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# $L_P$-GENERAL APPROXIMATIONS BY MULTIVARIATE SINGULAR INTEGRAL OPERATORS

**Abstract.** In this article, we study the $L_p$, $1 \leq p < \infty$ approximation properties of general multivariate singular integral operators over $R^N$, $N \geq 1$. We establish their convergence to the unit operator with rates. The established inequalities involve the multivariate higher order modulus of smoothness. We list the multivariate Picard, Gauss–Weierstrass, Poisson Cauchy and trigonometric singular integral operators where this theory can be applied directly.

## 1. Introduction

The rate of $L_p$, $1 \leq p < \infty$ convergence of univariate singular integral operators has been studied earlier in [1], [6], [7], [8], see also the related [10], [11], and these articles motivate the current work. Here, we consider some very general multivariate singular integral operators over $\mathbb{R}^N$, $N \geq 1$, and we study the degree of approximation to the unit operator with rates over smooth functions. The established related inequalities are involving the multivariate higher modulus of smoothness with respect to $\|\cdot\|_p$. See Theorems 4, 6, 8, 10. We mention particular operators that fulfill our theory. The discussed linear operators are not in general positive, see [2]. Other motivation comes from [3], [4].

## 2. Main results

Here $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, we define

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, \ldots, r, \\ 1 - \sum\limits_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 0, \end{cases} \tag{1}$$

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and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, \ldots, m \in \mathbb{N}. \tag{2}$$

See that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1, \tag{3}$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}. \tag{4}$$

Let $\mu_{\xi_n}$ be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$, $\xi_n > 0$, $n \in \mathbb{N}$.

We now define the multiple smooth singular integral operators

$$\begin{aligned} (5) \quad & \theta_{r,n}^{[m]} (f; x_1, \ldots, x_N) \\ & := \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f (x_1 + s_1 j, x_2 + s_2 j, \ldots, x_N + s_N j) \, d\mu_{\xi_n} (s), \end{aligned}$$

where $s := (s_1, \ldots, s_N)$, $x := (x_1, \ldots, x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function, and also $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers.

The operators $\theta_{r,n}^{[m]}$ preserve constants, see [2].

Here, we deal with $f \in C^m \left(\mathbb{R}^N\right)$, $m \in \mathbb{Z}^+$, with $f_\alpha \in L_p \left(\mathbb{R}^N\right)$, $|\alpha| = m \in \mathbb{Z}^+$, $p \geq 1$; where $f_\alpha$ denotes the mixed partial $\frac{\partial^{\widetilde{j}} f(\cdot,\ldots,\cdot)}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}$, $\alpha_j \in \mathbb{Z}^+$, $j = 1, \ldots, N : |\alpha| := \sum_{j=1}^{N} \alpha_j = \widetilde{j}$, $\widetilde{j} = 1, \ldots, m$.

We need

**Definition 1.** (see also [5]) We call

$$\begin{aligned} (6) \quad & \Delta_u^r f (x) := \Delta_{u_1, u_2, \ldots, u_N}^r f (x_1, \ldots, x_N) \\ & := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f (x_1 + j u_1, x_2 + j u_2, \ldots, x_N + j u_N). \end{aligned}$$

Let $p \geq 1$, the modulus of smoothness of order $r$ is given by

$$\omega_r (f; h)_p := \sup_{\|u\|_2 \leq h} \|\Delta_u^r (f)\|_p, \quad h > 0. \tag{7}$$

I) First we consider the case of $m \in \mathbb{N}$, $p > 1$.

We make

**Remark 2.** For $\widetilde{j} = 1, \ldots, m$, and $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = \widetilde{j}$, under the assumption of Theorem 4,

$$c_{\alpha,n,\widetilde{j}} := c_{\alpha,n} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} d\mu_{\xi_n}(s_1, \ldots, s_N) \in \mathbb{R}, \tag{8}$$

see also [2]. From [2], we get

$$E_{r,n}^{[m]}(x) := \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left( \sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n} f_\alpha(x)}{\prod_{i=1}^{N} \alpha_i!} \right) \tag{9}$$

$$= m \sum_{|\alpha|=m} \left( \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \left( \int_{\mathbb{R}^N} \left( \prod_{i=1}^{N} s_i^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left( \Delta_{\theta s}^{r} f_\alpha(x) \right) d\theta \right) d\mu_{\xi_n}(s) \right)$$

$$=: R_{r,n}^{[m]}(x), \quad \forall\, x \in \mathbb{R}^N. \tag{10}$$

Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left| E_{r,n}^{[m]}(x) \right|^p = \left| R_{r,n}^{[m]}(x) \right|^p$$

