subset \mathcal{Q}_2(a_2, b, \lambda, \gamma).$$

Proof. (i) Let $f \in \mathcal{Q}_2(a, b_2, \lambda, \gamma)$ and set

$$F_i(z) = \frac{z(\mathcal{I}_{a,b_i} f)' + \lambda z^2 (\mathcal{I}_{a,b_i} f)''}{(1 - \lambda)(\mathcal{I}_{a,b_i} f) + \lambda z (\mathcal{I}_{a,b_i} f)'}, \quad i = 1, 2.$$

From the definition of the class $\mathcal{Q}_2(a, b_2, \lambda, \gamma)$, we have $F_2(z) \prec p(z) = \frac{1+(1-2\gamma)z}{1-z}$. Thus, $F_2(z) = p(\omega(z))$ where $|\omega(z)| < 1$ and $\omega(0) = 0$. Note that

$$\begin{aligned}
F_1(z) &= \frac{z(\phi(b_1, a; z) * f(z))' + \lambda z^2 (\phi(b_1, a; z) * f(z))''}{(1 - \lambda)(\phi(b_1, a; z) * f(z)) + \lambda z (\phi(b_1, a; z) * f(z))'} \\
&= \frac{z(\phi(b_1, b_2; z) * \mathcal{I}_{a,b_2} f(z))' + \lambda z^2 (\phi(b_1, b_2; z) * \mathcal{I}_{a,b_2} f(z))''}{(1 - \lambda)(\phi(b_1, b_2; z) * \mathcal{I}_{a,b_2} f(z)) + \lambda z (\phi(b_1, b_2; z) * \mathcal{I}_{a,b_2} f(z))'} \\
&= \frac{\phi(b_1, b_2; z) * \left[z(\mathcal{I}_{a,b_2} f(z))' + \lambda z^2 (\mathcal{I}_{a,b_2} f(z))''\right]}{\phi(b_1, b_2; z) * [(1 - \lambda)(\mathcal{I}_{a,b_2} f(z)) + \lambda z (\mathcal{I}_{a,b_2} f(z))']} \\
&= \frac{\phi(b_1, b_2; z) * [F_2(z) \cdot q(z)]}{\phi(b_1, b_2; z) * q(z)} = \frac{\phi(b_1, b_2; z) * [p(\omega(z)) \cdot q(z)]}{\phi(b_1, b_2; z) * q(z)},
\end{aligned}$$

where $q(z)=(1-\lambda)(\mathcal{I}_{a,b_2}f(z))+\lambda z(\mathcal{I}_{a,b_2}f(z))'$. It follows from the definition of the class $\mathcal{Q}_2(a,b,\lambda,\gamma)$ that $\mathrm{Re}\frac{zq'(z)}{q(z)}=\mathrm{Re}F_2(z)>\gamma\geq 0$ that is, q is the starlike function. It is easily seen that, by Lemma 2.2, the function $\phi(b_1,b_2;z)$ is convex function, hence, from Lemma 2.1, we have

$$\frac{\phi(b_1,b_2;z)*[p(\omega(z))\cdot q(z)]}{\phi(b_1,b_2;z)*q(z)}\subset\overline{co}p(U)\subset p(U)$$

because p is convex univalent. Therefore $F_1\prec p$ and the result follows.

(ii) Since the desired inclusion relation follows, by applying the method used in the proof of the part (i), we omit the details.

The proof of Theorem 2.1 is completed. ∎

Theorem 2.2. *We have*
(i) *for* $b_1>0$ *and* $b_2\geq\max\{2,b_1\}$

$$\mathcal{T}_k(a,b_2,\lambda,\gamma)\subset\mathcal{T}_k(a,b_1,\lambda,\gamma),$$

(ii) *for* $a_1>0$ *and* $a_2\geq\max\{2,a_1\}$

$$\mathcal{T}_k(a_1,b,\lambda,\gamma)\subset\mathcal{T}_k(a_2,b,\lambda,\gamma).$$

