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INEQUALITIES FOR Γ_P FUNCTION AND GAMMA FUNCTION

Abstract. In this paper, we present a double inequality for the gamma function by estimating bounds of Γ_p function. Later, we also give a new inequality of gamma function.

1. Introduction

It is well known that the classical Euler gamma function may be defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad x > 0. \tag{1.1}$$

Later, Euler gave another equivalent definition for the $\Gamma(x)$ (see [1])

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\cdots(x+p)} = \frac{p^x}{x(1+\frac{x}{1})\cdots(1+\frac{x}{p})}, \tag{1.2}$$

where $\lim_{p\to\infty}\Gamma_p(x) = \Gamma(x)$. It is common knowledge that these functions are fundamental and have much extensive applications in mathematical science. In the past, several authors proved many remarkable inequalities for $\Gamma(x)$. In 1997, G. D. Anderson and S. L. Qiu [2] presented the following upper and lower bounds for $\Gamma(x)$:

$$x^{(1-C)x-1} < \Gamma(x) < x^{x-1} \quad x > 1. \tag{1.3}$$

Actually, the authors proved more. Next, H. Alzer proved a companion of (1.3) in [3]. He showed that if $x \in (1,\infty)$, then

$$x^{\alpha(x-1)-C} < \Gamma(x) < x^{\beta(x-1)-C} \tag{1.4}$$

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was valid with the best possible constants $\alpha = \frac{(\frac{\pi^2}{6}-C)}{2}$ and $\beta = 1$. This improved the bounds given in (1.3). Moreover, he showed that if $x \in (0,1)$, then (1.4) held with the best possible constants $\alpha = 1 - C$ and $\beta = \frac{(\frac{\pi^2}{6}-C)}{2}$. For potential availability to interested readers, we list the collection as references of this paper [4]–[8]. The aim of this article is to establish a new inequality for the gamma function by estimating bound of Γ_p. Finally, we also give an interesting inequality involving gamma function.

2. Main results

Lemma 2.1. *Let $x \in (0,1)$, then*

$$\frac{(x+n)^{x+n}}{x^x n^n e} < \frac{(x+n)(x+n-1)\cdots(x+1)}{n!} < \frac{(x+n)^{x+n}}{x^x n^n}. \tag{2.1}$$

Proof. The function

$$f(x) = (x+n)\ln(x+n) - x\ln x - n\ln n + \ln n! - \sum_{i=1}^{n} \ln(x+i) \tag{2.2}$$

is correctly defined. Simple computation yields

$$f'(x) = \ln(x+n) - \ln x - \sum_{i=1}^{n} \frac{1}{x+i} \tag{2.3}$$

and

$$\begin{aligned} f''(x) &= \frac{1}{x+n} - \frac{1}{x} + \sum_{i=1}^{n} \frac{1}{(x+i)^2} \\ &< \frac{1}{x+n} - \frac{1}{x} + \sum_{i=1}^{n} \frac{1}{(x+i-1)(x+i)} = 0. \end{aligned} \tag{2.4}$$

So the function $f'(x)$ is strictly decreasing on $(0,+\infty)$. Since $f'(\infty) = 0$, hence, $f'(x) > f'(\infty) = 0$. As the function $f(x)$ is strictly increasing on $(0,1)$, we have

$$0 = f(0) < f(x) < f(1) = n\ln(1+\frac{1}{n}) < 1. \tag{2.5}$$

The proof of Lemma 2.1 is complete. ∎

Lemma 2.2. *Let $x \in (0,1)$, then*

$$\frac{x^{x-1}x^{p+x}}{(x+p)^{x+p}} < \Gamma_p(x) < \frac{x^{x-1}x^{p+x}e}{(x+p)^{x+p}}. \tag{2.6}$$

Proof. Using Lemma 2.1 and (1.2), we easily obtain (2.6). ∎

Lemma 2.3. [9, p. 390, 3.6.48] *Let $x_i \in R^+, i = 1, 2 \cdots n$ and $\sum_{i=1}^{n} x_i = nx$, then*

$$\prod_{i=1}^{n} \Gamma(x_i) \geq (\Gamma(x))^n. \tag{2.7}$$

Theorem 2.1. *Let $x \in (0, 1)$, then*

$$x^{x-1}e^{-x} < \Gamma(x) < x^{x-1}e^{1-x}. \tag{2.8}$$

Proof. Let $p \to +\infty$ on the both parts of the inequality (2.6). Using the equality of limit $\lim_{p\to\infty} (\frac{x}{x+p})^{x+p} = e^{-x}$, we have the inequality (2.8). ∎

