DEMONSTRATIO MATHEMATICA
Vol. XLVI No 3 2013

10.1515/dema-2013-0476

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DIFFERENTIAL SUBORDINATIONS AND SUPERORDINATIONS FOR p-VALENT FUNCTIONS DEFINED BY FRACTIONAL DERIVATIVE OPERATOR

Abstract. In the present paper, we derive some subordination and superordination results for p-valent functions in the open unit disk by using certain fractional derivative operator. Relevant connections of the results, which are presented in the paper, with various known results are also considered.

1. Introduction and preliminaries

Let $\mathcal{H}(\mathcal{U})$ denote the class of analytic functions in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and let $\mathcal{H}[a,p]$ denote the subclass of the functions $f \in \mathcal{H}(\mathcal{U})$ of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \dots, \quad (a \in \mathbb{C},\ p \in \mathbf{N}).$$

Also, let $\mathcal{A}(p)$ be the class of functions $f \in \mathcal{H}(\mathcal{U})$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad p \in \mathbf{N} \tag{1.1}$$

and set $\mathcal{A} \equiv \mathcal{A}(1)$.

Let $f, g \in \mathcal{H}(\mathcal{U})$, we say that the function f is subordinate to g, if there exist a Schwarz function w, analytic in $\mathcal{U}$, with $w(0) = 0$ and $|w(z)| < 1\,(z \in \mathcal{U})$, such that $f(z) = g(w(z))$, for all $z \in \mathcal{U}$.

This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$. It is well known that, if the function g is univalent in $\mathcal{U}$ then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Let $p(z), h(z) \in \mathcal{H}(\mathcal{U})$, and let $\Phi(r,s,t;z) : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$. If $p(z)$ and $\Phi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions, and if $p(z)$ satisfies the

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2010 *Mathematics Subject Classification*: 30C45, 26A33.
Key words and phrases: p-valent function, differential subordination, differential superordination, fractional derivative operators.

second-order superordination

$$h(z) \prec \Phi(p(z), zp'(z), z^2p''(z); z) \tag{1.2}$$

then $p(z)$ is said to be a solution of the differential superordination (1.2). (If $f(z)$ is subordinatnate to $g(z)$, then $g(z)$ is called to be superordinate to $f(z)$). An analytic function $q(z)$ is called a subordinant if $q(z) \prec p(z)$, for all $p(z)$ satisfies (1.2). An univalent subordinant $\tilde{q}(z)$, that satisfies $q(z) \prec \tilde{q}(z)$, for all subordinants $q(z)$ of (1.2), is said to be the best subordinant.

Recently, Miller and Mocanu [6] obtained conditions on $h(z), q(z)$ and Φ for which the following implication holds true:

$$h(z) \prec \Phi(p(z), zp'(z), z^2p''(z); z) \Longrightarrow q(z) \prec p(z)$$

with the results of Miller and Mocanu [6], Bulboaca [3] invesetegated certain classes of first order differential superordinations as well as superordination-preserving integral operators [4]. Ali et al. [2] used the results obtained by Bulboaca [4] and gave the sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$. Shanmugam et al. [11] obtained sufficient conditions for normalized analytic functions to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z) \quad \text{and} \quad q_1(z) \prec \frac{z^2f'(z)}{(f(z))^2} \prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$.

Let ${}_2F_1(a, b; c; z)$ be the Gauss hypergeometric function defined for $z \in \mathcal{U}$ by (see Srivastava and Karlsson [13])

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \tag{1.3}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{when } n = 0, \\ \lambda(\lambda + 1)(\lambda + 2) \ldots (\lambda + n - 1), & \text{when } n \in \mathbf{N}, \end{cases} \tag{1.4}$$

for $\lambda \neq 0, -1, -2, \ldots$

We recall the following definitions of fractional derivative operators which were used by Owa [9], (see also [10]) as follows:

