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HERMITE–HADAMARD TYPE INEQUALITIES VIA m AND (α, m)-CONVEXITY

Abstract. In this paper, two new integral inequalities of Hadamard-type for product of m and (α, m)- convex functions and their applications for special means are given.

1. Introduction

The following definition is well known in the literature: A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds, for all $x, y \in I$ and $t \in [0,1]$. Geometrically, this means that if P, Q and R are three distinct points on the graph of f with Q between P and R, then Q is on or below chord PR. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as Hadamard's inequality for convex mapping. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave. The inequalities (1.1), which have many uses in a variety of settings, are of the cornerstones in mathematical analysis and optimization. New proofs, extensions, and considering its refinements, generalizations, numerous interpolations and applications in the theory of special means and information theory have been provided by many reports. See [1–7] for some results on generalizations, extensions and applications of the Hermite–Hadamard inequalities.

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Moreover, in [7], Toader introduced the class of m-convex functions as the following:

Definition 1. The function $f:[0,b]\to\mathbb{R}$ is said to be m-convex, where $m\in[0,1]$, if for every $x,y\in[0,b]$ and $t\in[0,1]$, we have

$$f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y). \tag{1.2}$$

Moreover, in [4], Miheşan introduced the class of (α,m)-convex functions as the following:

Definition 2. The function $f:[0,b]\to\mathbb{R}$ is said to be (α,m)-convex, where $(\alpha,m)\in[0,1]^2$, if for every $x,y\in[0,b]$ and $t\in[0,1]$, we have

$$f(tx+m(1-t)y)\leq t^{\alpha}f(x)+m(1-t^{\alpha})f(y). \tag{1.3}$$

Remark 1. Note that for $(\alpha,m)\in\{(0,0),(\alpha,0),(1,0),(1,m),(1,1),(\alpha,1)\}$ one obtains the following classes of functions: increasing, α-starshaped, starshaped, m-convex, convex and α-convex.

In [6], Pachpatte established two new Hadamard-type inequalities for products of convex functions.

Theorem 1. [6] *Let $f,g:[a,b]\to[0,\infty)$ be convex functions on $[a,b]\subset\mathbb{R}$, $a<b$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx\leq\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b) \tag{1.4}$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b) \tag{1.5}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

Up to today, there are many reports on the two convex functions, two s-convex functions, two m-convex functions or on the product of the s-convex function with an ordinary convex function. In the present study, in addition to previous literature, the some new inequalities on the product of classes of convex functions which are different from each other and from standard convex function will be obtained, and some applications in the special means for the obtained inequalities will be provided. The main purpose of this study is to establish new inequalities as in the theorem given above, but now we do aiming the product of different kinds of convex functions.

2. Main result

Theorem 2. *Let $a,b\in[0,\infty)$, $a<b$ and let $f,g:[a,b]\to\mathbb{R}$ be non-negative integrable functions, either increasing or decreasing synchronously*

and $f, g, fg \in L_1([a,b])$. *If* f, g *are* (α_1, m_1), (α_2, m_2)*-convex on* $[a,b]$ *for* (α_1, m_1), $(\alpha_2, m_2) \in [0,1] \times (\frac{a}{b}, 1]$, *then*

$$\frac{1}{m_1 b - a}\int_a^{m_1 b} f(x)dx.\frac{1}{m_2 b - a}\int_a^{m_2 b} g(x)dx \le \frac{1}{b-a}\int_a^b f(x)g(x)dx \tag{2.1}$$
$$\le Sf(a)g(a) + Ef(a)g(b) + Mf(b)g(a) + Af(b)g(b)$$

where $S = \frac{1}{\alpha_1+\alpha_2+1}$, $E = \frac{m_2\alpha_2}{(\alpha_1+\alpha_2+1)(\alpha_1+1)}$, $M = \frac{m_1\alpha_1}{(\alpha_1+\alpha_2+1)(\alpha_2+1)}$, $A = \frac{m_1 m_2 \alpha_1 \alpha_2(\alpha_1+\alpha_2+2)}{(\alpha_1+\alpha_2+1)(\alpha_1+1)(\alpha_2+1)}$.

