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MEAN-VALUE THEOREMS AND SOME SYMMETRIC MEANS

Abstract. Some variants of the Lagrange and Cauchy mean-value theorems lead to the conclusion that means, in general, are not symmetric. They are symmetric iff they coincide (respectively) with the Lagrange and Cauchy means. Under some regularity assumptions, we determine the form of all the relevant symmetric means.

Introduction

In [10], it has been proved that, under some natural conditions, if f and g are real differentiable functions in an interval $I \subset \mathbb{R}$, then there exists a unique mean : $I^2 \to I$ such that

$$\frac{f(M^{[f,g]}(x,y)) - f(x)}{g(y) - g(M^{[f,g]}(x,y))} = \frac{f'}{g'}(M^{[f,g]}(x,y)), \qquad x, y \in I,\ x \neq y.$$

If the mean $M^{[f,g]}$ is symmetric, then it coincides with the Cauchy mean $C^{[f,g]}$ generated by f and g (cf. Bullen, Mitrinović, Vasić [1], Bullen [2]). If $g = \mathrm{id}\,|_I$ we denote this mean by $M^{[f]}$. If $M^{[f]}$ is symmetric, then it is equal to the Lagrange mean $L^{[f]}$.

At this background, the following problem arises: determine all functions f and g such that the means $M^{[f]}$ and $M^{[f,g]}$ are symmetric.

In the present paper, we solve this problem assuming that the functions f and g are three times continously differentiable. In particular, we show that $M^{[f]}$ is symmetric iff f is a non-affine homographic function. This fact allows to establish the effective form of all symmetric $M^{[f,g]}$ means. Note that, contrary to the case of Cauchy means, it is rather difficult problem to find general form of $M^{[f,g]}$, even in the case when g is the identity function (cf. [10]).

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In section 1, for a given function f, we present a variant of the Lagrange mean-value theorem and a result on the existence and uniqueness of a relevant mean $M^{[f]}$ that slightly improve some results of [10]. In section 2, we show that (under the above mentioned regularity assumptions) $M^{[f]}$ is symmetric iff, for some real c, it is the translated c-geometric mean (Theorem 3). In section 3, for a given functions f and g, we prove a variant of the Cauchy mean-value theorem, and we give conditions guarantying the existence of a relevant unique strict and continuous mean $M^{[f,g]}$. According to the main result of section 4, the mean $M^{[f,g]}$ is symmetric iff it is g-conjugate of a c-translated geometric mean. We also note that, as $c \to \infty$, the pointwise limit of g-conjugate of c-translated quasi-geometric mean tends to the quasi-arithmetic mean generated by g.

Some symmetric counterparts of the Lagrange and Cauchy mean-value theorems is presented in [8] (cf. also [9] where some applications are given). The weighted extensions of the Cauchy means are considered in [7]. The mean-type mappings, their iterates, and invariant means are treated in [3]–[6].

1. A variant of the Lagrange mean-value theorem and the relevant means

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be differentiable on (a, b) and continuous at a and b. Then*

(i): ([10]) *there is $\eta \in (a, b)$ such that*

$$\frac{f(\eta) - f(a)}{b - \eta} = f'(\eta);$$

(ii): *if moreover the function*

$$(a, b) \ni x \to \frac{f(x) - f(a)}{b - x} - f'(x) \quad \text{is strictly monotonic,}$$

then the point $\eta \in (a, b)$ is unique.

Part (i), by the Rolle theorem, follows from the fact that $\varphi : [a, b] \to \mathbb{R}$,

$$\varphi(x) := [g(b) - g(x)]\,[f(x) - f(a)], \qquad x \in [a, b],$$

is continuous in $[a, b]$ and $\varphi(a) = 0 = \varphi(b)$ ([10], Corollary 1). Part (ii) is an immediate consequence of the strict monotonicity of the assumed function.

Remark 1. The assumption of part (ii) is satisfied if f is increasing and g' is strictly decreasing or if f is decreasing and g' is strictly increasing.

Let $I \subset \mathbb{R}$ be a nontrivial interval. Recall that a function $M : I^2 \to I$ is called a *mean in I* if it is *internal*, that is if

$$\min(x, y) \le M(x, y) \le \max(x, y), \qquad \text{for all} \quad x, y \in I.$$

The mean M is called *strict* if these inequalities are strict, for all $x, y \in I$, $x \neq y$, and *symmetric* if $M(x,y) = M(y,x)$, for all $x, y \in I$.