Definition 1.1. The fractional derivative operator of order λ is defined by

$$D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\lambda} d\xi, \tag{1.5}$$

where $0 \le \lambda < 1$, $f(z)$ is analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 1.2. Let $0 \le \lambda < 1$, and $\mu, \eta \in \mathbf{R}$. Then, in terms of the familiar Gauss's hypergeometric function ${}_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is

$$\begin{aligned} &J_{0,z}^{\lambda,\mu,\eta} f(z) \\ &= \frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-\xi)^{-\lambda} f(\xi)\, {}_2F_1\left(\mu-\lambda, 1-\eta; 1-\lambda; 1-\frac{\xi}{z}\right) d\xi \right), \end{aligned} \tag{1.6}$$

where $f(z)$ is analytic function in a simply-connected region of the z-plane containing the origin, with the order $f(z) = O(|z|^\varepsilon)$, $z \to 0$, where $\varepsilon > \max\{0, \mu-\eta\} - 1$ and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 1.3. Under the hypotheses of Definition 1.2, the fractional derivative operator $J_{0,z}^{\lambda+m,\mu+m,\eta+m}$ of a function $f(z)$ is defined by

$$J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z) = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\mu,\eta} f(z). \tag{1.7}$$

Notice that

$$J_{0,z}^{\lambda,\lambda,\eta} f(z) = D_z^\lambda f(z), \quad 0 \le \lambda < 1. \tag{1.8}$$

With the aid of the above definitions, we define a modification of the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}$ by

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\mu)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^\mu J_{0,z}^{\lambda,\mu,\eta} f(z), \tag{1.9}$$

for $f(z) \in \mathcal{A}(p)$ and $\lambda \ge 0; \mu < p+1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbf{N}$. Then it is observed that $M_{0,z}^{\lambda,\mu,\eta} f(z)$ maps $\mathcal{A}(p)$ onto itself as follows:

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = z^p + \sum_{n=1}^{\infty} \delta_n(\lambda,\mu,\eta,p) a_{p+n} z^{p+n} \tag{1.10}$$

where

$$\delta_n(\lambda,\mu,\eta,p) = \frac{(p+1)_n (p+1-\mu+\eta)_n}{(p+1-\mu)_n (p+1-\lambda+\eta)_n}. \tag{1.11}$$

It is easily verified from (1.10) that

$$z\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)' = (p-\mu)M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z) + \mu M_{0,z}^{\lambda,\mu,\eta}f(z). \tag{1.12}$$

Notice that

$$M_{0,z}^{0,0,\eta}f(z) = f(z) \quad \text{and} \quad M_{0,z}^{1,1,\eta}f(z) = \frac{zf'(z)}{p}.$$

The object of this paper is to derive several subordination results involving the fractional derivative operator. Furthermore, we obtain the previous results of Aouf et al. [1], Obradovic et al. [7], Obradovic and Owa [8], and Srivastava and Lashin [12] as special cases of some of the results presented here.

In order to prove our results, we mention to the following known results which shall be used in the sequel.

Lemma 1.4. [10] *Let $\lambda, \mu, \eta \in \mathbf{R}$, such that $\lambda \geq 0$ and $K > \max\{0, \mu-\eta\}-1$. Then*

$$J_{0,z}^{\lambda,\mu,\eta}z^k = \frac{\Gamma(k+1)\Gamma(k-\mu+\eta+1)}{\Gamma(k-\mu+1)\Gamma(k-\lambda+\eta+1)}z^{k-\mu}. \tag{1.13}$$

Definition 1.5. [6] Denote by Q the set of all functions f that are analytic and injective in $\bar{\mathcal{U}} - E(f)$, where

$$E(f) = \{\xi \in \partial\mathcal{U} : \lim_{z\to\infty} f(z) = \infty\}$$

and are such that $f'(\xi) \neq 0$ for $\xi \in \partial\mathcal{U} - E(f)$.