(set $c_1 := \left( m \sum_{|\alpha|=m} \left( \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \right)^p$)

$$= c_1 \left| \int_{\mathbb{R}^N} \left( \prod_{i=1}^{N} s_i^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left( \Delta_{\theta s}^{r} f_\alpha(x) \right) d\theta \right) d\mu_{\xi_n}(s) \right|^p \tag{11}$$

$$\leq c_1 \left( \int_{\mathbb{R}^N} \left( \prod_{i=1}^{N} |s_i|^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left| \Delta_{\theta s}^{r} f_\alpha(x) \right| d\theta \right) d\mu_{\xi_n}(s) \right)^p.$$

Hence we have

$$I_1 := \int_{\mathbb{R}^N} \left| E_{r,n}^{[m]}(x) \right|^p dx \tag{12}$$

$$\leq c_1 \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \left( \prod_{i=1}^{N} |s_i|^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left| \Delta_{\theta s}^{r} f_\alpha(x) \right| d\theta \right) d\mu_{\xi_n}(s) \right)^p dx$$

(call $0 \leq \gamma(s,x) := \prod_{i=1}^{N} |s_i|^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left| \Delta_{\theta s}^{r} f_\alpha(x) \right| d\theta$)

$$= c_1 \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \gamma(s,x)\, d\mu_{\xi_n}(s) \right)^p dx =: I_2. \tag{13}$$

Therefore, it holds

$$I_2 \le c_1 \int\limits_{\mathbb{R}^N} \left( \int\limits_{\mathbb{R}^N} \gamma^p (s,x)\, d\mu_{\xi_n}(s) \right) dx =: I_3. \tag{14}$$

But we have

$$\gamma(s,x) \le \prod_{i=1}^{N} |s_i|^{\alpha_i} \left( \int\limits_0^1 \left( (1-\theta)^{m-1} \right)^q d\theta \right)^{\frac{1}{q}} \left( \int\limits_0^1 \left| \Delta_{\theta s}^r f_\alpha (x) \right|^p d\theta \right)^{\frac{1}{p}} \tag{15}$$

$$= \prod_{i=1}^{N} |s_i|^{\alpha_i} \frac{1}{(q(m-1)+1)^{\frac{1}{q}}} \left( \int\limits_0^1 \left| \Delta_{\theta s}^r f_\alpha (x) \right|^p d\theta \right)^{\frac{1}{p}}.$$

Hence, we get

$$\gamma^p(s,x) \le \prod_{i=1}^{N} |s_i|^{\alpha_i p} \frac{1}{(q(m-1)+1)^{\frac{p}{q}}} \left( \int\limits_0^1 \left| \Delta_{\theta s}^r f_\alpha (x) \right|^p d\theta \right). \tag{16}$$

Thus, we obtain

$$I_3 \le c_2 \int\limits_{\mathbb{R}^N} \left( \int\limits_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i p} \left( \int\limits_0^1 \left| \Delta_{\theta s}^r f_\alpha (x) \right|^p d\theta \right) d\mu_{\xi_n}(s) \right) dx \tag{17}$$

(set $c_2 := c_1 \cdot \frac{1}{(q(m-1)+1)^{\frac{p}{q}}}$)

$$= c_2 \int\limits_{\mathbb{R}^N} \left( \int\limits_0^1 \left( \int\limits_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i p} \left| \Delta_{\theta s}^r f_\alpha (x) \right|^p dx \right) d\theta \right) d\mu_{\xi_n}(s)$$

$$= c_2 \int\limits_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i p} \left( \int\limits_0^1 \left( \int\limits_{\mathbb{R}^N} \left| \Delta_{\theta s}^r f_\alpha (x) \right|^p dx \right) d\theta \right) d\mu_{\xi_n}(s) =: I_4. \tag{18}$$

Consequently, we derive

$$I_4 \le c_2 \int\limits_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i p} \left( \int\limits_0^1 \left( \omega_r \left( f_\alpha; \theta \|s\|_2 \right)_p \right)^p d\theta \right) d\mu_{\xi_n}(s) \tag{19}$$

$$\le c_2 \int\limits_{\mathbb{R}^N} \left( \prod_{i=1}^{N} |s_i|^{\alpha_i p} \right) \omega_r \left( f_\alpha; \|s\|_2 \right)_p^p d\mu_{\xi_n}(s)$$

$$= c_2 \int\limits_{\mathbb{R}^N} \left( \prod_{i=1}^{N} |s_i|^{\alpha_i p} \right) \omega_r \left( f_\alpha; \xi_n \frac{\|s\|_2}{\xi_n} \right)_p^p d\mu_{\xi_n}(s)$$

