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TRANSLATIVE COVERING BY UNIT SQUARES

Abstract. Some results concerning translative coverings of squares and triangles by two, three and four unit squares are presented.

1. Introduction

The question of coverings by translations of the members of a given collection of convex bodies was described for the first time in [7] by Hlawka.

Let C be a convex body in Euclidean plane E^2, i.e., a compact convex set with nonempty interior and let (C_n) be a (finite or infinite) sequence of planar convex bodies. The sequence (C_n) is called a *covering* of D if $D \subset \bigcup C_n$. We say that (C_n) *permits a covering* of D, if there are rigid motions σ_n such that $(\sigma_n C_n)$ is a covering of D. If there are translations τ_n such that $(\tau_n C_n)$ is a covering of D, then we say that (C_n) *permits a translative covering* of D.

Various results concerning coverings and translative coverings are discussed in [1], [2], [3], [5] and [12].

Let I be a unit square, i.e., a square of side length 1. We will start with presenting a few results concerning coverings of I by rectangles with and without possibility of rotations.

About fifty years ago Leo Moser asked (see Problem LM5 in [11]): "Can any set of rectangles of largest edge 1 and total area 3 be used to cover a unit square (No rotations, please)?".

Moon and Moser [10] proved that I can be covered by any sequence of rectangles of side lengths not greater than 1 and with total area not smaller than 3. In the covering method presented in [10], a side of any rectangle, used for the covering of I, is parallel to a side of I. Groemer showed that any sequence of rectangles of diameters at most 1, whose total area is greater

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than $\frac{3}{2}(13+7\sqrt{3})$, permits a translative covering of I (see Proposition 1 in [6]).

The optimal bounds can be obtained in case of covering by sequences of squares. According to the hypothesis of Bognár, any sequence of squares with total area not smaller than 2, permits a covering of I (see Problem 108 (iv) in [11]). This conjecture was confirmed in [8]. Furthermore, I can be covered translatively by any sequence of squares whose total area is greater than or equal to 3 (see [9]).

In this note, we will cover translatively by congruent squares.

The question of coverings of squares and triangles by unit squares (with possibility of rotations) is a well-known problem. Results concerning coverings by unit squares can be found online in [4].

We will consider coverings without possibility of rotations.

Let the coordinate system in the plane be given. One of the coordinate system's axis is called x-axis. Denote by $I(t)$ a square of sides of length t and with a side parallel to the x-axis.

Let i be a positive integer, let $0 \leq \alpha_i < \pi/2$ and let $S(\alpha_i)$ be a unit square (in the plane) with an angle between the x-axis and a side of $S(\alpha_i)$ equal to α_i.

Denote by $\rho_I(n)$, where n is a positive integer, the greatest number t such that $I(t)$ can be covered translatively by any collection $S(\alpha_1), S(\alpha_2), \ldots, S(\alpha_n)$ of n unit squares. Obviously, $\rho_I(n) < \sqrt{n}$. On the other hand, $\rho_I(n) \geq \sqrt{n/3}$ (it follows by Theorem 1 of [9]). By Theorem 6 of [6] we deduce that $\lim_{n\to\infty} \rho_I(n)/\sqrt{n} = 1$. The problem is to find $\rho_I(n)$ for $n = 1, 2, 3, \ldots$.

We will also cover triangles. Let $T(t)$ be an equilateral triangle with sides of length t and with one side parallel to the x-axis and let $R(t)$ be an isosceles right triangle with legs of length t parallel to the axes.

Denote by $\rho_T(n)$ [by $\rho_R(n)$] the greatest number t such that any collection of n unit squares permits a translative covering of $T(t)$ (of $R(t)$, respectively).

Ten results, that will be proved in Sections 2, 3 and 4, are presented in the following table.

n	$\rho_I(n)$	$\rho_T(n)$	$\rho_R(n)$
1	$\sqrt{2}/2$	1	$\sqrt{2}/2$
2	$2\sqrt{5}/5$	$2\sqrt{3}/3$	1
3	1	?	$\sqrt{2}$
4	$\sqrt{2}$	?	1.844...