Proof. (i) Let us define $G_i(z)=(\mathcal{I}_{a,b_i}f)'+\lambda z(\mathcal{I}_{a,b_i}f)''$, $i=1,2$, and let $f\in\mathcal{T}_k(a,b_2,\lambda,\gamma)$. Thus $G_2\in\mathcal{P}_k(\gamma)$ or equivalently

$$G_2(z)=\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)p_2(z),$$

where $p_i\in\mathcal{P}(\gamma),\ i=1,2$. Note that

$$\begin{aligned}G_1(z)&=(\phi(b_1,a;z)*f(z))'+\lambda z\,(\phi(b_1,a;z)*f(z))''\\&=[\phi(b_1,b_2;z)*(\phi(b_2,a;z)*f(z))]'+\lambda z\,[\phi(b_1,b_2;z)*(\phi(b_2,a;z)*f(z))]''\\&=\frac{\phi(b_1,b_2;z)}{z}*G_2(z)=\frac{\phi(b_1,b_2;z)}{z}*\left[\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)p_2(z)\right]\\&=\left(\frac{k}{4}+\frac{1}{2}\right)\left(\frac{\phi(b_1,b_2;z)}{z}*p_1(z)\right)-\left(\frac{k}{4}-\frac{1}{2}\right)\left(\frac{\phi(b_1,b_2;z)}{z}*p_2(z)\right).\end{aligned}$$

By Lemma 2.2, we have that $\phi(b_1,b_2;z)\in\mathcal{K}$ so using the well known relation $f\in\mathcal{K}\Longrightarrow f\in\mathcal{S}^*\left(\frac{1}{2}\right)\Longrightarrow\mathrm{Re}\left(\frac{f(z)}{z}\right)>\frac{1}{2}$, we immediately obtain $\frac{\phi(b_1,b_2;z)}{z}\in\mathcal{P}\left(\frac{1}{2}\right)$. Therefore, from Lemma 2.3 with $k=2$, we conclude that $\frac{\phi(b_1,b_2;z)}{z}*p_i(z)\in\mathcal{P}(\delta)$ where $\delta=1-2(1-\frac{1}{2})(1-\gamma)=\gamma,\ i=1,2$, what means that $G_1\in\mathcal{P}_k(\gamma)$. The proof is thus completed.

(ii) The proof is similar to that of (i) so we omit the details. ∎

Remark 2.1. In [13], there are no results concerning inclusion relationships between the classes $\mathcal{Q}_k(a,b,\lambda,\gamma)$ and $\mathcal{T}_k(a,b,\lambda,\gamma)$ with respect to the parameter a. Thus, the results (ii) of Theorem 2.1 and (ii) of Theorem 2.2 become the essential supplement of the results of [13].

Theorem 2.3. *Let* $f \in \mathcal{T}_k(a,b,\lambda,\gamma)$, $g \in \mathcal{A}$ *and* $\mathrm{Re}\left(\frac{g(z)}{z}\right) > \frac{1}{2}$ *for* $z \in U$. *Then* $f * g \in \mathcal{T}_k(a,b,\lambda,\gamma)$.

Proof. We first suppose that $f \in \mathcal{T}_k(a,b,\lambda,\gamma)$, $g \in \mathcal{A}$ and $\mathrm{Re}\left(\frac{g(z)}{z}\right) > \frac{1}{2}$ in U. Let us define

$$H_1(z) = [\mathcal{I}_{a,b}f(z)]' + \lambda z\,[\mathcal{I}_{a,b}f(z)]''$$

and

$$H_2(z) = [\mathcal{I}_{a,b}(f*g)(z)]' + \lambda z\,[\mathcal{I}_{a,b}(f*g)(z)]''.$$

From the definition of the class $\mathcal{T}_k(a,b,\lambda,\gamma)$, we immediately get

$$H_1(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z),$$

where $p_i \in \mathcal{P}(\gamma)$, $i=1,2$. Note that

$$\begin{aligned} H_2(z) &= [\phi(b,a;z)*(f*g)(z)]' + \lambda z\,[\phi(b,a;z)*(f*g)(z)]'' \\ &= \frac{g(z)}{z} * \left[(\phi(b,a;z)*f(z))' + \lambda z(\phi(b,a;z)*f(z))''\right] = \frac{g(z)}{z} * H_1(z). \end{aligned}$$