(by $\omega_r(f, \lambda h)_p \leq (1+\lambda)^r \omega_r(f, h)_p$, for any $h, \lambda > 0$, $p \geq 1$)

$$\leq c_2 \omega_r(f_\alpha; \xi_n)_p^p \left( \int_{\mathbb{R}^N} \left( \prod_{i=1}^N |s_i|^{\alpha_i p} \right) \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^{rp} d\mu_{\xi_n}(s) \right). \tag{20}$$

We have proved that

$$\int_{\mathbb{R}^N} \left| E_{r,n}^{[m]}(x) \right|^p dx \leq \left[ \left( \sum_{|\alpha|=m} \left( \frac{1}{\prod_{i=1}^N \alpha_i!} \right) \right) \cdot \left( \frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \right]^p \cdot \omega_r(f_\alpha; \xi_n)_p^p \left( \int_{\mathbb{R}^N} \left[ \left( \prod_{i=1}^N |s_i|^{\alpha_i} \right) \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p d\mu_{\xi_n}(s) \right). \tag{21}$$

Thus, we get ($p > 1$)

$$\left\| E_{r,n}^{[m]}(x) \right\|_p \leq \left( \frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left( \sum_{|\alpha|=m} \left( \frac{1}{\prod_{i=1}^N \alpha_i!} \right) \right) \cdot \left[ \int_{\mathbb{R}^N} \left[ \left( \prod_{i=1}^N |s_i|^{\alpha_i} \right) \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p d\mu_{\xi_n}(s) \right]^{\frac{1}{p}} \omega_r(f_\alpha; \xi_n)_p. \tag{22}$$

We make

**Remark 3.** Notice that ($p > 1$)

$$\int_{\mathbb{R}^N} \left( \prod_{i=1}^N |s_i|^{\alpha_i} \right) \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \leq \left[ \int_{\mathbb{R}^N} \left( \left( \prod_{i=1}^N |s_i|^{\alpha_i} \right) \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) \right]^{\frac{1}{p}} < \infty, \tag{23}$$

by assumption of Theorem 4.

As in [2] then we get that

$$\int_{\mathbb{R}^N} \prod_{i=1}^N |s_i|^{\alpha_i} d\mu_{\xi_n}(s) < \infty. \tag{24}$$

Hence $c_{\alpha,n,\widetilde{j}} \in \mathbb{R}$.

From the above we have proved

**Theorem 4.** *Let $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, $N \geq 1$, with $f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Here $\mu_{\xi_n}$ is a Borel probability measure on $\mathbb{R}^N$, for $\xi_n > 0$, $(\xi_n)_{n \in \mathbb{N}}$ bounded sequence. Assume, for all*

$\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = m$ *that we have*

$$\int_{\mathbb{R}^N} \left( \left( \prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) < \infty. \tag{25}$$

*For* $\widetilde{j} = 1, \ldots, m$, *and* $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = \widetilde{j}$, *call*

$$c_{\alpha,n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} d\mu_{\xi_n}(s). \tag{26}$$

*Then*

$$\begin{aligned} \left\| E_{r,n}^{[m]} \right\|_p &= \left\| \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left( \sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n,\widetilde{j}} f_\alpha(x)}{\left(\prod_{i=1}^{N} \alpha_i!\right)} \right) \right\|_{p,x} \\ &\leq \left( \frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left( \sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \\ &\cdot \left[ \int_{\mathbb{R}^N} \left[ \left( \prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p d\mu_{\xi_n}(s) \right]^{\frac{1}{p}} \omega_r(f_\alpha, \xi_n)_p. \end{aligned} \tag{27}$$

*As* $n \to \infty$ *and* $\xi_n \to 0$, *by* (27), *we obtain that* $\left\| E_{r,n}^{[m]} \right\|_p \to 0$ *with rates.*

*One also gets by* (27) *that*

$$\left\| \theta_{r,n}^{[m]}(f;x) - f(x) \right\|_{p,x} \leq \sum_{\widetilde{j}=1}^{m} \left| \delta_{\widetilde{j},r}^{[m]} \right| \left( \sum_{|\alpha|=\widetilde{j}} \frac{\left| c_{\alpha,n,\widetilde{j}} \right|}{\prod_{i=1}^{N} \alpha_i!} \|f_\alpha\|_p \right) + R.H.S.(27),$$

*given that* $\|f_\alpha\|_p < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, \ldots, m$.