Theorem 2. *Suppose that $I \subset \mathbb{R}$ is an open interval, $f : I \to \mathbb{R}$ is continuously differentiable and $f'(x) \neq 0$, for all $x \in L$. Then*

(i): ([10]) *if for every $x, y \in I$ there is a unique mean-value $M(x,y)$ such that*

$$f(M(x,y)) - f(x) = f'(M(x,y))(y - M(x,y)), \qquad x, y \in I,$$

then M is continuous;

(ii): *if f is increasing (decreasing), and f' is strictly decreasing (strictly increasing) in I, then there exists a unique strict continuous mean $M^{[f]} : I^2 \to I$ such that*

$$\frac{f(M^{[f]}(x,y)) - f(x)}{y - M^{[f]}(x,y)} = f'(M^{[f]}(x,y)), \qquad x, y \in I,\ x \neq y.$$

Proof. Part (i) coincides with the first part of Theorem 3 in [10].

To prove part (ii) assume, for instance, that f increasing and f' is strictly decreasing. Then, for all arbitrary fixed $x, y \in I$, $x < y$, the function

$$[x,y] \ni t \to f(t) - f(x) \ \text{ is increasing,}$$

the function

$$[x,y) \ni t \to y - t \ \text{ is strictly decreasing and positive,}$$

and, consequently, for all $x, y \in I$, $x < y$, the function

$$[x,y) \ni t \to \frac{f(t) - f(x)}{y - t} - f'(t) \text{ is strictly increasing.}$$

This implies the uniqueness on the mean M in I. In view of part (i), f is continuous. ∎

Remark 2. This result improves Theorem 3 in [10] where, to guarantee the uniqueness of the mean, it is assumed that f is twice continuously differentiable and $f'f'' < 0$ in I.

The Lagrange mean-value theorem can be formulated in the following way. *If a function $f : I \to \mathbb{R}$ is differentiable, then there exists a strict symmetric mean $M : I^2 \to I$ such that, for all $x, y \in I, x \neq y$,*

$$\frac{f(x) - f(y)}{x - y} = f'(M(x,y)).$$

If f' is one-to-one then, obviously, M is uniquely determined. We denote this mean by $L^{[f]}$. Thus

$$L^{[f]}(x,y) := \begin{cases} (f')^{-1}\left(\frac{f(x)-f(y)}{x-y}\right), & \text{for} \quad x \neq y, \\ x, & \text{for} \quad x = y, \end{cases}$$

and it is called a *Lagrange mean* generated by f.

Example 1. For $f(x) = x^2$, we easily get

$$M^{[f]}(x,y) = \frac{1}{3}\left(\sqrt{3x^2 + y^2} + y\right), \qquad x, y \in \mathbb{R},$$

which shows that $M^{[f]}$ is not symmetric. Of course, this mean is not a Lagrange one.

2. Symmetric $M^{[f]}$ means

We begin this section with

Lemma 1. *Suppose that $f : I \to \mathbb{R}$ is differentiable, f' is one-to-one and the mean $M^{[f]} : I^2 \to I$ exists.*

(i): *The mean $M^{[f]}$ is symmetric if and only if $M^{[f]} = L^{[f]}$.*
(ii): *If $M^{[f]} = L^{[f]}$ then f satisfies the functional equation*

$$(1) \qquad (y-x)f(M^{[f]}(x,y)) + (f(y) - f(x))\, M^{[f]}(x,y) = yf(y) - xf(x), \qquad x, y \in I.$$

Proof. To prove the first part suppose that $M^{[f]}(x,y) = M^{[f]}(y,x)$, for all $x, y \in I$. Then we have

$$f\left(F^{[f]}(x,y)\right) - f(x) = f'\left(M^{[f]}(x,y)\right)\left[y - M^{[f]}(x,y)\right], \qquad x, y \in I,$$

and

$$f\left(M^{[f]}(x,y)\right) - f(y) = f'\left(M^{[f]}(x,y)\right)\left[x - M^{[f]}(x,y)\right], \qquad x, y \in I.$$

Subtracting the respective sides of these equalities, we get

$$f(y) - f(x) = f'\left(M^{[f]}(x,y)\right)(y-x), \qquad x, y \in I,$$

which proves that $M^{[f]} = L^{[f]}$. The converse implication is obvious.

