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ON $[\gamma, \gamma']$-PREOPEN SETS

Abstract. In this paper, we introduce and study the concept of $[\gamma, \gamma']$-preopen sets in topological space.

1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Kasahara [1] defined the concept of an operation on topological spaces. Maki and Noiri [3] introduced a new class of open sets called $[\gamma, \gamma']$-open sets into the field of General Topology. In this paper, we have introduced and studied the notion of $[\gamma, \gamma']$-preopen sets by using operations γ and γ' on a topological space (X, τ). We also introduced $([\gamma, \gamma'], [\beta, \beta'])$-preirresolute functions and $[\gamma, \gamma']$-prehomeomorphisms and investigate some important properties.

2. Preliminaries

Definition 2.1. [1] Let (X, τ) be a topological space. An operation γ on the topology τ is a function from τ to the power set $\mathcal{P}(X)$ of X such that $V \subset V^{\gamma}$, for each $V \in \tau$, where V^{γ} denotes the value of γ at V. It is denoted by $\gamma : \tau \to \mathcal{P}(X)$.

Definition 2.2. Let (X, τ) be a topological space. An operation γ is said to be regular [1] if, for every open neighborhood U and V of each $x \in X$, there exists an open neighborhoods W of x such that $W^{\gamma} \subset U^{\gamma} \cap V^{\gamma}$.

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Definition 2.3. A subset A of a topological space (X, τ) is said to be $[\gamma, \gamma']$-open set [3] if for each $x \in A$ there exist open neighborhoods U and V of x such that $U^\gamma \cap V^{\gamma'} \subset A$. The complement of $[\gamma, \gamma']$-open set is called $[\gamma, \gamma']$-closed. $\tau_{[\gamma,\gamma']}$ denotes set of all $[\gamma, \gamma']$-open sets in (X, τ).

Definition 2.4. [3] For a subset A of (X, τ), $\tau_{[\gamma,\gamma']}$-Cl(A) denotes the intersection of all $[\gamma, \gamma']$-closed sets containing A, that is, $\tau_{[\gamma,\gamma']}$-Cl(A) = $\bigcap\{F : A \subset F, X \backslash F \in \tau_{[\gamma,\gamma']}\}$.

Definition 2.5. Let A be any subset of X. The $\tau_{[\gamma,\gamma']}$-Int(A) is defined as $\tau_{[\gamma,\gamma']}$-Int$(A) = \bigcup\{U : U$ is a $[\gamma, \gamma']$-open set and $U \subset A\}$.

Definition 2.6. A topological space (X, τ) is said to be $[\gamma, \gamma']$-regular [3] if for each $x \in X$ and for every open neighborhood U of x there exist open neighborhoods W and S of x such that $W^\gamma \cap S^{\gamma'} \subset U$.

3. $[\gamma, \gamma']$-preopen sets

Definition 3.1. Let (X, τ) be a topological space and γ, γ' be operations on τ. A subset A of X is said to be $[\gamma, \gamma']$-preopen if $A \subset \tau_{[\gamma,\gamma']}$-Int$(\tau_{[\gamma,\gamma']}$-Cl$(A))$.

Remark 3.2. The set of all $[\gamma, \gamma']$-preopen sets of a topological space (X, τ) is denoted as $\tau_{[\gamma,\gamma']}$-$PO(X)$.

Example 3.3. Let $X = \{a, b, c\}$, and $\tau = \{\emptyset, X, \{a, b\}\}$, let $\gamma : \tau \to P(X)$ and $\gamma' : \tau \to P(X)$ be operators defined as follows: for every $A \in \tau$,

$$\gamma(A) = \begin{cases} \mathrm{Cl}(A), & \text{if A = \{a\}}, \\ A, & \text{if A} \neq \text{\{a\}}, \end{cases}$$

$$\gamma'(A) = \begin{cases} \mathrm{Cl}(A), & \text{if A = \{b\}}, \\ A, & \text{if A} \neq \text{\{b\}}. \end{cases}$$

Then $\tau_{[\gamma,\gamma']}$-$PO(X) = \mathcal{P}(X)\backslash\{\{c\}\}$.

Theorem 3.4. *If A is a $[\gamma, \gamma']$-open set in (X, τ), then it is $[\gamma, \gamma']$-preopen set.*

Proof. The proof is clear. ■

Remark 3.5. The converse of the above Theorem need not be true. From the Example 3.3, we have that $\{a\}$ is $[\gamma, \gamma']$-preopen set but it is not $[\gamma, \gamma']$-open.

Remark 3.6. By Theorem 3.4 and Remark 3.5, we have $\tau_{[\gamma,\gamma']} \subset \tau_{[\gamma,\gamma']}$-$PO(X, \tau)$.

Theorem 3.7. *Let γ and γ' be operations on τ and $\{A_\alpha\}_{\alpha\in\Delta}$ be the collection of $[\gamma,\gamma']$-preopen sets of (X,τ), then $\bigcup_{\alpha\in\Delta} A_\alpha$ is also a $[\gamma,\gamma']$-preopen set.*

Proof. Since each A_α is $[\gamma,\gamma']$-preopen and $A_\alpha \subset \bigcup_{\alpha\in\Delta} A_\alpha$, we have

$$\bigcup_{\alpha\in\Delta} A_\alpha \subset \tau_{[\gamma,\gamma']}\text{- Int}\Big(\tau_{[\gamma,\gamma']}\text{- Cl}\Big(\bigcup_{\alpha\in\Delta} A_\alpha\Big)\Big).$$

Hence $\bigcup_{\alpha\in\Delta} A_\alpha$ is also a $[\gamma,\gamma']$-preopen set in (X,τ). ∎

Remark 3.8. If A and B are any two $[\gamma,\gamma']$-preopen sets in (X,τ), then $A\cap B$ need not be $[\gamma,\gamma']$-preopen in (X,τ). From the Example 3.3, we have that $\{a,c\}$ and $\{b,c\}$ are $[\gamma,\gamma']$-preopen sets, but their intersection is not a $[\gamma,\gamma']$-preopen set in (X,τ).