2. Covering of squares

Proof of $\rho_I(1) = \sqrt{2}/2$. Obviously, a circle of unit diameter can be covered translatively by any unit square as well as any circle of unit diameter permits a translative covering of any square of side length $\sqrt{2}/2$ (see Fig. 1, left). This implies that $\rho_I(1) \geq \sqrt{2}/2$. On the other hand, $S(\pi/4)$ does not permit a translative covering of $I(\sqrt{2}/2 + \epsilon)$, for any $\epsilon > 0$, i.e., $\rho_I(1) \leq \sqrt{2}/2$. ∎

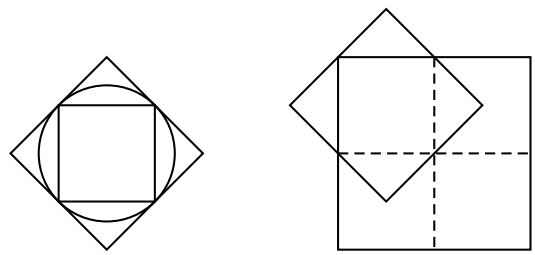

Fig. 1.

Proof of $\rho_I(2) = 2\sqrt{5}/5 \approx 0.89$. Let $s = 2\sqrt{5}/5$. We show that $\rho_I(2) \geq s$. Let $S(\alpha)$ and $S(\beta)$ be unit squares. We can assume that $\alpha \neq 0$ and $\beta \neq 0$, otherwise $I(s)$ can be covered translatively either by $S(\alpha)$ or by$S(\beta)$.

We cover by $S(\alpha)$ the left side of $I(s)$ as well as two segments contained in the top and in the bottom of $I(s)$ of lengths $\min(p, s)$ and $\min(q, s)$, respectively, where

$$p = \frac{1 - s\sin\alpha}{\cos\alpha}, \quad q = \frac{1 - s\cos\alpha}{\sin\alpha}.$$

Moreover, we cover by $S(\beta)$ the right side of $I(s)$ as well as two segments contained in the top and in the bottom of $I(s)$ of lengths $\min(u, s)$ and $\min(v, s)$, respectively (see Fig. 2, left and middle), where

$$u = \frac{1 - s\cos\beta}{\sin\beta}, \quad v = \frac{1 - s\sin\beta}{\cos\beta}.$$

It is easy to check that

$$\sin\alpha + \frac{1}{2}\cos\alpha \leq \sin(\arctan 2) + \frac{1}{2}\cos(\arctan 2) = \frac{1}{2}\sqrt{5}.$$

This implies that $p \geq \frac{1}{2}s$, for any $0 < \alpha < \pi/2$. For the same reason $q \geq \frac{1}{2}s$, $u \geq \frac{1}{2}s$ and $v \geq \frac{1}{2}s$. Consequently, $S(\alpha)$ and $S(\beta)$ permit a translative covering of $I(s)$.

Now we show that $\rho_I(2) \leq s$. Let

$$\gamma_1 = \arctan 2, \quad \gamma_2 = \arctan\frac{1}{2} \ = \frac{\pi}{2} - \gamma_1$$

and let $\epsilon > 0$. We show that $S(\gamma_1)$ and $S(\gamma_2)$ do not permit a translative

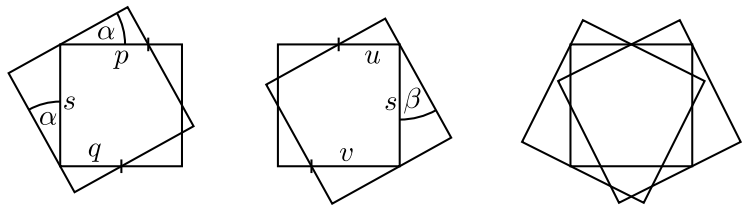

Fig. 2.