Lemma 1.6. [5] *Let the function q be univalent in the open unit disk $\mathcal{U}$, and θ and φ be analytic in a domain D containing $q(\mathcal{U})$ with $\varphi(w) \neq 0$ when $w \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\varphi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

(1) *Q is starlike univalent in $\mathcal{U}$, and*
(2) $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0$, *for $z \in \mathcal{U}$.*

If

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z))$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 1.7. [3] *Let the function q be univalent in the open unit disk $\mathcal{U}$, and θ and φ be analytic in a domain D containing $q(\mathcal{U})$ with $\varphi(w) \neq 0$ when $w \in q(\mathcal{U})$. Suppose that*

(1) $\operatorname{Re}\left(\frac{\theta'(q(z))}{\varphi(q(z))}\right) > 0$, *for $z \in \mathcal{U}$,*
(2) *$zq'(z)\varphi(q(z))$ is starlike univalent in $\mathcal{U}$.*

If $p(z) \in \mathcal{H}[q(0),1] \cap Q$ *with* $p(\mathcal{U}) \subseteq D$, *and* $\theta(p(z)) + zp'(z)\varphi(p(z))$ *is univalent in* $\mathcal{U}$, *and*

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z))$$

then $q(z) \prec p(z)$ *and* q *is the best subordinant.*

2. Subordination and superordination for p-valent functions

We begin with the following result involving differential subordination between analytic functions.

Theorem 2.1. *Let* $\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\gamma} \in \mathcal{H}(\mathcal{U})$ *and let the function* $q(z)$ *be analytic and univalent in* $\mathcal{U}$ *such that* $q(z) \neq 0, (z \in \mathcal{U})$. *Suppose that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* $\mathcal{U}$. *Let*

$$\operatorname{Re}\left\{1 + \frac{\xi}{\beta}q(z) + \frac{2\delta}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0, \tag{2.1}$$

$$(\alpha, \delta, \xi, \beta \in \mathbb{C}; \beta \neq 0)$$

and $f(z) \in \mathcal{A}(p)$, *and*

$$\Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z) = \alpha + \xi\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\gamma} + \delta\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{2\gamma} + \beta\gamma(p-\mu)\left[\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} - 1\right]. \tag{2.2}$$

If q *satisfies the following subordination:*

$$\Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z) \prec \alpha + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)},$$

$(\lambda \geq 0; \mu < p+1; \eta > \max(\lambda,\mu) - p - 1; p \in \mathbf{N}; \alpha,\delta,\xi,\gamma,\beta \in \mathbb{C}; \gamma \neq 0; \beta \neq 0)$

then

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\gamma} \prec q(z), \quad (\gamma \in \mathbb{C}\backslash\{0\}) \tag{2.3}$$

and q *is the best dominant.*

Proof. Let the function $p(z)$ be defined by

$$p(z) = \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\gamma}, \quad (z \in \mathcal{U}\backslash\{0\}; \gamma \in \mathbb{C}\backslash\{0\}).$$

So that, by a straightforward computation, we have

$$\frac{zp'(z)}{p(z)} = \gamma\left[\frac{z(M_{0,z}^{\lambda,\mu,\eta}f(z))'}{M_{0,z}^{\lambda,\mu,\eta}f(z)} - p\right].$$

By using the identity (1.12), we obtain

$$\frac{zp'(z)}{p(z)} = \gamma\left[(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} - (p-\mu)\right].$$

By setting $\theta(w) = \alpha + \xi w + \delta w^2$ and $\varphi(w) = \frac{\beta}{w}$, it can be easily verified that θ is analytic in $\mathbb{C}$, φ is analytic in $\mathbb{C}\backslash\{0\}$, and that $\varphi(w) \neq 0$ $(w \in \mathbb{C}\backslash\{0\})$. Also, by letting

$$Q(z) = zq'(z)\varphi(q(z)) = \beta\frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \alpha + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)},$$

we find that $Q(z)$ is starlike univalent in $\mathcal{U}$ and that

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{1 + \frac{\xi}{\beta}q(z) + \frac{2\delta}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0.$$

The assertion (2.3) of Theorem 2.1 now follows by an application of Lemma 1.6. ∎

Remark 1. For the choices $q(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$ and $q(z) = \left(\frac{1+z}{1-z}\right)^{\varepsilon}, 0 < \varepsilon \leq 1$, in Theorem 2.1, we get the following results (Corollary 2.2 and Corollary 2.3) below.