Now, if $M^{[f]} = L^{[f]}$ then

$$\frac{f(M^{[f]}(x,y)) - f(x)}{y - M^{[f]}(x,y)} = f'(M^{[f]}(x,y)) = \frac{f(x) - f(y)}{x - y}, \qquad x, y \in I, x \neq y,$$

whence we get equation (1). ■

Theorem 3. *Let a function $f : I \to \mathbb{R}$ be three times continuously differentiable and let f' be one-to-one. Suppose that $M^{[f]}$ exists. Then*

$$M^{[f]} = L^{[f]}$$

if and only if f is a non-affine homographic function, that is, if

$$f(x) = \frac{ax+b}{x+c}, \qquad x \in I,$$

for some $a, b, c \in \mathbb{R}$ such that $ac - b \neq 0$.

Proof. Suppose that $M^{[f]} = L^{[f]}$ in I. From (1), we get

$$(2)\quad (y-x)f(M(x,y)) + (f(y) - f(x))\, M(x,y) = yf(y) - xf(x), \qquad x, y \in I,$$

where

$$(3)\qquad M(x,y) := L^{[f]}(x,y) = (f')^{-1}\left(\frac{f(y)-f(x)}{y-x}\right), \qquad x, y \in I, x \neq y.$$

In the sequel, to make the notations shorter, we write M instead of $M(x,y)$.

There exists an $x_0 \in I$ such that $f''(x_0) \neq 0$. Indeed, in the opposite case we would have $f''(x) = 0$, for all $x \in I$, whence $f'(x) = a$, for some a and for all $x \in I$, which contradicts to the injectivity of f'. By the injectivity of f', there is at most one point $x \in I$ such that $f'(x) = 0$. Let $J \subset I$ be a maximal interval such that $x_0 \in J$ and

$$f'(x)f''(x) \neq 0, \quad \text{for all } x \in J.$$

Differentiating both sides of (2) twice with respect to y, we get, for all $x, y \in J, x \neq y$,

$$\begin{aligned} 2f'(M)\frac{\partial M}{\partial y} &+ (y-x)f''(M)\left(\frac{\partial M}{\partial y}\right)^2 + (y-x)f'(M)\frac{\partial^2 M}{\partial y^2} \\ &+ f''(y)M + 2f'(y)\frac{\partial M}{\partial y} + [f(y) - f(x)]\frac{\partial^2 M}{\partial y^2} = 2f'(y) + yf''(y), \end{aligned}$$

and differentiating twice with respect to x, we get, for all $x, y \in J$, $x \neq y$,

$$\begin{aligned} -2f'(M)\frac{\partial M}{\partial x} &+ (y-x)f''(M)\left(\frac{\partial M}{\partial x}\right)^2 + (y-x)f'(M)\frac{\partial^2 M}{\partial x^2} \\ &- f''(x)M - 2f'(x)\frac{\partial M}{\partial x} + [f(y) - f(x)]\frac{\partial^2 M}{\partial x^2} = -2f'(x) - xf''(x). \end{aligned}$$

Adding this two equations by sides and then dividing the both sides of the resulting equality by $y - x$ we obtain, for all $x, y \in J, x \neq y$,

$$(4)\qquad 2f'(M)\frac{\frac{\partial M}{\partial y}-\frac{\partial M}{\partial x}}{y-x}+f''(M)\left[\left(\frac{\partial M}{\partial y}\right)^2+\left(\frac{\partial M}{\partial x}\right)^2\right]$$
$$+f'(M)\left[\frac{\partial^2 M}{\partial y^2}+\frac{\partial^2 M}{\partial x^2}\right]+\frac{f''(y)-f''(x)}{y-x}M$$
$$+2\frac{f'(y)\frac{\partial M}{\partial y}-f'(x)\frac{\partial M}{\partial x}}{y-x}+\frac{f(y)-f(x)}{y-x}\left[\frac{\partial^2 M}{\partial y^2}+\frac{\partial^2 M}{\partial x^2}\right]$$
$$=2\frac{f'(y)-f'(x)}{y-x}+\frac{yf''(y)-xf''(x)}{y-x}.$$