Lemma 3.9. *Let (X,τ) be a topological space and γ, γ' operations on τ and A be the subset of X. Then the following properties hold:*

(i) $\tau_{[\gamma,\gamma']}\text{-Cl}(\tau_{[\gamma,\gamma']}\text{-Cl}(A)) = \tau_{[\gamma,\gamma']}\text{-Cl}(A)$.
(ii) $\tau_{[\gamma,\gamma']}\text{-Int}(\tau_{[\gamma,\gamma']}\text{-Int}(A)) = \tau_{[\gamma,\gamma']}\text{-Int}(A)$.
(iii) $\tau_{[\gamma,\gamma']}\text{-Cl}(A) = X\backslash\tau_{[\gamma,\gamma']}\text{-Int}(X\backslash A))$.
(iv) $\tau_{[\gamma,\gamma']}\text{-Int}(A) = X\backslash\tau_{[\gamma,\gamma']}\text{-Cl}(X\backslash A))$.

Proof. Straightforward. ∎

Theorem 3.10. *Let (X,τ) be a topological space. Every subset is $[\gamma,\gamma']$-preopen if and only if every $[\gamma,\gamma']$-open set in (X,τ) is $[\gamma,\gamma']$-closed.*

Proof. Let G be $[\gamma,\gamma']$-open. Then $X\backslash G = \tau_{[\gamma,\gamma']}\text{-Cl}(X\backslash G)$ which is $[\gamma,\gamma']$-preopen, so that $\tau_{[\gamma,\gamma']}\text{-Cl}(X\backslash G) \subset \tau_{[\gamma,\gamma']}\text{-Int}(\tau_{[\gamma,\gamma']}\text{-Cl}(\tau_{[\gamma,\gamma']}\text{-Cl}(X\backslash G))) = \tau_{[\gamma,\gamma']}\text{-Int}(\tau_{[\gamma,\gamma']}\text{-Cl}(X\backslash G)) = \tau_{[\gamma,\gamma']}\text{-Int}(X\backslash G)$. Thus, $X\backslash G = \tau_{[\gamma,\gamma']}\text{-Int}(X\backslash G)$, so that $X\backslash G$ is $[\gamma,\gamma']$-open, and G is $[\gamma,\gamma']$-closed.

Conversely, let A be any subset of X. Then $X\backslash\tau_{[\gamma,\gamma']}\text{-Cl}(A)$ is $[\gamma,\gamma']$-open, and hence $[\gamma,\gamma']$-closed. Thus, $X\backslash\tau_{[\gamma,\gamma']}\text{-Cl}(A) = \tau_{[\gamma,\gamma']}\text{-Cl}(X\backslash\tau_{[\gamma,\gamma']}\text{-Cl}(A)) = X\backslash\tau_{[\gamma,\gamma']}\text{-Int}(\tau_{[\gamma,\gamma']}\text{-Cl}(A))$, so that $A \subset \tau_{[\gamma,\gamma']}\text{-Cl}(A) = \tau_{[\gamma,\gamma']}\text{-Int}(\tau_{[\gamma,\gamma']}\text{-Cl}(A))$, and hence A is $[\gamma,\gamma']$-preopen. ∎

Theorem 3.11. *Let (X,τ) be a topological space, G be a $[\gamma,\gamma']$-open subset of X and b be a point of $\tau_{[\gamma,\gamma']}\text{-Cl}(G)\backslash G$. Then $\{b\}$ is not $[\gamma,\gamma']$-preopen in (X,τ).*

Proof. Suppose that $\{b\}$ is $[\gamma,\gamma']$-preopen, so that $\{b\} \subset \tau_{[\gamma,\gamma']}\text{-Int}(\tau_{[\gamma,\gamma']}\text{-Cl}(\{b\}))$. Thus, $G \cap \tau_{[\gamma,\gamma']}\text{-Int}(\tau_{[\gamma,\gamma']}\text{-Cl}(\{b\})) \neq \emptyset$. Let $c \in G \cap \tau_{[\gamma,\gamma']}\text{-Int}(\tau_{[\gamma,\gamma']}\text{-Cl}(\{b\}))$, so $c \in \tau_{[\gamma,\gamma']}\text{-Cl}(\{b\})$ and hence $\{b\}\cap(G\cap\tau_{[\gamma,\gamma']}\text{-Int}(\tau_{[\gamma,\gamma']}\text{-Cl}(\{b\})) \neq \emptyset$. This contradicts the fact that $\{b\}\cap G = \emptyset$. Hence $\{b\}$ is not $[\gamma,\gamma']$-preopen. ∎

Theorem 3.12. *Let (X,τ) be a topological space, G be a $[\gamma,\gamma']$-regular open subset of X and b be a point of $\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(G)\backslash G$. Then $G\cup\{b\}$ is not $[\gamma,\gamma']$-preopen in (X,τ).*

Proof. We have, since $\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(\{b\})\subset\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(G)$, so that $\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(G\cup\{b\}))=\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(G)\cup\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(\{b\}))=\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(G)=G$, and thus $G\cup\{b\}\nsubseteq\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(G\cup\{b\}))$. Hence $G\cup\{b\}$ is not $[\gamma,\gamma']$-preopen. ■

Lemma 3.13. *Let (X,τ) be a topological space and γ, γ' be regular operations on τ and A be the subset of X. Then*

(i) *for every $[\gamma,\gamma']$-open set G and every subset $A\subset X$, we have $\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(A)\cap G\subset\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(A\cap G)$,*
(ii) *for every $[\gamma,\gamma']$-closed set F and every subset $A\subset X$, we have $\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(A\cup F)\subset\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(A)\cup F$.*

Proof. (i). Let $x\in\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(A)\cap G$, then $x\in\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(A)$ and $x\in G$. Let V be the $[\gamma,\gamma']$-open set containing x. Then by Proposition 2.9 of [3], $V\cap G$ is also a $[\gamma,\gamma']$-open set containing x. Since $x\in\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(A)$, we have $(V\cap G)\cap A\neq\emptyset$. This implies that $V\cap(G\cap A)\neq\emptyset$. This is true for every $[\gamma,\gamma']$-open set V containing x, hence by Proposition 3.3 of [3] $x\in\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(A\cap G)$. Therefore, $\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(A)\cap G\subset\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(A\cap G)$.

(ii) Follows from (i) and Lemma 3.9(iv). ■

Theorem 3.14. *Let (X,τ) be a topological space and γ, γ' be regular operations on τ. Let A be a $[\gamma,\gamma']$-preopen and U be the $[\gamma,\gamma']$-open subset of X, then $A\cap U$ is also $[\gamma,\gamma']$-preopen set.*

Proof. Follows from the Proposition 2.9 of [3] and Lemma 3.9. ■

Definition 3.15. Let (X,τ) be a topological space, a subset A of X is said to be:

(i) $[\gamma,\gamma']$-dense if $\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(A)=X$.
(ii) $[\gamma,\gamma']$-nowhere dense if $\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(A))=\emptyset$.