Since $(\frac{g(z)}{z}) \in \mathcal{P}(\frac{1}{2})$ and $H_1 \in \mathcal{P}_k(\gamma)$ then putting $\alpha = \frac{1}{2}$ and $\beta = \gamma$ in Lemma 2.3, we conclude that $H_2 \in \mathcal{P}_k(\gamma)$. The proof of Theorem 2.3 is completed. ∎

Remark 2.2. We remark that the class of functions in $\mathcal{A}$ with $\mathrm{Re}\left(\frac{f(z)}{z}\right) > \frac{1}{2}$ is known to be equal to the closed convex hull of the convex functions ($\overline{co}\mathcal{K}$) ([6], p. 52). Thus, the previous theorem shows that the class $\mathcal{T}_k(a,b,\lambda,\gamma)$ is invariant under the convolution with functions of $\overline{co}\mathcal{K}$.

By applying the relation $f \in \mathcal{K} \Longrightarrow f \in \mathcal{S}^*\left(\frac{1}{2}\right) \Longrightarrow \mathrm{Re}\left(\frac{f(z)}{z}\right) > \frac{1}{2}$, we get the following

Corollary 2.1. *Let* $f \in \mathcal{T}_k(a,b,\lambda,\gamma)$. *We have*
(i) *if* $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$ *then* $(f*g) \in \mathcal{T}_k(a,b,\lambda,\gamma)$,
(ii) *if* $g \in \mathcal{K}$ *then* $(f*g) \in \mathcal{T}_k(a,b,\lambda,\gamma)$.

Remark 2.3. Let a and c be the complex numbers with $c \neq 0,-1,-2,\ldots$. We consider the function defined by

$$\Phi(a,c;z) = {}_1F_1(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{z^k}{k!}.$$

This function is called the confluent (or Kummer) hypergeometric function. If $\operatorname{Re} c > \operatorname{Re} a > 0$ then

$$\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{tz} dt,$$

where Γ denotes the Gamma function. Miller and Mocanu showed in [11] that for $a, c \in R$, $c \geq R(a)$ where

$$R(a) = \begin{cases} 2(1-a), & \text{if} \quad a < \frac{1}{4}, \\ (1-2a)^2 + \frac{5}{4}, & \text{if} \quad \frac{1}{4} \leq a < \frac{3}{4}, \\ 2a, & \text{if} \quad \frac{3}{4} \leq a, \end{cases} \tag{6}$$

the function $z\Phi(a, c; z)$ is starlike of order $\frac{1}{2}$ in U. Thus, we immediately obtain

Corollary 2.2. *If $a, c \in R$, and satisfy $c \geq R(a)$ where $R(a)$ is given by* (6) *then*

$$f \in \mathcal{T}_k(a, b, \lambda, \gamma) \Longrightarrow (z\Phi(a, c; z) * f(z)) \in \mathcal{T}_k(a, b, \lambda, \gamma).$$

Example 2.1. From the last result, we deduce that the class $\mathcal{T}_k(a, b, \lambda, \gamma)$ is invariant under convolution with the function

$$g_\delta(z) = z\Phi(1, \delta+1; z) = \delta z \int_0^1 (1-t)^{\delta-1} e^{tz} dt; \quad \delta \geq 1,$$

that is, if $f(z) = z + \sum_{k=1}^{\infty} a_{k+1} z^{k+1} \in \mathcal{T}_k(a, b, \lambda, \gamma)$ then $g(z) = z + \sum_{k=1}^{\infty} \frac{a_{k+1}}{(\delta+1)_k} z^{k+1}$ $\in \mathcal{T}_k(a, b, \lambda, \gamma)$.

Corollary 2.3. *Let $\alpha > 0$. If $\beta \geq \max\{2, \alpha\}$ then*

$$f \in \mathcal{T}_k(a, b, \lambda, \gamma) \Longrightarrow \mathcal{I}_{\beta,\alpha} f \in \mathcal{T}_k(a, b, \lambda, \gamma).$$

Proof. By applying Lemma 2.2 and the definition of $\mathcal{I}_{a,b}$, we immediately deduce desired assertion from (ii) of Corollary 2.1. ∎

Corollary 2.4. *Let, for $\mu \geq 0$*

$$F_\mu(f) = F_\mu(f)(z) = \frac{\mu+1}{z^\mu} \int_0^z t^{\mu-1} f(t) dt.$$

Then

$$f \in \mathcal{T}_k(a, b, \lambda, \gamma) \Longrightarrow F_\mu(f) \in \mathcal{T}_k(a, b, \lambda, \gamma).$$