covering of $I(s+\epsilon)$. Obviously, $S(\gamma_i)$ can cover translatively at the same time at most two vertices of $I(s+\epsilon)$, for $i \in \{1,2\}$. Without loss of generality, we can assume that $S(\gamma_1)$ covers the left vertices of $I(s+\epsilon)$ and that $S(\gamma_2)$ covers the right vertices (see Fig 2, right). Then $S(\gamma_1)$ covers the part of the top of $I(s+\epsilon)$ of length not greater than

$$p_1 = \frac{1 - \frac{2\sqrt{5}}{5}\sin\gamma_1}{\cos\gamma_1} = \frac{\sqrt{5}}{5}.$$

Moreover, $S(\gamma_2)$ covers the part of the top of $I(s+\epsilon)$ of length not greater than

$$u_1 = \frac{1 - \frac{2\sqrt{5}}{5}\cos\gamma_2}{\sin\gamma_2} = \frac{\sqrt{5}}{5}.$$

This means that these two squares do not permit a translative covering of $I(s+\epsilon)$. ∎

Proof of $\rho_I(3) = 1$. Three unit squares $S(0)$ do not permit a translative covering of $I(1+\epsilon)$, for any $\epsilon > 0$; the reason is that $S(0)$ can cover only one vertex of $I(1+\epsilon)$. Consequently, $\rho_I(3) \leq 1$. On the other hand, any collection of three unit squares permits a translative covering of $S(1)$ (see [9]). Thus $\rho_I(3) \geq 1$. ∎

Proof of $\rho_I(4) = \sqrt{2}$. Each unit square contains a square with sides parallel to the axes of the coordinate system and with side length $\sqrt{2}/2$. Four such squares permit a translative covering of $I(\sqrt{2})$. Consequently, $\rho_I(4) \geq \sqrt{2}$.

Let $\epsilon > 0$. Any square $S(\pi/4)$ can cover translatively a part of the boundary of $I(\sqrt{2}+\epsilon)$ of total length not greater than $\sqrt{2}$ (see Fig. 1, right). This implies that four squares $S(\pi/4)$ do not permit a translative covering of the boundary of $I(\sqrt{2}+\epsilon)$, i.e., that $\rho_I(4) \leq \sqrt{2}$. ∎

3. Covering of equilateral triangles

Proof of $\rho_T(1) = 1$. Obviously, $T(1+\epsilon)$ cannot be covered translatively by $S(0)$, for any $\epsilon > 0$. This means that $\rho_T(1) \leq 1$. On the other hand, any square $S(\alpha)$ permits a translative covering of $T(1)$. Three cases: $\alpha < \pi/6$, $\pi/6 \leq \alpha \leq \pi/3$ and $\alpha > \pi/3$ are presented in Fig. 3. ∎

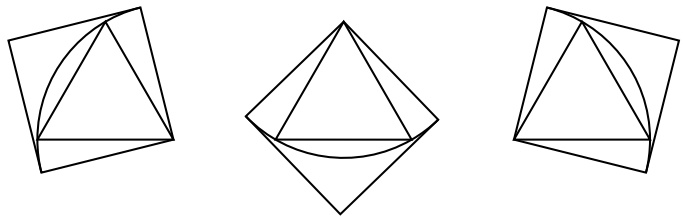

Fig. 3.

Proof of $\rho_T(2) = 2\sqrt{3}/3 \approx 1.15$**.** Let $\zeta = 2\sqrt{3}/3$. To show that $\rho_T(2) \leq \zeta$, it suffices to observe that two squares $S(0)$ do not permit a translative covering of $T(\zeta + \epsilon)$, for any $\epsilon > 0$. The reason is that $S(0)$ can cover only one vertex of $T(\zeta + \epsilon)$ (see Fig 4, left).

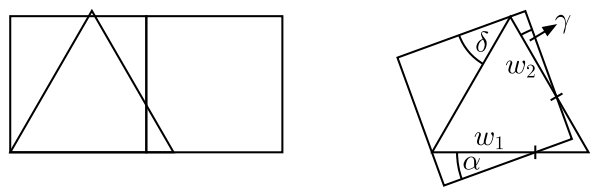

Fig. 4.