Theorem 3.16. *Let (X,τ) be a topological space and γ, γ' be regular operations on τ. Then, a subset N of X is $[\gamma,\gamma']$-nowhere dense if and only if any one of the following conditions holds:*

(i) $\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(X\backslash\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(N))=X$.
(ii) $N\subset\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(X\backslash\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(N))$.
(iii) *Every nonempty $[\gamma,\gamma']$-open set U contains a nonempty $[\gamma,\gamma']$-open set A disjoint with N.*

Proof. (i) $\tau_{[\gamma,\gamma']}$-Int$(\tau_{[\gamma,\gamma']}$-Cl$(N)) = \emptyset$ if and only if $X\backslash(\tau_{[\gamma,\gamma']}$-Cl$(X\backslash(\tau_{[\gamma,\gamma']}$-Cl$(N))) = \emptyset$ if and only if $X \subset \tau_{[\gamma,\gamma']}$-Cl$(X\backslash(\tau_{[\gamma,\gamma']}$-Cl$(N)))$ if and only if $X = \tau_{[\gamma,\gamma']}$-Cl$(X\backslash(\tau_{[\gamma,\gamma']}$-Cl$(N)))$.

(ii) $N \subset X = \tau_{[\gamma,\gamma']}$-Cl$(X\backslash(\tau_{[\gamma,\gamma']}$-Cl$(N)))$ by (i). Conversely, $N \subset \tau_{[\gamma,\gamma']}$-Cl$(X\backslash(\tau_{[\gamma,\gamma']}$-Cl$(N)))$, implies $\tau_{[\gamma,\gamma']}$-Cl$(N) \subset \tau_{[\gamma,\gamma']}$-Cl$(X\backslash(\tau_{[\gamma,\gamma']}$-Cl$(N)))$. Since $X = \tau_{[\gamma,\gamma']}$-Cl$(N) \cup (X\backslash(\tau_{[\gamma,\gamma']}$-Cl$(N)))$, implies $X \subset \tau_{[\gamma,\gamma']}$-Cl$(X\backslash(\tau_{[\gamma,\gamma']}$-Cl$(N))) \cup (X\backslash(\tau_{[\gamma,\gamma']}$-Cl$(N))) = \tau_{[\gamma,\gamma']}$-Cl$(X\backslash(\tau_{[\gamma,\gamma']}$-Cl$(N)))$. Hence $X = \tau_{[\gamma,\gamma']}$-Cl$(X\backslash(\tau_{[\gamma,\gamma']}$-Cl$(N)))$.

(iii) Let N be a $[\gamma,\gamma']$-nowhere dense subset of X, then $\tau_{[\gamma,\gamma']}$-Int$(X\backslash\tau_{[\gamma,\gamma']}$ -Cl$(N)) = \emptyset$. This implies that $\tau_{[\gamma,\gamma']}$-Cl(N) does not contain any $[\gamma,\gamma']$-open set. Hence for any nonempty $[\gamma,\gamma']$-open set U, $U\backslash(\tau_{[\gamma,\gamma']}$-Cl$(N)) \neq \emptyset$. Thus by Proposition 2.9 of [3], $A = U\backslash\tau_{[\gamma,\gamma']}$-Cl$(N)$ is a nonempty $[\gamma,\gamma']$-open set contained in U and disjoint with N. Conversely, if for any given nonempty $[\gamma,\gamma']$-open set U, there exists a nonempty $[\gamma,\gamma']$-open set A such that $A \subset U$ and $A \cap N = \emptyset$, then $N \subset X\backslash A$, $\tau_{[\gamma,\gamma']}$-Cl$(N) \subset \tau_{[\gamma,\gamma']}$-Cl$(X\backslash A) = X\backslash A$. Therefore, $U\backslash\tau_{[\gamma,\gamma']}$-Cl$(N) \supset U\backslash(X\backslash A) = U \cap A = A \neq \emptyset$. Thus, $\tau_{[\gamma,\gamma']}$-Cl(N) does not contain any nonempty $[\gamma,\gamma']$-open set. This implies that $\tau_{[\gamma,\gamma']}$-Int$(X\backslash\tau_{[\gamma,\gamma']}$-Cl$(N)) = \emptyset$. Hence N is $[\gamma,\gamma']$-nowhere dense set in X. ■

Theorem 3.17. *Let (X,τ) be a topological space and γ, γ' operations on τ. Then for every $x \in X$, $\{x\}$ is either $[\gamma,\gamma']$-preopen or $[\gamma,\gamma']$-nowhere dense set.*

Proof. Suppose that $\{x\}$ is not $[\gamma,\gamma']$-preopen, then $\tau_{[\gamma,\gamma']}$-Int$(X\backslash\tau_{[\gamma,\gamma']}$-Cl$(\{x\})) = \emptyset$. This implies that $\{x\}$ is $[\gamma,\gamma']$-nowhere dense set in X. ■

Theorem 3.18. *For any subset of a space (X,τ), the following are equivalent:*

(i) $S \in [\gamma,\gamma']$-$PO(X)$.
(ii) *There is a $[\gamma,\gamma']$-regular open set $G \subset X$ such that $S \subset G$ and $\tau_{[\gamma,\gamma']}$-Cl$(S) = \tau_{[\gamma,\gamma']}$-Cl$(G)$.*
(iii) *S is the intersection of a $[\gamma,\gamma']$-regular open set and a $[\gamma,\gamma']$-dense set.*
(iv) *S is the intersection of a $[\gamma,\gamma']$-open set and a $[\gamma,\gamma']$-dense set.*

Proof. (i)⇒ (ii): Let $S \in [\gamma,\gamma']$-$PO(X)$. Then $S \subset \tau_{[\gamma,\gamma']}$-Int$(\tau_{[\gamma,\gamma']}$-Cl$(S))$. Let $G = \tau_{[\gamma,\gamma']}$-Int$(S)$. Then G is $[\gamma,\gamma']$-regular open with $S \subset G$ and $S = G$.

(ii)⇒ (iii): Let $D = S \cup (X\backslash G)$. Then D is $[\gamma,\gamma']$-dense and $S = G \cap D$.

(iii)⇒ (iv): This is trivial.