Theorem 2.2. *Let $x_i, y_i, z_i, \omega_i \in R^+, i = 1, 2 \cdots n, \alpha > 0, \beta > 0$ such that*

$$\sum_{i=1}^{n} x_i = nx, \sum_{i=1}^{n} y_i = ny, \sum_{i=1}^{n} \omega_i = n\omega,$$

$$\Gamma(z_i) \geq \Gamma(\omega_i), \sum_{i=1}^{n} \Gamma(z_i) = n\Gamma^*(z).$$

Then

$$\sum_{i=1}^{n} \frac{(\Gamma(x_i) + \Gamma(y_i))^\alpha}{(\Gamma(z_i) - \Gamma(\omega_i))^\beta} \geq n\frac{(\Gamma(x) + \Gamma(y))^\alpha}{(\Gamma^*(z) - \Gamma(\omega))^\beta}. \tag{2.9}$$

Proof. First, we prove the following inequality

$$\frac{\sqrt[n]{\prod_{i=1}^{n} (\Gamma(x_i) + \Gamma(y_i))^\alpha}}{\sqrt[n]{\prod_{i=1}^{n} (\Gamma(z_i) - \Gamma(\omega_i))^\beta}} \geq \frac{\left(\sqrt[n]{\prod_{i=1}^{n} \Gamma(x_i)} + \sqrt[n]{\prod_{i=1}^{n} \Gamma(y_i)}\right)^\alpha}{\left(\sqrt[n]{\prod_{i=1}^{n} \Gamma(z_i)} - \sqrt[n]{\prod_{i=1}^{n} \Gamma(\omega_i)}\right)^\beta}. \tag{2.10}$$

In fact, the inequality (2.10) is equivalent to the following two inequalities:

$$\sqrt[n]{\prod_{i=1}^{n} (\Gamma(x_i) + \Gamma(y_i))^\alpha} \geq \left(\sqrt[n]{\prod_{i=1}^{n} \Gamma(x_i)} + \sqrt[n]{\prod_{i=1}^{n} \Gamma(y_i)}\right)^\alpha \tag{2.11}$$

and

$$\sqrt[n]{\prod_{i=1}^{n}(\Gamma(z_i)-\Gamma(\omega_i))^{\beta}} \le \left(\sqrt[n]{\prod_{i=1}^{n}\Gamma(z_i)} - \sqrt[n]{\prod_{i=1}^{n}\Gamma(\omega_i)}\right)^{\beta}. \tag{2.12}$$

It is easily known that

$$\begin{aligned}(2.11) \Leftrightarrow \sqrt[n]{\prod_{i=1}^{n}(\Gamma(x_i)+\Gamma(y_i))} &\ge \sqrt[n]{\prod_{i=1}^{n}\Gamma(x_i)} + \sqrt[n]{\prod_{k=1}^{n}\Gamma(y_i)} \\ \Leftrightarrow 1 &\ge \sqrt[n]{\frac{\prod\limits_{i=1}^{n}\Gamma(x_i)}{\prod\limits_{i=1}^{n}(\Gamma(x_i)+\Gamma(y_i))}} + \sqrt[n]{\frac{\prod\limits_{i=1}^{n}\Gamma(y_i)}{\prod\limits_{i=1}^{n}(\Gamma(x_i)+\Gamma(y_i))}}.\end{aligned} \tag{2.13}$$

Using the AGM inequality, we easily obtain

$$\sqrt[n]{\frac{\prod\limits_{i=1}^{n}\Gamma(x_i)}{\prod\limits_{i=1}^{n}(\Gamma(x_i)+\Gamma(y_i))}} \le \frac{\sum\limits_{i=1}^{n}\frac{\Gamma(x_i)}{\Gamma(x_i)+\Gamma(y_i)}}{n} \tag{2.14}$$

and

$$\sqrt[n]{\frac{\prod\limits_{i=1}^{n}\Gamma(y_i)}{\prod\limits_{i=1}^{n}(\Gamma(x_i)+\Gamma(y_i))}} \le \frac{\sum\limits_{i=1}^{n}\frac{\Gamma(y_i)}{\Gamma(x_i)+\Gamma(y_i)}}{n}. \tag{2.15}$$