Corollary 2.2. *Assume that* (2.1) *holds. If* $f \in \mathcal{A}(p)$ *and*

$$\Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z) \prec \alpha + \xi\frac{1+Az}{1+Bz} + \delta\left(\frac{1+Az}{1+Bz}\right)^2 + \frac{\beta(A-B)z}{(1+Az)(1+Bz)},$$

$(\lambda \geq 0; \mu < p+1; \eta > \max(\lambda,\mu) - p - 1; p \in \mathbf{N}; \alpha,\delta,\xi,\gamma,\beta \in \mathbb{C}; \gamma \neq 0; \beta \neq 0)$ *where* $\Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z)$ *is as defined in* (2.2)*, then*

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p}\right)^{\gamma} \prec \frac{1+Az}{1+Bz}, \quad (\gamma \in \mathbb{C}\backslash\{0\})$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Corollary 2.3. *Assume that* (2.1) *holds. If* $f \in \mathcal{A}(p)$ *and*

$$\Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z) \prec \alpha + \xi\left(\frac{1+z}{1-z}\right)^{\varepsilon} + \delta\left(\frac{1+z}{1-z}\right)^{2\varepsilon} + \frac{2\beta\varepsilon z}{1-z^2},$$

$(\lambda \geq 0; \mu < p+1; \eta > \max(\lambda, \mu)-p-1; p \in \mathbf{N}; \alpha, \delta, \xi, \gamma, \beta \in \mathbb{C}; \gamma \neq 0; \beta \neq 0)$ *where* $\Psi_{\lambda,\mu,\eta}(\gamma, \xi, \beta, \delta, f)(z)$ *is as defined in* (2.2), *then*

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\gamma} \prec \left(\frac{1+z}{1-z}\right)^{\varepsilon}, \quad (\gamma \in \mathbb{C}\backslash\{0\})$$

and $\left(\frac{1+z}{1-z}\right)^{\varepsilon}$ *is the best dominant.*

Remark 2. For the choice $q(z) = \frac{1}{(1-z)^{2ab}}$, $(a, b \in \mathbb{C}\backslash\{0\}); \lambda = \mu = \delta = \xi = 0; p = \alpha = 1; \gamma = a$ and $\beta = \frac{1}{ab}$ in Theorem 2.1, we obtain the following known result due to Obradovic et al. [7].

Corollary 2.4. *Let* $a, b \in \mathbb{C}\backslash\{0\}$ *such that* $|2ab-1| \leq 1$ *or* $|2ab+1| \leq 1$. *If* $f \in \mathcal{A}$ *and*

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1+z}{1-z}$$

then

$$\left(\frac{f(z)}{z}\right)^{a} \prec \frac{1}{(1-z)^{2ab}}$$

and $\frac{1}{(1-z)^{2ab}}$ *is the best dominant.*

Remark 3. For $a = 1$, Corollary 2.4 reduces to the recent result of Srivastava and Lashin [12].

Remark 4. For the choice $q(z) = (1+Bz)^{\frac{\gamma(A-B)}{B}}$; $\lambda = \mu = \delta = \xi = 0$; $p = \alpha = 1$ and $\beta = 1$ in Theorem 2.1, we obtain the following known result due to Obradovic and Owa [8].