(iv)⇒ (i): Suppose $S = G \cap D$ with G is $[\gamma,\gamma']$-open and D $[\gamma,\gamma']$-dense. Then $S = G$, hence $S \subset G \subset \tau_{[\gamma,\gamma']}$-Cl$(G) = S$. ■

Theorem 3.19. *If every subset of X is either $[\gamma,\gamma']$-open or $[\gamma,\gamma']$-closed, then every $[\gamma,\gamma']$-preopen set in X is $[\gamma,\gamma']$-open.*

Proof. Let A be a $[\gamma,\gamma']$-preopen in X. If A is not $[\gamma,\gamma']$-open, then A is $[\gamma,\gamma']$-closed by hypothesis. Hence $A=\tau_{[\gamma,\gamma']}$-Cl$(A)$, and $\tau_{[\gamma,\gamma']}$-Int$(\tau_{[\gamma,\gamma']}$-Cl$(A))=\tau_{[\gamma,\gamma']}$-Int$(A)$ is a proper subset of A. Thus, $A\not\subseteq\tau_{[\gamma,\gamma']}$-Int$(\tau_{[\gamma,\gamma']}$-Cl$(A))$, so that A is not $[\gamma,\gamma']$-preopen, contradiction. ∎

Theorem 3.20. *Let (X,τ) be a topological space in which every $[\gamma,\gamma']$-preopen set in X is $[\gamma,\gamma']$-open. Then each singleton in X is either $[\gamma,\gamma']$-open or $[\gamma,\gamma']$-closed.*

Proof. Let $x\in X$, and suppose that $\{x\}$ is not $[\gamma,\gamma']$-open. Then $\{x\}$ is not $[\gamma,\gamma']$-preopen. Hence $\{x\}\not\subseteq\tau_{[\gamma,\gamma']}$-Int$(\tau_{[\gamma,\gamma']}$-Cl$(\{x\})$, so that $\tau_{[\gamma,\gamma']}$-Int$(\tau_{[\gamma,\gamma']}$-Cl$(\{x\})=\emptyset$. We have that $\tau_{[\gamma,\gamma']}$-Int$(\tau_{[\gamma,\gamma']}$-Cl$(X\backslash\{x\})\supset\tau_{[\gamma,\gamma']}$-Int$(\tau_{[\gamma,\gamma']}$-Cl$(X\backslash(\tau_{[\gamma,\gamma']}$-Cl$(\{x\})))=\tau_{[\gamma,\gamma']}$-Int$(X\backslash(\tau_{[\gamma,\gamma']}$-Int$(\tau_{[\gamma,\gamma']}$-Cl$(\{x\}))))$ $=X\supset X\backslash\{x\}$. Thus, $X\backslash\{x\}$ is $[\gamma,\gamma']$-preopen and hence $[\gamma,\gamma']$-open. Therefore, $\{x\}$ is $[\gamma,\gamma']$-closed. ∎

Theorem 3.21. *For a topological space (X,τ) and γ, γ' regular operations on τ, the following are equivalent:*

(i) *Every $[\gamma,\gamma']$-preopen set is $[\gamma,\gamma']$-open.*
(ii) *Every $[\gamma,\gamma']$-dense set is $[\gamma,\gamma']$-open.*

Proof. (i)⇒(ii): Let A be a $[\gamma,\gamma']$-dense subset of X. Then $\tau_{[\gamma,\gamma']}$-Int$(\tau_{[\gamma,\gamma']}$-Cl$(A)=X$, so that $A\subset\tau_{[\gamma,\gamma']}$-Int$(\tau_{[\gamma,\gamma']}$-Cl$(A))$ and A is $[\gamma,\gamma']$-preopen. Hence A is $[\gamma,\gamma']$-open.

(ii)⇒(i): Let B be a $[\gamma,\gamma']$-preopen subset of X, so that $B\subset\tau_{[\gamma,\gamma']}$-Int$(\tau_{[\gamma,\gamma']}$-Cl$(B))=G$, say. Then $\tau_{[\gamma,\gamma']}$-Cl$(B)=\tau_{[\gamma,\gamma']}$-Cl$(G)$, so that $\tau_{[\gamma,\gamma']}$-Cl$(X\backslash G)\cup B)=\tau_{[\gamma,\gamma']}$-Cl$(X\backslash G)\cup\tau_{[\gamma,\gamma']}$-Cl$(B)=(X\backslash G)\cup\tau_{[\gamma,\gamma']}$-Cl$(G)=X$, and thus $(X\backslash G)\cup B$ is $[\gamma,\gamma']$-dense in X. Thus, $(X\backslash G)\cup B$ is $[\gamma,\gamma']$-open. Now, $B=(X\backslash G)\cup B)\cap G$, the intersection of two $[\gamma,\gamma']$-open sets is $[\gamma,\gamma']$-open ([3], Proposition 2.9), so that B is $[\gamma,\gamma']$-open. ∎

Definition 3.22. A topological space X is said to be $[\gamma,\gamma']$-submaximal if every $[\gamma,\gamma']$-dense subset of X is $[\gamma,\gamma']$-open.

Theorem 3.23. *Let (X,τ) be a topological space in which every $[\gamma,\gamma']$-preopen set is $[\gamma,\gamma']$-open, then (X,τ) is $[\gamma,\gamma']$-submaximal.*

Proof. Let A be a $[\gamma,\gamma']$-dense subset of (X,τ). Then $A\subset\tau_{[\gamma,\gamma']}$-Int$(\tau_{[\gamma,\gamma']}$-Cl$(A))$; hence A is a $[\gamma,\gamma']$-preopen set. Therefore, from the assumption it is $[\gamma,\gamma']$-open. Hence (X,τ) is $[\gamma,\gamma']$-submaximal. ∎

Definition 3.24. Let A be a subset of a topological space (X,τ) and γ, γ' be operations on τ. Then a subset A of X is said to be $[\gamma,\gamma']$-preclosed if and only if $X\backslash A$ is $[\gamma,\gamma']$-preopen, equivalently a subset A of X is $[\gamma,\gamma']$-preclosed if and only if $\tau_{[\gamma,\gamma']}$-Cl$(\tau_{[\gamma,\gamma']}$-Int$(A))\subset A$.

Remark 3.25. The set of all $[\gamma,\gamma']$-preclosed sets of a topological space (X,τ) is denoted as $\tau_{[\gamma,\gamma']}$-$PC(X)$.