Since, for all $x\in J$,

$$\lim_{y\to x} M(x,y)=x$$

and, by (3),

$$\lim_{y\to x}\frac{\partial M}{\partial y}=\lim_{y\to x}\frac{1}{f''(M(x,y))}\frac{f'(y)(y-x)-f(y)+f(x)}{(y-x)^2}=\frac{1}{2},$$
$$\lim_{y\to x}\frac{\partial M}{\partial x}=\lim_{y\to x}\frac{1}{f''(M(x,y))}\frac{-f'(x)(y-x)+f(y)-f(x)}{(y-x)^2}=\frac{1}{2},$$
$$\lim_{y\to x}\frac{\frac{\partial M}{\partial y}-\frac{\partial M}{\partial x}}{y-x}=\lim_{y\to x}\frac{1}{f''(M(x,y))}\frac{[f'(y)+f'(x)](y-x)-2[f(y)-f(x)]}{(y-x)^2}$$
$$=\frac{1}{6}\frac{f'''(x)}{f''(x)},$$
$$\lim_{y\to x}\frac{f'(y)\frac{\partial M}{\partial y}-f'(x)\frac{\partial M}{\partial x}}{y-x}$$
$$=\lim_{y\to x}\frac{f'(y)[f'(y)(y-x)-f(y)+f(x)]+f'(x)[f'(x)(y-x)-f(y)+f(x)]}{f''(M(x,y))(y-x)^3}$$
$$=\frac{1}{2}\frac{f'''(x)}{f''(x)}+\frac{1}{6}\frac{f'(x)f'''(x)}{f''(x)}.$$

Since

$$\frac{\partial^2 M}{\partial y^2}=\frac{-f'''(M)}{[f''(M(x,y))]^3}\left[\frac{f'(y)(y-x)-f(y)+f(x)}{(y-x)^2}\right]^2$$
$$+\frac{f''(y)(y-x)^2-2[f'(y)(y-x)-f(y)+f(x)]}{f''(M(x,y))(y-x)^3},$$

and

$$\frac{\partial^2 M}{\partial x^2}=\frac{-f'''(M)}{[f''(M(x,y))]^3}\left[\frac{-f'(x)(y-x)+f(y)-f(x)}{(y-x)^2}\right]^2$$
$$+\frac{-f''(x)(y-x)^2-2[-f'(x)(y-x)+f(y)-f(x)]}{f''(M(x,y))(y-x)^3},$$

we have

$$\lim_{y\to x}\frac{\partial^2 M}{\partial y^2}=\lim_{y\to x}\frac{\partial^2 M}{\partial x^2}=\frac{1}{12}\frac{f'''(x)}{f''(x)}.$$

Now, letting $y\to x$ in equality (4), we obtain, for all $x\in J$,

$$\begin{aligned}2f'(x)\frac{1}{6}\frac{f'''(x)}{f''(x)}+f''(x)\left[\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2\right]+f'(x)\left[\frac{1}{12}\frac{f'''(x)}{f''(x)}+\frac{1}{12}\frac{f'''(x)}{f''(x)}\right]\\ +f'''(x)x+2\left(\frac{1}{2}\frac{f'''(x)}{f''(x)}+\frac{1}{6}\frac{f'(x)f'''(x)}{f''(x)}\right)+f'(x)\left[\frac{1}{12}\frac{f'''(x)}{f''(x)}+\frac{1}{12}\frac{f'''(x)}{f''(x)}\right]\\ =2f''(x)+f''(x)+xf'''(x),\end{aligned}$$

which reduces to the differential equation

$$2f'(x)f'''(x)-3[f''(x)]^2=0,\qquad x\in J,$$

that is

$$\left(\log\left|\frac{[f''(x)]^2}{[f'(x)]^3}\right|\right)'=0,\qquad x\in J,$$

whence

$$\frac{[f''(x)]^2}{[f'(x)]^3}=A,\qquad x\in J,$$

for some real constant $A\neq 0$. It follows that

$$\text{either } f''(x)=[Af'(x)]^{3/2},\ \text{ for all }\ x\in J,$$
$$\text{or }\ f''(x)=-[Af'(x)]^{3/2},\ \text{for all }\ x\in J.$$

Solving each of these two equations, we obtain

$$|f'(x)|=\frac{1}{(Cx+D)^2},\qquad x\in J,$$

for some $C,D\in\mathbb{R}$, $C\neq 0$. (Since f' does not vanish in J, the Darboux property of derivative implies that f' has a constant sign in J.) Consequently, after simple calculations,

$$f(x)=\frac{ax+b}{x+c},\qquad x\in J,$$

for some $a,b,c\in\mathbb{R}$ such that $ac-b\neq 0$. Since $-c\notin I$ and

$$\left(\frac{ax+b}{x+c}\right)'\left(\frac{ax+b}{x+c}\right)''=\frac{ac-b}{(x+c)^2}\cdot\frac{2(b-ac)}{(x+c)^3}\neq 0,\qquad x\in\mathbb{R},\ x\neq -c,$$

from the maximality of J we conclude that $I=J$. This completes the proof. ■

Define $\mathcal{G} : \left[(0,\infty)^2 \cup (-\infty,0)^2\right] \to \mathbb{R}\backslash\{0\}$ by

$$\mathcal{G}(x,y) := \begin{cases} \sqrt{xy}, & \text{for } (x,y) \in (0,\infty)^2, \\ -\sqrt{xy}, & \text{for } (x,y) \in (-\infty,0)^2. \end{cases}$$