Adding by sides inequalities (2.14) to (2.15), we obtain (2.13). Considering condition $\Gamma(z_i) \ge \Gamma(\omega_i)$,

$$\begin{aligned}(2.12) \Leftrightarrow \sqrt[n]{\prod_{i=1}^{n}(\Gamma(z_i)-\Gamma(\omega_i))} &\le \sqrt[n]{\prod_{i=1}^{n}\Gamma(z_i)} - \sqrt[n]{\prod_{k=1}^{n}\Gamma(\omega_i)} \\ \Leftrightarrow \sqrt[n]{\prod_{i=1}^{n}(1-\frac{\Gamma(\omega_i)}{\Gamma(z_i)})} &\le 1 - \sqrt[n]{\prod_{i=1}^{n}\frac{\Gamma(\omega_i)}{\Gamma(z_i)}}.\end{aligned} \tag{2.16}$$

Then, using the AGM inequality, we easily obtain

$$\sqrt[n]{\prod_{i=1}^{n}\left(1-\frac{\Gamma(\omega_i)}{\Gamma(z_i)}\right)} \leq \left(\frac{\sum_{i=1}^{n}(1-\frac{\Gamma(\omega_i)}{\Gamma(z_i)})}{n}\right)^n = \left(\frac{n-\sum_{i=1}^{n}\frac{\Gamma(\omega_i)}{\Gamma(z_i)}}{n}\right)^n \tag{2.17}$$

$$\leq \left(\frac{n-n\sqrt[n]{\prod_{i=1}^{n}\frac{\Gamma(\omega_i)}{\Gamma(z_i)}}}{n}\right)^n = \left(1-\sqrt[n]{\prod_{i=1}^{n}\frac{\Gamma(\omega_i)}{\Gamma(z_i)}}\right)^n.$$

Finally, using the inequality (2.10) and Lemma 2.3, we have

$$\sum_{i=1}^{n}\frac{(\Gamma(x_i)+\Gamma(y_i))^\alpha}{(\Gamma(z_i)-\Gamma(\omega_i))^\beta} \geq n\frac{\sqrt[n]{\prod_{i=1}^{n}(\Gamma(x_i)+\Gamma(y_i))^\alpha}}{\sqrt[n]{\prod_{i=1}^{n}(\Gamma(z_i)-\Gamma(\omega_i))^\beta}}$$

$$\geq n\frac{\left(\sqrt[n]{\prod_{i=1}^{n}\Gamma(x_i)}+\sqrt[n]{\prod_{i=1}^{n}\Gamma(y_i)}\right)^\alpha}{\left(\sqrt[n]{\prod_{i=1}^{n}\Gamma(z_i)}-\sqrt[n]{\prod_{i=1}^{n}\Gamma(\omega_i)}\right)^\beta}$$

$$\geq n\frac{(\Gamma(x)+\Gamma(y))^\alpha}{(\frac{\sum_{i=1}^{n}\Gamma(z_k)}{n}-\Gamma(\omega))^\beta} = n\frac{(\Gamma(x)+\Gamma(y))^\alpha}{(\Gamma^*(z)-\Gamma(\omega))^\beta}.$$

The proof is complete. ■

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References

[1] V. Krasniqi, F. Merovci, *Inequalities and monotonicity for the ratio of Γ_p functions,* International Journal of Open Problems in Computer Science and Mathematics 3(1) (2010), 1–8.

[2] G. D. Anderson, S. L. Qiu, *A monotonicity property of the gamma function,* Proc. Amer. Math. Soc. 125 (1997), 3355–3362.

[3] H. Alzer, *Inequalities for the gamma function,* Proc. Amer. Math. Soc. 128(1) (1999), 141–147.

[4] H. Alzer, *On some inequalities for the gamma function and psi function,* Math. Comp. 66 (1997), 373–389.
[5] P. Ivády, *A note on a gamma function inequality,* J. Math. Inequal. 3(2) (2009), 227–236.
[6] F. Qi, B. N. Guo, *An elegant refinement of a double inequality for the gamma function,* available online at http://arxiv.org/abs/0902.2509.6.
[7] B. N. Guo, F. Qi, *Two new proofs of the complete monotonicity of a function involving the psi function,* Bull. Korean Math. Soc. 47(1) (2010), 103–111.
[8] H. Alzer, N. Batir, *Monotonicity properties of the gamma function,* Appl. Math. Lett. 20(7) (2007), 778–781.
[9] D. S. Mitrinovic, *Analytic Inequalities,* Springer-Verlag, New York, 1970.

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