Definition 3.26. Let A be a subset of a topological space (X,τ) and let γ, γ' be operations on τ. Then

(i) the $\tau_{[\gamma,\gamma']}$-preclosure of A is defined as intersection of all $[\gamma,\gamma']$-preclosed sets containing A. That is, $\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(A)=\bigcap\{F : F$ is $[\gamma,\gamma']$-preclosed and $A\subset F\}$,

(ii) the $\tau_{[\gamma,\gamma']}$-preinterior of A is defined as union of all $[\gamma,\gamma']$-preopen sets contained in A. That is, $\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Int}(A)=\bigcup\{U : U$ is $[\gamma,\gamma']$-preopen and $U\subset A\}$.

The proof of the following theorem is obvious and therefore is omitted.

Theorem 3.27. *Let A be a subset of a topological space (X,τ) and γ, γ' be operations on τ. Then*

(i) $\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Int}(A)$ *is a $[\gamma,\gamma']$-preopen set contained in A.*
(ii) $\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(A)$ *is a $[\gamma,\gamma']$-preclosed set containing A.*
(iii) *A is $[\gamma,\gamma']$-preclosed if and only if $\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(A)=A$.*
(iv) *A is $[\gamma,\gamma']$-preopen if and only if $\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Int}(A)=A$.*

Remark 3.28. From the definitions, we have $A\subset\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(A)\subset\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(A)$, for any subset A of (X,τ).

Theorem 3.29. *Let $A\subseteq X$, a point $x\in X$ belongs to $\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(A)$ if and only if $V\cap A\neq\emptyset$, for all $[\gamma,\gamma']$-preopen set V of X containing x.*

Proof. Let E be the set of all $y\in X$ such that $V\cap A\neq\emptyset$, for every $V\in\tau_{[\gamma,\gamma']}$-$PO(X)$ and $y\in V$. Now, to prove the theorem it is enough to prove that $E=\tau_{[\gamma,\gamma']}-p\,\mathrm{Cl}(A)$. Let $x\in\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(A)$ and $x\notin E$. Then there exists a $[\gamma,\gamma']$-preopen set U of x such that $U\cap A=\emptyset$. This implies $A\subset U^c$. Hence $\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(A)\subset U^c$. Therefore $x\notin\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(A)$. This is a contradiction. Hence $\tau_{[\gamma,\gamma']}-p\,\mathrm{Cl}(A)\subset E$.

Conversely, let F be a set such that $A\subset F$ and $X\backslash F\in\tau_{[\gamma,\gamma']}$-$PO(X)$. Let $x\notin F$, then we have $x\in X\backslash F$ and $(X\backslash F)\cap A=\emptyset$. This implies $x\notin E$. Hence $E\subset\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(A)$. ■

Theorem 3.30. *Let (X,τ) be a topological space and γ, γ' be regular operations on τ and A be a subset of X. Then the following properties hold:*

(i) $\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(A)=A\cup\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(A))$.
(ii) $\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Int}(A)=A\cap\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(A))$.

Proof. (i). $\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(A\cup\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(A)))\subset\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(A)\cup\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(\tau_{[\gamma,\gamma']}$-Cl-$\mathrm{Int}(A)))\subset\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(A))\cup\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(\tau_{[\gamma,\gamma']}$ -$\mathrm{Int}(A))\subset A\cup(\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(A))$. Hence $A\cup\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(A))$ is a $[\gamma,\gamma']$-preclosed set containing A. Therefore, $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A) \subset A \cup \tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(A))$.

Conversely, $\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(A)) \subset \tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)) \subset \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)$ since $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)$ is a $[\gamma,\gamma']$-preclosed set. Hence $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A) = A \cup \tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(A))$.

(ii) Follows from (i) and Lemma 3.9(i). ∎

Corollary 3.31. *Let (X,τ) be a topological space and γ, γ' be regular operations on τ and A be a subset of X. Then the following properties hold:*

(i) $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(A)) = \tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(A))$.
(ii) $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(A)) = \tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(A))$.
(iii) $\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)) = \tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(A)))$.
(iv) $\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Int}(A)) = \tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(A)))$.

Proof. (i) By Theorem 3.30(ii), $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(A)) = \tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(A) \cap \tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(A))) = \tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(A) \cap \tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(A)) = \tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(A))$.

(ii) By Theorem 3.30(i), $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(A)) = \tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(A) \cup \tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(A))) = \tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(A))$.

(iii) and (iv) follows from (i) and (ii), respectively. ∎

Theorem 3.32. *Let (X,τ) be a topological space and γ, γ' be regular operations on τ and A be a subset of X. Then $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Int}(A)) = \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Int}(A) \cup \tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(A)))$.*

Proof. Since $\tau_{[\gamma,\gamma']} \subset \tau_{[\gamma,\gamma']}\text{-}PO(X)$, implies $\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(A) \subset \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Int}(A) \subset A$. Hence $\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Int}(A)) = \tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(A)$. By Theorem 3.30(i), $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Int}(A)) = \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Int}(A) \cap \tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Int}(A)) = \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Int}(A) \cup \tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(A))$. ∎

Theorem 3.33. *Let (X,τ) be a topological space and γ, γ' be operations on τ and A be a subset of X. Then V is $[\gamma,\gamma']$-preopen if and only if $V \subset \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(V))$.*

Proof. (i) Let V be $[\gamma,\gamma']$-preopen. Then $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Int}(V) = V$ and also $V \subset \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(V))$. Conversely, let $V \subset \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(V))$. Then $V \subset \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(V)) = \tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(V) \cap \tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(V))) = \tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(V))$. Hence, V is $[\gamma,\gamma']$-preopen. ∎

Theorem 3.34. *Let (X,τ) be a topological space. Then every singleton of X is either $[\gamma,\gamma']$-open or $[\gamma,\gamma']$-preclosed.*

Proof. If $\{x\}$ is not $[\gamma,\gamma']$-open, then $\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(\{x\}) = \emptyset$. Thus, $\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Int}(\{x\})) = \emptyset$; hence $\{x\}$ is $[\gamma,\gamma']$-preclosed. The proof of the second part is straightforward. ∎

Definition 3.35. A subset A of (X,τ) is said to be $[\gamma,\gamma']$-pregeneralized closed (briefly, $[\gamma,\gamma']$-pg.closed) if $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is $[\gamma,\gamma']$-preopen in (X,τ).