Thus $\mathcal{G}$ restricted to $(0,\infty)^2$ is the usual geometric mean in $(0,\infty)$, and $\mathcal{G}$ restricted to $(-\infty,0)^2$ is mean in $(-\infty,0)$ and, in the sequel, it is called the *negative geometric mean.*

Consider the following important

Example 2. Let $a,b,c \in \mathbb{R}$ be such that $ac-b \neq 0$ and let $f(x) := \frac{ax+b}{x+c}$ as in the conclusion of Theorem 3. For this function, we have either $I \subseteq (-c,\infty)$ or $I \subseteq (-\infty,-c)$.

Taking $I = (-c,\infty)$, by simple calculations, we obtain

$$M^{[f]}(x,y) = \sqrt{(x+c)(y+c)} - c = \mathcal{G}(x+c,y+c) - c, \qquad x,y \in (-c,\infty),$$

that is $M^{[f]}$ is the c-translated geometric mean; and, taking $I = (-\infty,c)$, we obtain

$$M^{[f]}(x,y) = -\sqrt{(x+c)(y+c)} - c \ = \mathcal{G}(x+c,y+c) - c, \quad x,y \in (-\infty,-c),$$

that is $M^{[f]}$ is c-translated negative geometric mean.

Applying this example and Lemma 2, we obtain the main result of this section.

Theorem 4. *Let $f : I \to \mathbb{R}$ be three times continuously differentiable in an interval I and let f' be one-to-one. Suppose that $M^{[f]}$ exists. Then the following conditions are equivalent*

(i): *$M^{[f]}$ is a symmetric mean;*
(ii): *$M^{[f]} = L^{[f]}$;*
(iii): *there is $c \in \mathbb{R}$ such that either $M^{[f]}$ is a c-translated geometric mean or it is a c-translated negative geometric mean.*

This result determines effectively all symmetric means of the type $M^{[f]}$. Let us note that, for a given generating function f, in general, it is a difficult problem to find the effective formula of the mean $M^{[f]}$ (cf. [10]).

3. A variant of the Cauchy mean-value theorem and the relevant means

We begin this section with the following

Theorem 5. *Let $f,g : [a,b] \to \mathbb{R}$ be differentiable on (a,b) and continuous at the points a and b. Then*

(i): *there is $\eta \in (a,b)$ such that*

$$g'(\eta)\left[f(\eta) - f(a)\right] = f'(\eta)\left[g(b) - g(\eta)\right],$$

(ii): *if $g'(x) \neq 0$ for all $x \in (a,b)$, then there exists $\eta \in (a,b)$ such that*

$$\frac{f(\eta) - f(a)}{g(b) - g(\eta)} = \frac{f'(\eta)}{g'(\eta)};$$

if moreover the function

$$(a,b) \ni x \to \frac{f(x) - f(a)}{g(b) - g(x)} - \frac{f'(x)}{g'(x)} \quad \text{is strictly monotonic,}$$

then the point $\eta \in (a,b)$ is unique.

Proof. The function $\varphi : [a,b] \to \mathbb{R}$,

$$\varphi(x) := [g(b) - g(x)]\,[f(x) - f(a)]\,, \qquad x \in [a,b],$$

is continuous in $[a,b]$. Since $\varphi(a) = 0 = \varphi(b)$, there exists a point $\eta \in (a,b)$ such that $\varphi'(\eta) = 0$. As

$$\varphi'(t) = -g'(t)\,[f(t) - f(a)] + [g(b) - g(t)]f'(t), \qquad t \in [a,b],$$

we hence get $-g'(\eta)\,[f(\eta) - f(a)] + [g(b) - g(\eta)]f'(\eta) = 0$. Now the second part is obvious. ∎

Remark 3. Suppose that $g'(x) \neq 0$ for $x \in I$. Then, by the Darboux property of the derivative, g is invertible. To prove the second part it is enough to apply Theorem 1 to the function $f \circ (g^{-1})$ defined in the interval $g(I)$.