Proof. Since f, g are (α_1, m_1), (α_2, m_2)-convex on $[a,b]$ for (α_1, m_1), $(\alpha_2, m_2) \in [0,1] \times (\frac{a}{b}, 1]$ and as they are the functions either increasing or decreasing synchronously, we have,

$$f(ta + m_1(1-t)b) \le t^{\alpha_1} f(a) + m_1(1-t^{\alpha_1})f(b),$$
$$g(ta + m_2(1-t)b) \le t^{\alpha_2} g(a) + m_2(1-t^{\alpha_2})g(b).$$

Multiplying the inequalities above on either side (i.e. from left to left and right to right), we get

$$\begin{aligned} &f(ta + m_1(1-t)b)g(ta + m_2(1-t)b) \\ &\le [t^{\alpha_1} f(a) + m_1(1-t^{\alpha_1})f(b)][t^{\alpha_2} g(a) + m_2(1-t^{\alpha_2})g(b)] \\ &= t^{\alpha_1}t^{\alpha_2} f(a)g(a) + m_2 t^{\alpha_1}(1-t^{\alpha_2})f(a)g(b) \\ &\quad + m_1 t^{\alpha_2}(1-t^{\alpha_1})f(b)g(a) + m_1 m_2(1-t^{\alpha_1})(1-t^{\alpha_2})f(b)g(b). \end{aligned} \tag{2.2}$$

Since $f, g : [a,b] \to \mathbb{R}$ are integrable functions, either increasing or decreasing synchronously, by using the following Chebyshev inequality (see [5, 8])

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \ge \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx \tag{2.3}$$

and generalization Szegö and Weinberger, we can write

$$\begin{aligned} &\int_0^1 f(ta + m_1(1-t)b)g(ta + m_2(1-t)b)dt \\ &\qquad \ge \int_0^1 f(ta + m_1(1-t)b)dt \int_0^1 g(ta + m_2(1-t)b)dt \\ &\qquad = \frac{1}{m_1 b - a}\int_a^{m_1 b} f(x)dx.\frac{1}{m_2 b - a}\int_a^{m_2 b} g(x)dx. \end{aligned}$$

On the other hand, integrating both sides of the inequality (2.2) according to t over $[0,1]$, we obtain

$$\frac{1}{m_1b-a}\int_a^{m_1b} f(x)dx.\frac{1}{m_2b-a}\int_a^{m_2b} g(x)dx$$

$$\leq f(a)g(a)\int_0^1 t^{\alpha_1+\alpha_2}dt+m_2f(a)g(b)\int_0^1 t^{\alpha_1}(1-t^{\alpha_2})dt$$

$$+m_1f(b)g(a)\int_0^1 t^{\alpha_2}(1-t^{\alpha_1})dt+m_1m_2f(b)g(b)\int_0^1(1-t^{\alpha_1})(1-t^{\alpha_2})dt$$

$$=\frac{f(a)g(a)}{\alpha_1+\alpha_2+1}+f(a)g(b)\frac{m_2\alpha_2}{(\alpha_1+\alpha_2+1)(\alpha_1+1)}$$

$$+f(b)g(a)\frac{m_1\alpha_1}{(\alpha_1+\alpha_2+1)(\alpha_2+1)}+f(b)g(b)\frac{m_1m_2\alpha_1\alpha_2(\alpha_1+\alpha_2+2)}{(\alpha_1+\alpha_2+1)(\alpha_1+1)(\alpha_2+1)}.$$

The proof is complete. ■

Remark 2. In Theorem 2, if we particularly choose $m_1=m_2=\alpha_1=\alpha_2=1$ then (2.1) is reduced to (1.4). In addition, if we choose $g(x)=1$, then we have the right side of Hermite–Hadamard inequality.

Theorem 3. *Let $a,b\in[0,\infty)$, $a<b$ such that $f,g:[a,b]\to\mathbb{R}$ be non-negative integrable functions, either increasing or decreasing synchronously, and $f,g,fg\in L_1([a,b])$. If f is m_1-convex , g is (α,m_2)-convex on $[a,b]$, for $\alpha\in[0,1]$ and $m_{1,2}\in(0,1]$ then*

$$\frac{1}{(b-a)^2}\int_a^b f(x)dx.\int_a^b g(x)dx\leq\min\{E,L,I,F\} \tag{2.4}$$

where

$$E=f(a)g(a)\frac{1}{\alpha+2}+m_2f(a)g\left(\frac{b}{m_2}\right)\frac{\alpha}{2(\alpha+2)}$$
$$+m_1g(a)f\left(\frac{b}{m_1}\right)\frac{1}{(\alpha+1)(\alpha+2)}$$
$$+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)},$$
$$L=f(b)g(b)\frac{1}{\alpha+2}+m_2f(b)g\left(\frac{a}{m_2}\right)\frac{\alpha}{2(\alpha+2)}$$
$$+m_1g(b)f\left(\frac{a}{m_1}\right)\frac{1}{(\alpha+1)(\alpha+2)}$$
$$+m_1m_2f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)},$$