Theorem 3.36. *If A is a $[\gamma,\gamma']$-preopen and $[\gamma,\gamma']$-pg.closed subset of (X,τ), then A is $[\gamma,\gamma']$-preclosed.*

Proof. Since A is $[\gamma,\gamma']$-preopen and $[\gamma,\gamma']$-pg.closed, $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A) \subset A$ and hence $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A) = A$. This implies that A is $[\gamma,\gamma']$-preclosed. ∎

Theorem 3.37. *A subset A of (X,τ) is $[\gamma,\gamma']$-pg.closed if and only if $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(\{x\}) \cap A \neq \emptyset$ holds for every $x \in \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)$.*

Proof. Let U be any $[\gamma,\gamma']$-preopen set such that $A \subset U$. Let $x \in \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)$. By assumption there exists $z \in \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(\{x\})$ and $z \in A \subset U$. It follows from Theorem 3.29 that $U \cap \{x\} \neq \emptyset$. Hence $x \in U$. This implies $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A) \subset U$. Therefore A is a $[\gamma,\gamma']$-pg.closed set in (X,τ).

Conversely, suppose $x \in \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)$ such that $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(\{x\}) \cap A = \emptyset$. Since $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(\{x\})$ is $[\gamma,\gamma']$-preclosed set in (X,τ), $(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(\{x\}))^c$ is a $[\gamma,\gamma']$-preopen set of (X,τ). Since $A \subset (\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(\{x\}))^c$ and A is $[\gamma,\gamma']$-pg.closed, $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A) \subset (\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(\{x\}))^c$. This implies that $x \notin \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)$. This is a contradiction. Hence $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(\{x\}) \cap A \neq \emptyset$. ∎

Theorem 3.38. *If A is a $[\gamma,\gamma']$-pg.closed subset of a topological space (X,τ), then $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)\backslash A$ does not contain a nonempty $[\gamma,\gamma']$-preclosed set.*

Proof. Suppose there exists a nonempty $[\gamma,\gamma']$-preclosed set F such that $F \subset \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)\backslash A$. Let $x \in F$, $x \in \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)$ holds. Then $F \cap A = \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(F) \cap A \supset \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(\{x\}) \cap A \neq \emptyset$. Hence $F \cap A \neq \emptyset$. This is a contradiction. ∎

Theorem 3.39. *For each $x \in X$, $\{x\}$ is $[\gamma,\gamma']$-preclosed or $\{x\}^c$ is $[\gamma,\gamma']$-pg.closed set in (X,τ).*

Proof. Suppose that $\{x\}$ is not $[\gamma,\gamma']$-preclosed. Then $X\backslash\{x\}$ is not $[\gamma,\gamma']$-preopen. Let U be any $[\gamma,\gamma']$-preopen set such that $\{x\}^c \subset U$. It follows that $U = X$, $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(\{x\}^c) \subset U$. Therefore, $\{x\}^c$ is $[\gamma,\gamma']$-pg.closed. ∎

Theorem 3.40. *If A is a $[\gamma,\gamma']$-pg.closed subset of (X,τ) such that $A \subset B \subset \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)$, then B is $[\gamma,\gamma']$-pg.closed subset of (X,τ).*

Proof. Let U be a $[\gamma,\gamma']$-preopen set in (X,τ) such that $B \subset U$. Then $A \subset U$. Since A is $[\gamma,\gamma']$-pg.closed, then $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A) \subset U$. Now, since $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)$ is $[\gamma,\gamma']$-preclosed, $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(B) \subset \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A))$ $= \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A) \subset U$. Therefore, B is also a $[\gamma,\gamma']$-pg.closed. ∎

Theorem 3.41. *A set A, in a topological space (X,τ), is $[\gamma,\gamma']$-pg.open if and only if $F \subset \tau_{[\gamma,\gamma']}\text{-}p\operatorname{Int}(A)$ whenever F is $[\gamma,\gamma']$-preclosed in (X,τ) and $F \subset A$.*

Proof. Let A be $[\gamma,\gamma']$-pg.open. Let F be $[\gamma,\gamma']$-preclosed and $F \subset A$. Then $X\backslash A \subset X\backslash F$, where $X\backslash F$ is $[\gamma,\gamma']$-preopen. $[\gamma,\gamma']$-pg.closedness of $X\backslash A$ implies $\tau_{[\gamma,\gamma']}\text{-}p\operatorname{Cl}(X\backslash A) \subset X\backslash F$. By Theorem 3.27, $X\backslash\tau_{[\gamma,\gamma']}\text{-}p\operatorname{Int}(A) \subset X\backslash F$. That is, $F \subset \tau_{[\gamma,\gamma']}\text{-}p\operatorname{Int}(A)$.

Conversely, suppose that F is $[\gamma,\gamma']$-preclosed and $F \subset A$ implies $F \subset \tau_{[\gamma,\gamma']}\text{-}p\operatorname{Int}(A)$. Let $X\backslash A \subset U$ where U is $[\gamma,\gamma']$-preopen. Then $X\backslash U \subset A$ where $X\backslash U$ is $[\gamma,\gamma']$-preclosed. By supposition, $X\backslash U \subset \tau_{[\gamma,\gamma']}\text{-}p\operatorname{Int}(A)$. That is, $X\backslash\tau_{[\gamma,\gamma']}\text{-}p\operatorname{Int}(A) \subset U$. Then $\tau_{[\gamma,\gamma']}\text{-}p\operatorname{Cl}(X\backslash A) \subset U$. This implies $X\backslash A$ is $[\gamma,\gamma']$-pg.closed and hence A is $[\gamma,\gamma']$-pg.open. ∎

Theorem 3.42. *If $\tau_{[\gamma,\gamma']}\text{-}p\operatorname{Int}(A) \subset B \subset A$ and A is $[\gamma,\gamma']$-pg.open, then B is $[\gamma,\gamma']$-pg.open.*

Proof. Easily follows from Theorems 3.27 and 3.40. ∎

Theorem 3.43. *If a set A is $[\gamma,\gamma']$-pg.open in a topological space (X,τ), then $G = X$ whenever G is $[\gamma,\gamma']$-preopen in (X,τ) and $\tau_{[\gamma,\gamma']}\text{-}p\operatorname{Int}(A) \cup X\backslash A \subset G$.*

Proof. Suppose that G is $[\gamma,\gamma']$-preopen and $\tau_{[\gamma,\gamma']}\text{-}p\operatorname{Int}(A) \cup X\backslash A \subset G$. Now $X\backslash G \subset \tau_{[\gamma,\gamma']}\text{-}p\operatorname{Cl}(X\backslash A) \cap A = \tau_{[\gamma,\gamma']}\text{-}p\operatorname{Cl}(X\backslash A)\backslash X\backslash A$. Since $X\backslash G$ is $[\gamma,\gamma']$-preclosed and $X\backslash A$ is $[\gamma,\gamma']$-pg.closed, $X\backslash G = \varnothing$ and hence $G = X$. ∎