As a consequence of this result, we obtain the following

Theorem 6. *Suppose that $I \subset \mathbb{R}$ is an open interval, $f, g : I \to \mathbb{R}$ are continuously differentiable and $f'(x)g'(x) \neq 0$ for all $x \in L$. Then*

(i): *if for every $x, y \in I$ there is a unique mean-value $M(x,y)$ such that*

$$g'(M(x,y))\,[f(M(x,y)) - f(x)] \\ = f'(M(x,y))\,[g(y) - g(M(x,y))]\,, \qquad x, y \in I,$$

then M is continuous;

(ii): *if f' is increasing (decreasing), and $\frac{f'}{g'}$ is strictly decreasing (strictly increasing) in I, then there exists a unique strict continuous mean $M^{[f,g]} : I^2 \to I$ such that*

$$\frac{f(M^{[f,g]}(x,y)) - f(x)}{g(y) - g\left(M^{[f,g]}(x,y)\right)} = \frac{f'(M^{[f,g]}(x,y))}{g'(M^{[f\cdot g]}(x,y))}, \qquad x, y \in I,\ x \neq y. \tag{5}$$

Proof. The function $f \circ g^{-1}$ is continuously differentiable in the open interval $g(I)$ and

$$\left(f \circ g^{-1}\right)'(u) = \frac{f'}{g'} \circ g^{-1}(u) \neq 0, \quad u \in g(I).$$

According to part (i) of Theorem 2, if for any $u, v \in g(I)$ there is a unique mean-value $N(u,v)$ such that

$$f \circ g^{-1}(N(u,v)) - f \circ g^{-1}(u) = \left(f \circ g^{-1}\right)'(N(u,v))\,[v - N(u,v)]\,,$$

then N is continuous. Now, setting

$$M(x,y) := g^{-1}\left(N(g(x), g(y))\right), \qquad x, y \in I,$$

and taking into account that

$$\left(f \circ g^{-1}\right)'(g(x)) = \frac{f'(x)}{g'(x)}, \qquad x \in I,$$

we hence get (i).

The function g is strictly monotonic as $g'(x) \neq 0$, for all $x \in I$. Assume for instance that g is strictly increasing. If f is increasing (decreasing) and $\frac{f'}{g'}$ is strictly decreasing (strictly increasing), then $f \circ g^{-1}$ is increasing (decreasing) and $\left(f \circ g^{-1}\right)' = \frac{f'}{g'} \circ g^{-1}$ is strictly decreasing (strictly increasing). Thus, the function $f \circ g^{-1}$ satisfies the assumptions of part (ii) of Theorem 2. In the same way, we can check that $f \circ g^{-1}$ satisfies these assumptions in the case when g is strictly decreasing. In view of part (ii) of Theorem 2, there exists a unique strict and continuous mean $N^{[f \circ g^{-1}]} : g(I)^2 \to g(I)$ such that

$$\frac{f \circ g^{-1}(N^{[f \circ g^{-1}]}(u,v)) - f \circ g^{-1}(u)}{v - N^{[f \circ g^{-1}]}(u,v)} = \left(f \circ g^{-1}\right)'(N^{[f \circ g^{-1}]}(u,v)), \qquad u, v \in g(I),\ u \neq v.$$

It follows that the mean $M^{[f,g]} : I^2 \to I$, defined by

$$M^{[f,g]}(x,y) := g^{-1}(N^{[f \circ g^{-1}]}(g(x), g(x))), \qquad x, y \in I,$$

is strict, continuous and satisfies (5). ∎

The Cauchy mean-value theorem can be formulated as follows. If $f, g : I \to R$ are differentiable, $g'(x) \neq 0$ for all $x \in I$, then there exists a mean $M : I^2 \to I$ such that for all $x, y \in I, x \neq y$,

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(M(x,y))}{g'(M(x,y))}.$$

If $\frac{f'}{g'}$ is one-to-one then $C^{[f,g]} := M$ is unique, strict, continuous, symmetric and it is called a *Cauchy mean* generated by f and g. Moreover, we have

$$C^{[f,g]}(x,y) = \left(\frac{f'}{g'}\right)^{-1}\left(\frac{f(x) - f(y)}{g(x) - g(y)}\right), \qquad x, y \in I,\ x \neq y.$$