$$I = f(b)g(a)\frac{1}{\alpha+2} + m_2 f(b) g\left(\frac{b}{m_2}\right)\frac{\alpha}{2(\alpha+2)}$$
$$+ m_1 g(a) f\left(\frac{a}{m_1}\right)\frac{1}{(\alpha+1)(\alpha+2)}$$
$$+ m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{b}{m_2}\right)\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)},$$
$$F = f(a)g(b)\frac{1}{\alpha+2} + m_2 f(a) g\left(\frac{a}{m_2}\right)\frac{\alpha}{2(\alpha+2)}$$
$$+ m_1 g(b) f\left(\frac{b}{m_1}\right)\frac{1}{(\alpha+1)(\alpha+2)}$$
$$+ m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{a}{m_2}\right)\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}.$$

Proof. Since f is m_1-convex , g is (α, m_2)-convex on $[a, b]$, for $\alpha \in [0, 1]$ and $m_{1,2} \in (0, 1]$, we have

$$f(ta + (1-t)b) = f\left(ta + m_1(1-t)\frac{b}{m_1}\right) \le tf(a) + m_1(1-t) f\left(\frac{b}{m_1}\right),$$
$$g(ta + (1-t)b) = g\left(ta + m_2(1-t)\frac{b}{m_2}\right) \le t^\alpha g(a) + m_2(1-t^\alpha) g\left(\frac{b}{m_2}\right).$$

Since f and g are nonnegative for $\forall t \in [0, 1]$, we get

$$\begin{aligned} &f(ta + (1-t)b)g(ta + (1-t)b) \\ &\le f(a)g(a)t^{\alpha+1} + m_2 f(a) g\left(\frac{b}{m_2}\right) t(1-t^\alpha) \\ &\quad + m_1 g(a) f\left(\frac{b}{m_1}\right) t^\alpha(1-t) + m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)(1-t)(1-t^\alpha). \end{aligned} \tag{2.5}$$

Integrating both sides of the inequality (2.5) according to t over $[0, 1]$, we obtain

$$\int_0^1 f(ta + (1-t)b)g(ta + (1-t)b)dt$$
$$\le f(a)g(a)\int_0^1 t^{\alpha+1}dt + m_2 f(a) g\left(\frac{b}{m_2}\right)\int_0^1 t(1-t^\alpha)dt$$
$$+ m_1 g(a) f\left(\frac{b}{m_1}\right)\int_0^1 t^\alpha(1-t)dt$$
$$+ m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\int_0^1 (1-t)(1-t^\alpha)dt$$

$$= f(a)g(a)\frac{1}{\alpha+2} + m_2 f(a) g\left(\frac{b}{m_2}\right)\frac{\alpha}{2(\alpha+2)}$$
$$+ m_1 g(a) f\left(\frac{b}{m_1}\right)\frac{1}{(\alpha+1)(\alpha+2)}$$
$$+ m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}.$$

Similarly, we can write

$$\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt$$
$$\leq f(b)g(b)\frac{1}{\alpha+2} + m_2 f(b) g\left(\frac{a}{m_2}\right)\frac{\alpha}{2(\alpha+2)}$$
$$+ m_1 g(b) f\left(\frac{a}{m_1}\right)\frac{1}{(\alpha+1)(\alpha+2)}$$
$$+ m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right)\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}$$

and

$$\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt$$
$$\leq f(b)g(a)\frac{1}{\alpha+2} + m_2 f(b) g\left(\frac{b}{m_2}\right)\frac{\alpha}{2(\alpha+2)}$$
$$+ m_1 g(a) f\left(\frac{a}{m_1}\right)\frac{1}{(\alpha+1)(\alpha+2)}$$
$$+ m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{b}{m_2}\right)\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}$$

and

$$\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt$$
$$\leq f(a)g(b)\frac{1}{\alpha+2} + m_2 f(a) g\left(\frac{a}{m_2}\right)\frac{\alpha}{2(\alpha+2)}$$
$$+ m_1 g(b) f\left(\frac{b}{m_1}\right)\frac{1}{(\alpha+1)(\alpha+2)}$$
$$+ + m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{a}{m_2}\right)\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}.$$

By applying Chebychev integral inequality

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \geq \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$$

on the four different inequalities given above, we complete the proof. ∎

Remark 3. In Theorem 3, if we choose $m_1 = m_2 = \alpha = 1$ then (2.4) is reduced to

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \min\left\{\frac{M(a,b)+N(a,b)}{3}, \frac{M(a,b)}{6} + \frac{N(a,b)}{3}\right\}.$$

In addition, if we choose $g(x) = 1$, then (2.4) is reduced to

$$\frac{1}{b-a}\int_a^b f(x)dx \le \min\left\{2\frac{f(a)+f(b)}{3}, \frac{f(a)+f(b)}{2}\right\}.$$

3. Applications to some special means

We shall consider the means as arbitrary positive real numbers a, b, $a \ne b$. In the resources following are included also see [2, pp.12],

The quadratic mean: $K = K(a,b) = \sqrt{\frac{a^2+b^2}{2}}$, $a, b > 0$.