Proof. It is sufficient to note that $F_\mu(f) = \mathcal{L}(\mu+1, \mu+2)f$. Thus the desired implication follows directly from Corollary 2.3. ∎

Corollary 2.5. *Let f has the series expansion $f(z) = \sum\limits_{k=2}^{\infty} a_k z^k$ and let*

$$s_n(z) := z + \sum_{k=2}^{n} a_k z^k \quad (n \in N \setminus \{1\}).$$

Then

$$f \in \mathcal{T}_k(a,b,\lambda,\gamma) \Longrightarrow \frac{1}{r_n} s_n(r_n z) \in \mathcal{T}_k(a,b,\lambda,\gamma)$$

where

$$\text{(7)} \qquad r_n = \sup\left\{ r : \operatorname{Re}\Big(\sum_{k=0}^{n-1} z^k\Big) > \frac{1}{2}, \quad (|z| < 1) \right\}, \quad (n \in N \setminus \{1\}).$$

Proof. Let $f \in \mathcal{T}_k(a,b,\lambda,\gamma)$. Putting $h_n(z) = \sum\limits_{k=1}^{n} z^k$, we can write $s_n(z) = (f * h_n)(z)$. Thus from (7), we get $\operatorname{Re}\left(\frac{h_n(r_n z)}{r_n z}\right) > \frac{1}{2}$, for $z \in U$, $n \in N \setminus \{1\}$. Hence, by applying Theorem 2.3, we have

$$\frac{1}{r_n} s_n(r_n z) = \frac{1}{r_n}(f * h_n)(r_n z) = f(z) * \frac{1}{r_n} h_n(r_n z) \in \mathcal{T}_k(a,b,\lambda,\gamma).$$

The proof is thus completed. ∎

Theorem 2.4. *Let $f_1 \in \mathcal{T}_2(a,b,\lambda,\gamma_1)$ and $f_2 \in \mathcal{T}_k(a,b,\lambda,\gamma_2)$, $(0 \le \gamma_i < 1,\ i = 1,2)$. If $f \in \mathcal{A}$ is defined by*

$$\mathcal{I}_{a,b} f(z) = \int_0^z (\mathcal{I}_{a,b} f_1(z))' * (\mathcal{I}_{a,b} f_2(z))' \, dt$$

then $f \in \mathcal{T}_k(a,b,\lambda,\kappa)$ where $\kappa = 1 - 8(1-\eta)^2(1-\gamma_1)(1-\gamma_2)$ and η is given by (5).

Proof. Let $f_1 \in \mathcal{T}_2(a,b,\lambda,\gamma_1)$ and $f_2 \in \mathcal{T}_k(a,b,\lambda,\gamma_2)$. It follows from Lemma 2.4 that $(\mathcal{I}_{a,b} f_1(z))' \in \mathcal{P}_2(\delta_1)$ and $(\mathcal{I}_{a,b} f_2(z))' \in \mathcal{P}_k(\delta_2)$ where $\delta_i = \gamma_i + (1-\gamma_i)(2\eta - 1)$ and the parameter η is given by (5). Using the definition of the class $\mathcal{P}_k(\gamma)$, we thus obtain that

$$(\mathcal{I}_{a,b} f_1(z))' = p(z), \quad p \in \mathcal{P}(\delta_1)$$

and

$$(\mathcal{I}_{a,b} f_2(z))' = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad p_i \in \mathcal{P}(\delta_2), \quad i = 1,2,$$

which leads to

$$\begin{aligned}(\mathcal{I}_{a,b}f(z))' &= p(z) * \left[\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z)\right] \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)(p(z)*p_1(z)) - \left(\frac{k}{4}-\frac{1}{2}\right)(p(z)*p_2(z)).\end{aligned}$$