*Assuming that* $c_{\alpha,n,\widetilde{j}} \to 0$, $\xi_n \to 0$, *as* $n \to \infty$, *we get* $\left\| \theta_{r,n}^{[m]}(f) - f \right\|_p \to 0$, *that is* $\theta_{r,n}^{[m]} \to I$ *the unit operator, in* $L_p$ *norm, with rates.*

II) Case of $m = 0$, $p > 1$.

We make

**Remark 5.** In [2] we proved that

$$\theta_{r,n}^{[0]}(f;x) - f(x) = \int_{\mathbb{R}^N} (\Delta_s^r f(x))\, d\mu_{\xi_n}(s). \tag{28}$$

Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Hence

$$\left|\theta_{r,n}^{[0]}(f;x) - f(x)\right|^p \leq \left(\int_{\mathbb{R}^N} |\Delta_s^r f(x)| \, d\mu_{\xi_n}(s)\right)^p \leq \int_{\mathbb{R}^N} |\Delta_s^r f(x)|^p \, d\mu_{\xi_n}(s). \tag{29}$$

And it holds

$$\int_{\mathbb{R}^N} \left|\theta_{r,n}^{[0]}(f;x) - f(x)\right|^p dx \leq \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |\Delta_s^r f(x)|^p \, d\mu_{\xi_n}(s)\right) dx \tag{30}$$

$$= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |\Delta_s^r f(x)|^p \, dx\right) d\mu_{\xi_n}(s) \leq \int_{\mathbb{R}^N} \omega_r\left(f, \|s\|_2\right)_p^p d\mu_{\xi_n}(s) \tag{31}$$

$$= \int_{\mathbb{R}^N} \omega_r\left(f, \xi_n \frac{\|s\|_2}{\xi_n}\right)_p^p d\mu_{\xi_n}(s) \leq \omega_r(f, \xi_n)_p^p \int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^{rp} d\mu_{\xi_n}(s). \tag{32}$$

Therefore, we get

$$\left\|\theta_{r,n}^{[0]}(f) - f\right\|_p \leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^{rp} d\mu_{\xi_n}(s)\right)^{\frac{1}{p}} \omega_r(f, \xi_n)_p. \tag{33}$$

We proved

**Theorem 6.** *Let $f \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$; $N \geq 1$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $\mu_{\xi_n}$ probability Borel measures on $\mathbb{R}^N$, $(\xi_n)_{n\in\mathbb{N}} > 0$ and bounded. Also suppose*

$$\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^{rp} d\mu_{\xi_n}(s) < \infty. \tag{34}$$

*Then*

$$\left\|\theta_{r,n}^{[0]}(f) - f\right\|_p \leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^{rp} d\mu_{\xi_n}(s)\right)^{\frac{1}{p}} \omega_r(f, \xi_n)_p. \tag{35}$$

*As $\xi_n \to 0$, when $n \to \infty$, we obtain $\left\|\theta_{r,n}^{[0]}(f) - f\right\|_p \to 0$, i.e. $\theta_{r,n}^{[0]} \to I$, the unit operator, in $L_p$ norm.*

III) Next follows the case $m = 0$, $p = 1$.

We make

**Remark 7.** As before we have

$$\left|\theta_{r,n}^{[0]}(f;x)-f(x)\right|\leq\int_{\mathbb{R}^N}\left|\Delta_s^r f(x)\right|d\mu_{\xi_n}(s). \tag{36}$$

Hence

$$\left\|\theta_{r,n}^{[0]}(f)-f\right\|_1=\int_{\mathbb{R}^N}\left|\theta_{r,n}^{[0]}(f;x)-f(x)\right|dx \tag{37}$$

$$\leq\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\left|\Delta_s^r f(x)\right|d\mu_{\xi_n}(s)\right)dx \tag{38}$$

$$=\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\left|\Delta_s^r f(x)\right|dx\right)d\mu_{\xi_n}(s)\leq\int_{\mathbb{R}^N}\omega_r\left(f,\|s\|_2\right)_1 d\mu_{\xi_n}(s)$$

$$=\int_{\mathbb{R}^N}\omega_r\left(f,\xi_n\frac{\|s\|_2}{\xi_n}\right)_1 d\mu_{\xi_n}(s)\leq\omega_r(f,\xi_n)_1\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s). \tag{39}$$