Corollary 2.5. *Let* $-1 \leq A < B \leq 1$ *with* $B \neq 0$ *and suppose that* $|\frac{\gamma(A-B)}{B} - 1| \leq 1$ *or* $|\frac{\gamma(A-B)}{B} + 1| \leq 1, \gamma \in \mathbb{C}\backslash\{0\}$. *If* $f \in \mathcal{A}$ *and*

$$1 + \gamma\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1+[B+\gamma(A-B)]z}{1+Bz}$$

then

$$\left(\frac{f(z)}{z}\right)^{\gamma} \prec (1+Bz)^{\frac{\gamma(A-B)}{B}}$$

and $(1+Bz)^{\frac{\gamma(A-B)}{B}}$ *is the best dominant.*

Remark 5. For the choice $q(z) = \frac{1}{(1-z)^{2abe^{-i\tau}\cos\tau}}$, $(b \in \mathbb{C}\backslash\{0\}); |\tau| < \pi/2; \lambda = \mu = \delta = \xi = 0; p = \alpha = 1; \gamma = a$ and $\beta = \frac{e^{i\tau}}{ab\cos\tau}$ in Theorem 2.1, we get the following result due to Aouf et al. [1].

Corollary 2.6. *Let* $a, b \in \mathbb{C}\backslash\{0\}$ *and* $|\tau| < \pi/2$, *and suppose that* $|2abe^{-i\tau}\cos\tau - 1| \leq 1$ *or* $|2abe^{-i\tau}\cos\tau + 1| \leq 1$. *Let* $f \in \mathcal{A}$, *and suppose* *that* $\frac{f(z)}{z} \neq 0$, *for all* $z \in \mathcal{U}$. *If*

$$1 + \frac{e^{i\tau}}{b \cos \tau} \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{1+z}{1-z}$$

then

$$\left(\frac{f(z)}{z} \right)^{a} \prec \frac{1}{(1-z)^{2abe^{-i\tau} \cos \tau}}$$

and $\frac{1}{(1-z)^{2abe^{-i\tau} \cos \tau}}$ *is the best dominant.*

Next, by appealing to Lemma 1.7 from the preceding section, we prove the following.

Theorem 2.7. *Let* q *be analytic and univalent in* $\mathcal{U}$ *such that* $q(z) \neq 0$ *and* $\frac{zq'(z)}{q(z)}$ *be starlike univalent in* $\mathcal{U}$. *Further, let us assume that*

$$\text{Re} \left\{ \frac{2\delta}{\beta} (q(z))^2 + \frac{\xi}{\beta} q(z) \right\} > 0, \quad (\delta, \xi, \beta \in \mathbb{C}; \beta \neq 0). \tag{2.4}$$

If $f(z) \in \mathcal{A}(p)$,

$$0 \neq \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p} \right)^{\gamma} \in \mathcal{H}[q(0), 1] \cap Q$$

and $\Psi_{\lambda,\mu,\eta}(\gamma, \xi, \beta, \delta, f)(z)$ *is univalent in* $\mathcal{U}$, *then*

$$\alpha + \xi q(z) + \delta (q(z))^2 + \beta \frac{zq'(z)}{q(z)} \prec \Psi_{\lambda,\mu,\eta}(\gamma, \xi, \beta, \delta, f)(z),$$

$(\lambda \geq 0; \mu < p+1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbf{N}; \alpha, \delta, \xi, \gamma, \beta \in \mathbb{C}; \gamma \neq 0; \beta \neq 0)$

implies

$$q(z) \prec \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p} \right)^{\gamma}, \quad (\gamma \in \mathbb{C} \backslash \{0\}) \tag{2.5}$$

and q *is the best subordinant where* $\Psi_{\lambda,\mu,\eta}(\gamma, \xi, \beta, \delta, f)(z)$ *is as defined in* (2.2).

Proof. By setting $\theta(w) = \alpha + \xi w + \delta w^2$ and $\varphi(w) = \frac{\beta}{w}$, it can be easily observed that θ is analytic in $\mathbb{C}$, φ is analytic in $\mathbb{C} \backslash \{0\}$, and that $\varphi(w) \neq 0 \; (w \in \mathbb{C} \backslash \{0\})$. Since q is convex (univalent) function it follows that,

$$\text{Re} \left\{ \frac{\theta'(q(z))}{\varphi(q(z))} \right\} = \text{Re} \left\{ \frac{2\delta}{\beta} (q(z))^2 + \frac{\xi}{\beta} q(z) \right\} > 0, \quad (\delta, \xi, \beta \in \mathbb{C}; \beta \neq 0).$$

The assertion (2.5) of Theorem 2.7 now follows by an application of Lemma 1.7. ∎

Combining Theorem 2.1 and Theorem 2.7, we get the following sandwich theorem.