Applying Lemma 2.3, we deduce that $(\mathcal{I}_{a,b}f(z))' \in \mathcal{P}_k(\kappa)$ where

$$\begin{aligned}\kappa &= 1-2(1-\delta_1)(1-\delta_2) \\ &= 1-2[1-\gamma_1-(1-\gamma_1)(2\eta-1)]\cdot[1-\gamma_2-(1-\gamma_2)(2\eta-1)] \\ &= 1-8(1-\eta)^2(1-\gamma_1)(1-\gamma_2)\end{aligned}$$

where the parameter η is given in (5). The proof of theorem is thus completed. ∎

Theorem 2.5. *For a fixed number n, $n \in N$, let*

$$f(z) = z + \sum_{k=1}^{\infty} a_{kn+1} z^{kn+1}. \tag{8}$$

If $f \in \mathcal{T}_k(a,b,0,\gamma)$, then $f \in \mathcal{T}_k(a,b,\lambda,\gamma)$, for $|z| < r_n$, where $r_n = \left(\sqrt{1+(\lambda n)^2} - \lambda n\right)^{\frac{1}{n}}$. The result is best possible.

Proof. Under the hypothesis that f of the form (8) is in $\mathcal{T}_k(a,b,0,\gamma)$, we have

$$(\mathcal{I}_{a,b}f(z))' \in \mathcal{P}_k(\gamma)$$

and consequently

$$(\mathcal{I}_{a,b}f(z))' = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z), \tag{9}$$

where $p_i \in \mathcal{P}(\gamma),\ i = 1,2$. Let us put

$$F(z) = [\mathcal{I}_{a,b}f(z)]' + \lambda z\,[\mathcal{I}_{a,b}f(z)]''.$$

We can easily prove that

$$F(z) = [\mathcal{I}_{a,b}f(z)]' * \frac{g_n(z)}{z} \tag{10}$$

where

$$g_n(z) = (1-n\lambda)\frac{z}{1-z^n} + n\lambda\frac{z}{(1-z^n)^2} = z + \sum_{k=1}^{\infty}(1+kn\lambda)z^{kn+1}.$$

Let us set $z^n = 1-\rho e^{i\theta}$, $\quad(\rho > 0)$ and $|z| = r < 1$. It was shown in [22] that

$$\mathrm{Re}\frac{g_n(z)}{z} \geq \frac{1}{2} + \frac{1}{2\rho^2}(1-2n\lambda r^n - r^{2n}).$$

Thus, for $|z| < r_n = \left(\sqrt{1+(\lambda n)^2} - \lambda n\right)^{\frac{1}{n}}$, we have $\frac{g_n(z)}{z} \in \mathcal{P}(\frac{1}{2})$. It follows from (9) and (10) that

$$F(z) = \left(\frac{k}{4} + \frac{1}{2}\right)\left(p_1(z) * \frac{g_n(z)}{z}\right) - \left(\frac{k}{4} - \frac{1}{2}\right)\left(p_2(z) * \frac{g_n(z)}{z}\right),$$

where $p_i \in \mathcal{P}(\gamma),\ i = 1, 2$. Hence, an application of Lemma 2.3 leads to the result $F \in \mathcal{P}_k(\gamma)$ and consequently, $f \in \mathcal{T}_k(a, b, \lambda, \gamma)$ for $|z| < r_n$. The proof is completed. ∎

If $n = 1$ then from Theorem 2.5, we deduce

Corollary 2.6. *Let* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. *If* $f \in \mathcal{T}_k(a, b, 0, \gamma)$, *then* $f \in \mathcal{T}_k(a, b, \lambda, \gamma)$, *for* $|z| < \sqrt{1+\lambda^2} - \lambda$.

Remark 2.4. In [13], the following result was proved (Theorem 4.5): Let $(\mathcal{I}_{a,b} f(z))' \in \mathcal{P}_k(\gamma)$. Then $f \in \mathcal{T}_k(a, b, \lambda, \gamma)$ for $|z| < r_\lambda$ where

$$r_\lambda = \frac{1}{2\lambda + \sqrt{4\lambda^2 - 2\lambda + 1}}; \quad \lambda \neq \frac{1}{2}.$$

The result is best possible.

In the light of the Corollary 2.6, the previous result of [13] seems to be not the best possible because, for example, for $\lambda = 1$, we have

$$\sqrt{1+\lambda^2} - \lambda > \frac{1}{2\lambda + \sqrt{4\lambda^2 - 2\lambda + 1}}.$$

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