Proposition 3.44. *Let (X,τ) be a topological space and $A, B \subset X$. If B is $[\gamma,\gamma']$-pg.open and if $A \supset \tau_{[\gamma,\gamma']}\text{-}p\operatorname{Int}(B)$, then $A \cap B$ is $[\gamma,\gamma']$-pg.open.*

Proof. Since B is $[\gamma,\gamma']$-pg.open and $A \supset \tau_{[\gamma,\gamma']}\text{-}p\operatorname{Int}(B)$, $\tau_{[\gamma,\gamma']}\text{-}p\operatorname{Int}(B) \subset A \cap B \subset B$. By Theorem 3.42, $A \cap B$ is $[\gamma,\gamma']$-pg.open. ∎

Proposition 3.45. *Let $[\gamma,\gamma']\text{-}PO(X)$, the family of all $[\gamma,\gamma']$-preopen subsets of (X,τ), be closed under finite intersections i.e., let $[\gamma,\gamma']\text{-}PO(X)$ be the topology on X. If A and B are $[\gamma,\gamma']$-pg.open in (X,τ), then $A \cap B$ is $[\gamma,\gamma']$-pg.open.*

Proof. Let $X\backslash(A \cap B) = (X\backslash A) \cup (X\backslash B) \subset U$, where U is $[\gamma,\gamma']$-preopen. Then $X\backslash A \subset U$ and $X\backslash B \subset U$. Since A and B are $[\gamma,\gamma']$-pg.open, $\tau_{[\gamma,\gamma']}\text{-}p\operatorname{Cl}(X\backslash A) \subset U$ and $\tau_{[\gamma,\gamma']}\text{-}p\operatorname{Cl}(X\backslash B) \subset U$. By hypothesis, $\tau_{[\gamma,\gamma']}\text{-}p\operatorname{Cl}((X\backslash A) \cup (X\backslash B)) = \tau_{[\gamma,\gamma']}\text{-}p\operatorname{Cl}(X\backslash A) \cup \tau_{[\gamma,\gamma']}\text{-}p\operatorname{Cl}(X\backslash B) \subset U$. That is, $\tau_{[\gamma,\gamma']}\text{-}p\operatorname{Cl}(X\backslash(A \cap B)) \subset U$. This shows that $A \cap B$ is $[\gamma,\gamma']$-pg.open. ∎

Theorem 3.46. *If $A \subset X$ is $[\gamma,\gamma']$-pg.closed, then $\tau_{[\gamma,\gamma']}\text{-}p\operatorname{Cl}(A)\backslash A$ is $[\gamma,\gamma']$-pg.open.*

Proof. Let A be $[\gamma,\gamma']$-pg.closed. Let F be a $[\gamma,\gamma']$-preclosed set such that $F \subset \tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(A)\backslash A$. Then by Theorem 3.38, $F = \emptyset$. So, $F \subset \tau_{[\gamma,\gamma']}$-$p\,\mathrm{Int}(\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(A)\backslash A)$. This shows $\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(A)\backslash A$ is $[\gamma,\gamma']$-pg.open. ∎

Definition 3.47. A topological space (X,τ) is said to be $[\gamma,\gamma']$-pre-$T_{1/2}$ if every $[\gamma,\gamma']$-pregeneralized closed set in (X,τ) is $[\gamma,\gamma']$-preclosed.

Theorem 3.48. *A topological space (X,τ) is $[\gamma,\gamma']$-pre-$T_{1/2}$ space if and only if every singleton subset of X is $[\gamma,\gamma']$-preclosed or $[\gamma,\gamma']$-preopen in (X,τ).*

Proof. Let $x \in X$. Suppose $\{x\}$ is not $[\gamma,\gamma']$-preclosed. Then, it follows from assumption and Theorem 3.46 that $\{x\}$ is $[\gamma,\gamma']$-preopen.

Conversely, let F be a $[\gamma,\gamma']$-pregeneralized closed set in (X,τ). Let x be any point in $\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(F)$, then by assumption $\{x\}$ is $[\gamma,\gamma']$-preopen or $[\gamma,\gamma']$-preclosed.

Case (i): Suppose $\{x\}$ is $[\gamma,\gamma']$-preopen. Then by Theorem 3.37, we have $\{x\}\cap F \neq \emptyset$. This implies $\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(F) = F$; hence (X,τ) is $[\gamma,\gamma']$-pre-$T_{1/2}$.

Case (ii): Suppose $\{x\}$ is $[\gamma,\gamma']$-preclosed. Assume $x \notin F$, then $x \in \tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(F)\backslash F$. This is not possible by Theorem 3.38. Thus, we have $x \in F$. Therefore, $\tau_{[\gamma,\gamma']}$-$p\,\mathrm{Cl}(F) = F$ and hence F is $[\gamma,\gamma']$-preclosed. It follows that (X,τ) is $[\gamma,\gamma']$-pre-$T_{1/2}$. ∎

Theorem 3.49. *Every topological space (X,τ) with operations γ and γ' on τ is $[\gamma,\gamma']$-pre-$T_{1/2}$.*

Proof. Let $x \in X$. We prove (X,τ) is $[\gamma,\gamma']$-pre-$T_{1/2}$, it is sufficient to show that $\{x\}$ is $[\gamma,\gamma']$-preopen or $[\gamma,\gamma']$-preclosed. Now, if $\{x\}$ is $[\gamma,\gamma']$-open, then it is obviously $[\gamma,\gamma']$-preopen. If $\{x\}$ is not $[\gamma,\gamma']$-open, then $\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(\{x\}) = \emptyset$; hence $\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(\tau_{[\gamma,\gamma']}$-$\mathrm{Int}(\{x\})) = \emptyset \subset \{x\}$. Therefore, $\{x\}$ is $[\gamma,\gamma']$-preclosed. ∎

4. $([\gamma,\gamma'],[\beta,\beta'])$-preirresolute functions

Throughout this section let (X,τ) and (Y,σ) be two topological spaces and let $\gamma,\gamma' : \tau \to \mathcal{P}(X)$ and $\beta,\beta' : \sigma \to \mathcal{P}(Y)$ be operations on τ and σ, respectively.

Definition 4.1. A function $f : (X,\tau)\to(Y,\sigma)$ is said to be $([\gamma,\gamma'],[\beta,\beta'])$-preirresolute if for each $x \in X$ and each $[\beta,\beta']$-preopen set V containing $f(x)$ there exists a $[\gamma,\gamma']$-preopen set U such that $x \in U$ and $f(U) \subset V$.