We have proved

**Theorem 8.** *Let $f\in\left(C\left(\mathbb{R}^N\right)\cap L_1\left(\mathbb{R}^N\right)\right)$, $N\geq 1$. Assume, $\mu_{\xi_n}$ probability Borel measures on $\mathbb{R}^N$, $(\xi_n)_{n\in\mathbb{N}}>0$ and bounded. Also suppose*

$$\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s)<\infty. \tag{40}$$

*Then*

$$\left\|\theta_{r,n}^{[0]}(f)-f\right\|_1\leq\left(\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s)\right)\omega_r(f,\xi_n)_1. \tag{41}$$

*As $\xi_n\to 0$, we get $\theta_{r,n}^{[0]}\to I$ in $L_1$ norm.*

IV) Case of $m\in\mathbb{N}$, $p=1$.
We make

**Remark 9.** We have

$$\left\|E_{r,n}^{[m]}\right\|_1=\int_{\mathbb{R}^N}\left|E_{r,n}^{[m]}(x)\right|dx\leq m\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^N\alpha_i!}\right) \tag{42}$$

$$\cdot\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\left(\prod_{i=1}^N|s_i|^{\alpha_i}\int_0^1(1-\theta)^{m-1}\left|\Delta_{\theta s}^r f_\alpha(x)\right|d\theta\right)d\mu_{\xi_n}(s)\right)dx$$

(set $\overline{c_1} := m \sum\limits_{|\alpha|=m} \left( \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right)$)

$$= \overline{c_1} \int\limits_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i} \left( \int\limits_0^1 (1-\theta)^{m-1} \left( \int\limits_{\mathbb{R}^N} \left| \Delta_{\theta s}^r f_\alpha (x) \right| dx \right) d\theta \right) d\mu_{\xi_n}(s)$$

$$\le \overline{c_1} \int\limits_{\mathbb{R}^N} \left( \prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left( \int\limits_0^1 (1-\theta)^{m-1} \omega_r \left( f_\alpha, \theta \|s\|_2 \right)_1 d\theta \right) d\mu_{\xi_n}(s) \tag{43}$$

$$\le \frac{\overline{c_1}}{m} \int\limits_{\mathbb{R}^N} \left( \prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \omega_r \left( f_\alpha, \|s\|_2 \right)_1 d\mu_{\xi_n}(s)$$

$$= \sum_{|\alpha|=m} \left( \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \int\limits_{\mathbb{R}^N} \left( \prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \omega_r \left( f_\alpha, \xi_n \frac{\|s\|_2}{\xi_n} \right)_1 d\mu_{\xi_n}(s) \tag{44}$$

$$\le \sum_{|\alpha|=m} \left( \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \omega_r \left( f_\alpha, \xi_n \right)_1 \tag{45}$$

$$\cdot \int\limits_{\mathbb{R}^N} \left( \prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s).$$

We have proved

**Theorem 10.** *Let $f \in C^m\left(\mathbb{R}^N\right)$, $m, N \in \mathbb{N}$, with $f_\alpha \in L_1\left(\mathbb{R}^N\right)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Here $\mu_{\xi_n}$ is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ is a bounded sequence. Assume, for all $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum\limits_{i=1}^{N} \alpha_i = m$ that we have*

$$\int\limits_{\mathbb{R}^N} \left( \left( \prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^r \right) d\mu_{\xi_n}(s) < \infty. \tag{46}$$

*For $\widetilde{j} = 1, \ldots, m$, and $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum\limits_{i=1}^{N} \alpha_i = \widetilde{j}$, call*

$$c_{\alpha, n, \widetilde{j}} := \int\limits_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} d\mu_{\xi_n}(s). \tag{47}$$

*Then*

$$\left\| E_{r,n}^{[m]} \right\|_1 = \left\| \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left( \sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n,\widetilde{j}} f_\alpha(x)}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_{1,x} \tag{48}$$

$$\leq \sum_{|\alpha|=m} \left( \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \omega_r(f_\alpha, \xi_n)_1 \int_{\mathbb{R}^N} \left( \prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s).$$

*As* $\xi_n \to 0$, *we get* $\left\| E_{r,n}^{[m]} \right\|_1 \to 0$ *with rates. From* (48), *we get*

$$\left\| \theta_{r,n}^{[m]} f - f \right\|_1 \leq \sum_{\widetilde{j}=1}^{m} \left| \delta_{\widetilde{j},r}^{[m]} \right| \left( \sum_{|\alpha|=\widetilde{j}} \frac{\left| c_{\alpha,n,\widetilde{j}} \right|}{\prod_{i=1}^{N} \alpha_i!} \|f_\alpha\|_1 \right) + R.H.S.(48),$$

*given that* $\|f_\alpha\|_1 < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, \dots, m$.