4. Symmetric $M^{[f,g]}$ means

Lemma 2. *Let $f, g : I \to \mathbb{R}$ be differentiable, $g'(x) \neq 0$, for all $x \in I$, and let $\frac{f'}{g'}$ be one-to-one. Suppose that the mean $M^{[f,g]}$ exists. The mean $M^{[f,g]}$ is symmetric if and only if*

$$M^{[f,g]} = C^{[f,g]}.$$

Proof. Assume that $M = M^{[f,g]}$ is symmetric, i.e. that

$$M(x, y) = M(y, x), \qquad x, y \in I,\ x \neq y.$$

Hence, changing the roles x and y in (5) (with $M^{[f,g]} = M$), we get

$$\frac{f(M(x,y)) - f(y)}{g(x) - g(M(x,y))} = \frac{f'(M(x,y))}{g'(M(x,y))}, \qquad x, y \in I,\ x \neq y.$$

From (5) and from above equality, we have

$$f(M)g'(M) - f(x)g'(M) - f'(M)g(y) + f'(M)g(M) = 0,$$
$$f(M(x,y))g'(M) - f(y)g'(M) - f'(M)g(x) + f'(M)g(M) = 0,$$

where $M = M(x, y) = M(y, x)$. Subtracting by sides, we hence get

$$f(y)g'(M) + f'(M)g(x) - f(x)g'(M) - f'(M)g(y) = 0,$$

whence

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(M(x,y))}{g'(M(x,y))}, \qquad x, y \in I,\ x \neq y.$$

Since, by assumption, $\frac{f'}{g'}$ is one-to-one, the mean M is uniquely determined and

$$M(x, y) = \left(\frac{f'}{g'}\right)^{-1} \left(\frac{f(y) - f(x)}{g(y) - g(x)}\right) = C^{[f,g]}(x, y), \qquad x, y \in I,\ x \neq y.$$

The converse implication is easy to prove. ∎

Let us note the following easy to verify

Remark 4. Let $I \subset \mathbb{R}$ be an interval. Assume that $g : I \to \mathbb{R}$ is continuous and strictly monotonic and let $c \in \mathbb{R}$ be such that the function $g + c$ is of a constant sign in I. Then the function $\mathcal{G}_c^{[g]} : I^2 \to I$ defined by

$$\mathcal{G}_c^{[g]}(x, y) = g^{-1}\left(\operatorname{sgn}(g + c)\sqrt{(g(x) + c)(g(y) + c)} - c\right), \qquad x, y \in I,$$

is a mean in I. Moreover,

(1) if $g + c \geq 0$ in I, then

$$\mathcal{G}_c^{[g]}(x, y) = g^{-1}\left(\sqrt{(g(x) + c)(g(y) + c)} - c\right), \qquad x, y \in I,$$

i.e. $\mathcal{G}_c^{[g]}$ is g-conjugate of the c-translated geometric mean in I;

furthermore, the pointwise limit $\lim_{c\to+\infty}\mathcal{G}_c^{[g]}$ exists, and

$$\lim_{c\to+\infty}\mathcal{G}_c^{[g]}(x,y)=\mathcal{A}^{[g]}(x,y),\qquad x,y\in I,$$

where $\mathcal{A}^{[g]}:I^2\to I$ defined by

$$\mathcal{A}^{[g]}(x,y):=g^{-1}\left(\frac{g(x)+g(y)}{2}\right),\qquad x,y\in I,$$

is a quasi-arithmetic mean of a generator g;

(2) if $g+c\le 0$ in I then

$$\mathcal{G}_c^{[g]}(x,y)=g^{-1}\left(\sqrt{(g(x)+c)(g(y)+c)}-c\right),\qquad x,y\in I,$$

i.e. $\mathcal{G}_c^{[g]}$ is g-conjugate of the c-translated negative geometric mean in I. Moreover

$$\lim_{c\to-\infty}\mathcal{G}_c^{[g]}(x,y)=\mathcal{A}^{[g]}(x,y),\qquad x,y\in I.$$

Remark 5. Since

$$\mathcal{A}^{[g]}=\mathcal{G}_0^{[\exp\circ g]},$$

the quasi-arithmetic mean of a generator g is a $\log\circ g$-conjugate geometric mean in I. The g-conjugate geometric mean can be also called *quasi-geometric mean* of a generator g.

The main result of this section reads as follows.