Theorem 4.2. *Let $f : (X,\tau) \to (Y,\sigma)$ be $([\gamma,\gamma'],[\beta,\beta'])$-preirresolute. Then the following hold:*

(i) $f(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)) \subset \sigma_{[\beta,\beta']}\text{-}p\,\mathrm{Cl}(f(A))$ *holds for every subset* A *of* (X,τ),

(ii) *for every* $[\beta,\beta']$*-preclosed set* B *of* (Y,σ), $f^{-1}(B)$ *is* $[\gamma,\gamma']$*-preclosed in* (X,τ).

Proof. (i) Let $y \in f(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A))$ and V be any $[\beta,\beta']$-preopen set containing y. Then there exists $x \in X$ and a $[\gamma,\gamma']$-preopen set U such that $f(x) = y$ and $x \in U$ and $f(U) \subset V$. Since $x \in \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)$, we have $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U \cap A) \subset f(U) \cap f(A) \subset V \cap f(A)$. This implies that $x \in \sigma_{[\beta,\beta']}\text{-}p\,\mathrm{Cl}(f(A))$. Therefore, we have $f(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)) \subset \sigma_{[\beta,\beta']}\text{-}p\,\mathrm{Cl}(f(A))$.

(ii) Let B be a $[\beta,\beta']$-preclosed set in (Y,σ). Therefore, $\sigma_{[\beta,\beta']}\text{-}p\,\mathrm{Cl}(B) = B$. By using (i), we have $f(\tau_{[\gamma,\gamma']}\text{-}b\,\mathrm{Cl}(f^{-1}(B)) \subset \sigma_{[\beta,\beta']}\text{-}p\,\mathrm{Cl}(B) = B$. Therefore, we have $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(f^{-1}(B)) = f^{-1}(B)$. Hence $f^{-1}(B)$ is $[\gamma,\gamma']$-preclosed. ■

Definition 4.3. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be $[\gamma,\gamma']$-preclosed if for any $[\gamma,\gamma']$-preclosed set A of (X,τ), $f(A)$ is a $[\beta,\beta']$-closed in Y.

Theorem 4.4. *Suppose that* f *is* $([\gamma,\gamma'],[\beta,\beta'])$*-preirresolute and* $[\gamma,\gamma']$*-preclosed function. Then,*

(i) *for every* $[\gamma,\gamma']$*-pg.closed set* A *of* (X,τ), *the image* $f(A)$ *is* $[\beta,\beta']$*-pg.closed.*

(ii) *for every* $[\beta,\beta']$*-pg.closed set* B *of* (Y,σ), $f^{-1}(B)$ *is* $[\gamma,\gamma']$*-pg.closed.*

Proof. (i). Let V be any $[\beta,\beta']$-preopen set in (Y,σ) such that $f(A) \subset V$. By Theorem 4.2(ii), $f^{-1}(V)$ is $[\gamma,\gamma']$-preopen in (X,τ). Since A is $[\gamma,\gamma']$-pg.closed and $A \subset f^{-1}(V)$, we have $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A) \subset f^{-1}(V)$, and hence $f(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)) \subset V$. It follows that $f(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A))$ is a $[\beta,\beta']$-preclosed set in Y. Therefore, $\sigma_{[\beta,\beta']}\text{-}p\,\mathrm{Cl}(f(A)) \subset \sigma_{[\beta,\beta']}\text{-}p\,\mathrm{Cl}(f(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A))) = f(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(A)) \subset V$. This implies that $f(A)$ is $[\beta,\beta']$-pg.closed.

(ii) Let U be a $[\gamma,\gamma']$-preopen set of (X,τ) such that $f^{-1}(B) \subset U$. Put $F = \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(f^{-1}(B)) \cap U^c$. It follows that F is $[\gamma,\gamma']$-preclosed set in (X,τ). Since f is $[\gamma,\gamma']$-preclosed, $f(F)$ is $[\beta,\beta']$-preclosed in (Y,σ). Then $f(F) \subset f(\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(f^{-1}(B) \cap X\backslash U)) \subset \sigma_{[\beta,\beta']}\text{-}\mathrm{Cl}(f(f^{-1}(B)) \cap f(X\backslash U)) \subset \tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(B)\backslash B$. This implies $f(F) = \emptyset$, and hence $F = \emptyset$. Therefore, $\tau_{[\gamma,\gamma']}\text{-}p\,\mathrm{Cl}(f^{-1}(B)) \subset U$. Hence $f^{-1}(B)$ is $[\gamma,\gamma']$-pregeneralized closed in (X,τ). ■

Theorem 4.5. *Let* $f : (X,\tau) \to (Y,\sigma)$ *be* $([\gamma,\gamma'],[\beta,\beta'])$*-preirresolute and* $[\gamma,\gamma']$*-preclosed function. Then,*

(i) *if* f *is injective and* (Y,σ) *is* $[\beta,\beta']$*-pre-*$T_{1/2}$, *then* (X,τ) *is* $[\gamma,\gamma']$*-pre-*$T_{1/2}$ *space.*

(ii) *if f is surjective and (X,τ) is $[\gamma,\gamma']$-pre-$T_{1/2}$, then (Y,σ) is $[\beta,\beta']$-pre-$T_{1/2}$.*

Proof. Straightforward. ∎

Definition 4.6. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be $[\gamma,\gamma']$-prehomeomorphism, if f is bijective, $([\gamma,\gamma'],[\beta,\beta'])$-preirresolute and f^{-1} is $[\beta,\beta']$-preirresolute.

Theorem 4.7. *Let $f:(X,\tau)\to(Y,\sigma)$ be $[\gamma,\gamma']$-prehomeomorphism. If (X,τ) is $[\gamma,\gamma']$-pre-$T_{1/2}$, then (Y,σ) is $[\beta,\beta']$-pre-$T_{1/2}$.*

Proof. Let $\{y\}$ be a singleton set of (Y,σ). Then there exists a point x of X such that $y=f(x)$ and, by Theorem 3.48, $\{x\}$ is $[\gamma,\gamma']$-preopen or $[\gamma,\gamma']$-preclosed. Then by Theorem 4.4(i), $\{y\}$ is $[\beta,\beta']$-preclosed or $[\beta,\beta']$-preopen. By Theorem 3.48, (Y,σ) is $[\beta,\beta']$-pre-$T_{1/2}$ space. ∎

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