The p-quadratic mean: $K_p = K_p(a,b) = \sqrt{\frac{a^{2p}+b^{2p}}{2}}$, $a, b > 0$.

The geometric mean: $G = G(a,b) = \sqrt{ab}$, $a, b > 0$.

The p-logarithmic mean: $L_p = L_p(a,b) = \left\{\begin{array}{ll} a & \text{if } a = b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a \ne b \end{array}\right\}$, $a \ne b$, $a, b > 0$.

Now, we present some applications of the result in Section 2 to the special means of real numbers. The following propositions hold:

Proposition 1. *Let $0 < a < b < \infty$, and then we have*

$$L_p^p(a, m_1 b)L_p^p(a, m_2 b) \le L_{2p}^{2p}(a,b) \le Sa^{2p} + (E+M)(ab)^p + Ab^{2p}, \tag{3.1}$$

where S, E, M, A is as in the (2.1).

Proof. If we choose in (2.1), $f, g : [0,\infty) \to [0,\infty)$, $f(x) = g(x) = x^p$ with $p \ge 1$, then we obtain

$$\begin{aligned}
&\frac{1}{m_1 b - a}\int_a^{m_1 b} x^p dx . \frac{1}{m_2 b - a}\int_a^{m_2 b} x^p dx \\
&= \frac{1}{m_1 b - a}\left[\frac{x^{p+1}}{p+1}\right]_a^{m_1 b} . \frac{1}{m_2 b - a}\left[\frac{x^{p+1}}{p+1}\right]_a^{m_2 b} \\
&= \frac{(m_1 b)^{p+1} - a^{p+1}}{(m_1 b - a)(p+1)} \frac{(m_2 b)^{p+1} - a^{p+1}}{(m_2 b - a)(p+1)} = L_p^p(a, m_1 b)L_p^p(a, m_2 b)
\end{aligned}$$

$$\leq \frac{1}{b-a}\int_a^b x^{2p}dx = \frac{1}{b-a}\left[\frac{x^{2p+1}}{2p+1}\right]_a^b = \frac{b^{2p+1}-a^{2p+1}}{2p+1} = L_{2p}^{2p}(a,b)$$
$$\leq Sa^{2p} + (E+M)(ab)^p + Ab^{2p}$$

and the proof is complete. ∎

Remark 4. In Proposition 1, if we choose $m_1 = m_2 = \alpha_1 = \alpha_2 = 1$ then (3.1) is reduced to

$$L_p^{2p}(a,b) \leq L_{2p}^{2p}(a,b) \leq \frac{a^{2p}+a^pb^p+b^{2p}}{3}.$$

Proposition 2. *Let $0 < a < b < \infty$, and then we have*

$$L_p^{2p}(a,b) \leq \frac{1}{3}(2K_p^2(a,b) + G^{2p}(a,b)). \tag{3.2}$$

Proof. If we choose in (2.4), $f, g : [0,\infty) \to [0,\infty)$, $f(x) = g(x) = x^p$ with $p \geq 1$ and $\alpha = m_1 = m_2 = 1$, we have that E;

$$\frac{1}{(b-a)^2}\int_a^b x^p dx \int_a^b x^p dx = L_p^{2p}(a,b)$$
$$\leq a^{2p}\frac{1}{\alpha+2} + m_2^{1-p}(ab)^p\frac{\alpha}{2(\alpha+2)} + m_1^{1-p}(ab)^p\frac{1}{(\alpha+1)(\alpha+2)}$$
$$+ (m_1m_2)^{1-p}b^{2p}\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}$$
$$= \frac{a^{2p}}{3} + \frac{(ab)^p}{3} + \frac{b^{2p}}{3} = \frac{1}{3}(a^{2p} + (ab)^p + b^{2p})$$
$$= \frac{1}{3}(2\frac{a^{2p}+b^{2p}}{2} + (ab)^p) = \frac{1}{3}(2K_p^2(a,b) + G^{2p}(a,b))$$

and the proof is complete. ∎

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