Theorem 7. *Let $f,g:I\to\mathbb{R}$ be three times differentiable in an interval I, $g'(x)\neq 0$ for all $x\in I$, and $\frac{f'}{g'}$ be one-to-one. Suppose that the mean $M^{[f,g]}$ exists. Then the following conditions are equivalent*

- (i): *$M^{[f,g]}$ is a symmetric mean;*
- (ii): $M^{[f,g]}=C^{[f,g]}$;
- (iii): *there is $c\in\mathbb{R}$ such that either $M^{[f,g]}$ is g-conjugate of the c-translated geometric mean in I or $M^{[f,g]}$ is g-conjugate of the c-translated negative geometric mean in I.*

Proof. In view of Lemma 3, the first two conditions are equivalent. Assume that $M=M^{[f,g]}$ is symmetric. By the definition of $M^{[f,g]}$, we have

$$\frac{f(M(x,y))-f(x)}{g(y)-g(M(x,y))}=\frac{f'(M(x,y))}{g'(M(x,y))},\qquad x,y\in I,\ x\neq y.$$

Since g is continuous and strictly monotonic, we can write this equality in the form

$$\frac{f\circ g^{-1}\left[g(M(g^{-1}(u),g^{-1}(v)))\right]-f\circ g^{-1}(u)}{v-g(M(g^{-1}(u),g^{-1}(v)))} = \frac{f'\circ g^{-1}\left[g(M(g^{-1}(u),g^{-1}(v)))\right]}{g'\circ g^{-1}\left[g(M(g^{-1}(u),g^{-1}(v)))\right]},$$

for all $u,v\in g(I)$, $u\neq v$. Setting here

$$M^*(u,v) := g(M(g^{-1}(u),g^{-1}(v))), \qquad u,v\in g(I),$$

we get

$$\frac{f\circ g^{-1}(M^*(u,v))-f\circ g^{-1}(u)}{v-M^*(u,v)} = \frac{f'\circ g^{-1}(M^*(u,v))}{g'\circ g^{-1}(M^*(u,v))}, \qquad u,v\in g(I),\ u\neq v,$$

whence, as $\left(f\circ g^{-1}\right)' = \frac{f'\circ g^{-1}}{g'\circ g^{-1}}$, we obtain

$$\frac{f\circ g^{-1}(M^*(u,v))-f\circ g^{-1}(u)}{v-M^*(u,v)} = \left(f\circ g^{-1}\right)'(M^*(u,v)), \quad u,v\in g(I), u\neq v,$$

where, obviously, M^* is a strict and symmetric mean in $g(I)$. Applying Lemma 2 with f replaced by $f\circ g^{-1}$, we conclude that

$$f\circ g^{-1}(u) = \frac{au+b}{u+c}, \qquad u\in g(I),$$

whence

$$f(x) = \frac{ag(x)+b}{g(x)+c}, \qquad x\in I, \tag{6}$$

and, of course,

$$g(x)+c\neq 0, \qquad x\in I.$$

Consequently, by the continuity of g, the function $g+c$ is of a constant sign in I,

$$f'(x) = \frac{(ac-b)g'(x)}{(g(x)+c)^2}, \qquad x\in I,$$

$$\frac{f'}{g'}(x) = \frac{ac-b}{(g(x)+c)^2} \qquad x\in I,$$

and

$$\left(\frac{f'}{g'}\right)^{-1}(u) = g^{-1}\left(\operatorname{sgn}(g+c)\sqrt{\frac{ac-b}{u}}-c\right), \qquad u\in\frac{f'}{g'}(I).$$

Since, in view of previous result, $F^{[f,g]} = C^{[f,g]}$ where $C^{[f,g]}$ is the Cauchy mean, we hence get

$$\begin{aligned} M^{[f,g]}(x,y) &= \left(\frac{f'}{g'}\right)^{-1}\left(\frac{f(y)-f(x)}{g(y)-g(x)}\right) \\ &= g^{-1}\left(\operatorname{sgn}(g+c)\sqrt{\frac{ac-b}{\frac{\frac{ag(x)+b}{g(x)+c}-\frac{ag(y)+b}{g(y)+c}}{g(x)-g(y)}}}-c\right) \\ &= g^{-1}\left(\operatorname{sgn}(g+c)\sqrt{(g(x)+c)(g(y)+c)}-c\right) = \mathcal{G}_c^{[g]}(x,y), \end{aligned}$$

for all $x, y \in I$, $x \neq y$.

To finish the proof it is enough to observe that $\mathcal{G}_c^{[g]}$ is symmetric $M^{[f,g]}$-type mean with f given by (6). ∎

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