*As* $n \to \infty$, *assuming* $\xi_n \to 0$ *and* $c_{\alpha,n,\widetilde{j}} \to 0$, *we get* $\left\| \theta_{r,n}^{[m]} f - f \right\|_1 \to 0$, *that is* $\theta_{r,n}^{[m]} \to I$ *in* $L_1$ *norm, with rates.*

## 3. Applications

Let all entities as in Section 2. We define the following specific operators:

i) The general multivariate Picard singular integral operators:

$$P_{r,n}^{[m]}(f;x_1,\dots,x_N) := \frac{1}{(2\xi_n)^N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \tag{49}$$

$$\cdot \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, \dots, x_N + s_N j)\, e^{-\frac{\left(\sum_{i=1}^{N} |s_i|\right)}{\xi_n}} ds_1 \dots ds_N.$$

Notice that

$$\frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} e^{-\frac{\left(\sum_{i=1}^{N} |s_i|\right)}{\xi_n}} ds_1 \dots ds_N = 1, \tag{50}$$

see [1].

ii) The general multivariate Gauss–Weierstrass singular integral operators:

$$W_{r,n}^{[m]}(f;x_1,\dots,x_N) := \frac{1}{\left(\sqrt{\pi \xi_n}\right)^N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \tag{51}$$

$$\cdot \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, \dots, x_N + s_N j)\, e^{-\frac{\left(\sum_{i=1}^{N} s_i^2\right)}{\xi_n}} ds_1 \dots ds_N.$$

Notice that

$$\frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\int\limits_{\mathbb{R}^N} e^{-\frac{\left(\sum_{i=1}^N s_i^2\right)}{\xi_n}} ds_1 \dots ds_N = 1, \tag{52}$$

see [6].

iii) The general multivariate Poisson–Cauchy singular integral operators:

$$U_{r,n}^{[m]}\left(f; x_1, \dots, x_N\right) := W_n^N \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \tag{53}$$

$$\cdot \int\limits_{\mathbb{R}^N} f\left(x_1 + s_1 j, \dots, x_N + s_N j\right) \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^\beta} ds_1 \dots ds_N,$$

with $\alpha \in \mathbb{N}$, $\beta > \frac{1}{2\alpha}$, and

$$W_n := \frac{\Gamma(\beta)\,\alpha\xi_n^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta - \frac{1}{2\alpha}\right)}, \tag{54}$$

see [7].

Notice that

$$W_n^N \int\limits_{\mathbb{R}^N} \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^\beta} ds_1 \dots ds_N = 1, \tag{55}$$

see [7], [12], p. 397, formula 595.

iv) The general multivariate trigonometric singular integral operators:

$$T_{r,n}^{[m]}\left(f; x_1, \dots, x_N\right) := \lambda_n^{-N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \tag{56}$$

$$\cdot \int\limits_{\mathbb{R}^N} f\left(x_1 + s_1 j, \dots, x_N + s_N j\right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1 \dots ds_N,$$

where $\beta \in \mathbb{N}$, and

$$\lambda_n := 2\xi_n^{1-2\beta}\pi\left(-1\right)^\beta \beta \sum_{k=1}^{\beta} \left(-1\right)^k \frac{k^{2\beta-1}}{(\beta-k)!\,(\beta+k)!}, \tag{57}$$

see [8], [9], p. 210, item 1033.

Notice that

$$\lambda_n^{-N} \int\limits_{\mathbb{R}^N} \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1 \dots ds_N = 1, \tag{58}$$

see also [8], [9], p. 210, item 1033.

One can apply Theorems 4, 6, 8, 10 to operators $P_{r,n}^{[m]}$, $W_{r,n}^{[m]}$, $U_{r,n}^{[m]}$, $T_{r,n}^{[m]}$ (special cases of $\theta_{r,n}^{[m]}$) and derive interesting results. We intend to do that in a future article.

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