Theorem 2.8. *Let q_1 and q_2 be univalent in $\mathcal{U}$ such that $q_1(z) \neq 0$ and $q_2(z) \neq 0, (z \in \mathcal{U})$ with $\frac{zq_1'(z)}{q_1(z)}$ and $\frac{zq_2'(z)}{q_2(z)}$ being starlike univalent. Suppose that q_1 satisfies (2.4) and q_2 satisfies (2.1). If $f(z) \in \mathcal{A}(p)$,*

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\gamma} \in \mathcal{H}[q(0), 1] \cap Q$$

and $\Psi_{\lambda,\mu,\eta}(\gamma, \xi, \beta, \delta, f)(z)$ is univalent in $\mathcal{U}$, then

$$\alpha + \xi q_1(z) + \delta(q_1(z))^2 + \beta \frac{zq_1'(z)}{q_1(z)} \prec \Psi_{\lambda,\mu,\eta}(\gamma, \xi, \beta, \delta, f)(z)$$

$$\prec \alpha + \xi q_2(z) + \delta(q_2(z))^2 + \beta \frac{zq_2'(z)}{q_2(z)},$$

$(\lambda \geq 0; \mu < p+1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbf{N}; \alpha, \delta, \xi, \gamma, \beta \in \mathbb{C}; \gamma \neq 0; \beta \neq 0)$ *implies*

$$q_1(z) \prec \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\gamma} \prec q_2(z), \quad (\gamma \in \mathbb{C}\backslash\{0\}) \tag{2.6}$$

and q_1 and q_2 are respectively the best subordinant and the best dominant where $\Psi_{\lambda,\mu,\eta}(\gamma, \xi, \beta, \delta, f)(z)$ is as defined in (2.2).

Remark 6. For $\lambda = \mu = 0$ in Theorem 2.8, we get the following result.

Theorem 2.9. *Let q_1 and q_2 be univalent in $\mathcal{U}$ such that $q_1(z) \neq 0$ and $q_2(z) \neq 0, (z \in \mathcal{U})$ with $\frac{zq_1'(z)}{q_1(z)}$ and $\frac{zq_2'(z)}{q_2(z)}$ being starlike univalent. Suppose that q_1 satisfies (2.4) and q_2 satisfies (2.1). If $f(z) \in \mathcal{A}(p)$,*

$$\left(\frac{f(z)}{z^p}\right)^{\gamma} \in \mathcal{H}[q(0), 1] \cap Q$$

and let

$$\Psi_1(\gamma, \xi, \beta, \delta, f)(z) = \alpha + \xi \left(\frac{f(z)}{z^p}\right)^{\gamma} + \delta \left(\frac{f(z)}{z^p}\right)^{2\gamma} + \beta\gamma \left(\frac{zf'(z)}{f(z)} - p\right)$$

is univalent in $\mathcal{U}$, then

$$\alpha + \xi q_1(z) + \delta(q_1(z))^2 + \beta \frac{zq_1'(z)}{q_1(z)} \prec \Psi_1(\gamma, \xi, \beta, \delta, f)(z)$$

$$\prec \alpha + \xi q_2(z) + \delta(q_2(z))^2 + \beta \frac{zq_2'(z)}{q_2(z)},$$

$$(p \in \mathbf{N}; \alpha, \delta, \xi, \beta \in \mathbb{C}; \gamma \neq 0; \beta \neq 0)$$

implies

$$q_1(z) \prec \left(\frac{f(z)}{z^p}\right)^\gamma \prec q_2(z) \quad (\gamma \in \mathbb{C}\backslash\{0\}) \tag{2.7}$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

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Received June 6, 2011.