Now we show that $\rho_T(2) \geq \zeta$. Let $S(\alpha)$ and $S(\beta)$ be unit squares.

Observe that at least one angle between a side of $S(\alpha)$ and a side of $T(\zeta)$ is smaller than $\pi/6$. If $\pi/6 \leq \alpha < \pi/3$, then $\alpha_1 = \alpha - \pi/6 < \pi/6$ (see Fig. 5, left). If $\alpha \geq \pi/3$, then $\alpha_2 = \alpha - \pi/3 < \pi/6$ (see Fig. 5, right). We can assume that $\alpha < \pi/6$.

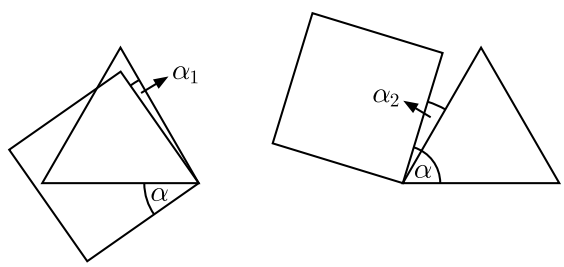

Fig. 5.

Denote by L the part of $T(\zeta)$ lying to the left of the straight line going through the center of the bottom and the center of the right side of $T(\zeta)$. We show that L can be covered translatively by $S(\alpha)$. If $\alpha = 0$, then $S(\alpha)$ permits a translative covering of L. Assume that

$$0 < \alpha < \frac{\pi}{6}.$$

We cover the part L of $T(\zeta)$ as in the right-hand picture in Fig. 4. In this figure $\gamma = \pi/6 - \alpha$ and $\delta = \pi/3 - \alpha$. Put

$$f_1(\alpha) = \frac{1 - \cos\alpha + \frac{\sqrt{3}}{3}\sin\alpha}{\sin\alpha}$$

and

$$f_2(\alpha) = \frac{2(1 - \frac{\sqrt{3}}{3}\cos\alpha - \sin\alpha)}{\cos\alpha - \sqrt{3}\sin\alpha}.$$

It is easy to check that

$$w_1 = \frac{1 - \zeta\sin\delta}{\sin\alpha} = f_1(\alpha)$$

and that

$$w_2 = \frac{1 - \zeta\cos\delta}{\sin\gamma} = f_2(\alpha).$$

Since $f_1'(\alpha) = \frac{1-\cos\alpha}{\sin^2\alpha} > 0$, it follows that

$$w_1 > \lim_{\alpha\to 0} f_1(\alpha) = \frac{\sqrt{3}}{3}.$$

Moreover,

$$f_2'(\alpha) = \frac{2(\sin\alpha + \sqrt{3}\cos\alpha - 2)}{(\cos\alpha - \sqrt{3}\sin\alpha)^2} < 0.$$

Hence,

$$w_2 > \lim_{\alpha\to\pi/6} f_2(\alpha) = \frac{\sqrt{3}}{3}.$$

This implies that L can be covered translatively by $S(\alpha)$.

By $\rho_T(1) = 1$, we conclude that $T(\zeta) \setminus L$ can be covered translatively by $S(\beta)$. Thus $\rho_T(2) \geq \zeta$. ∎

4. Covering of isosceles right triangles

Proof of $\rho_R(1) = \sqrt{2}/2$. A circle of unit diameter can be covered translatively by any unit square as well as any circle with unit diameter permits a translative covering of any isosceles right triangle of legs of length $\sqrt{2}/2$ (see Fig. 6, left). Consequently, $\rho_R(1) \geq \sqrt{2}/2$. On the other hand, $S(\pi/4)$ does not permit a translative covering of $R(\sqrt{2}/2 + \epsilon)$, for any $\epsilon > 0$. ∎

Proof of $\rho_R(2) = 1$. Let $S(\alpha)$ and $S(\beta)$ be unit squares. Moreover, let R_1 and R_2 be right triangles of legs of length $\sqrt{2}/2$ presented in Fig. 6. By $\rho_1 = \sqrt{2}/2$ we conclude that $S(\alpha)$ permits a translative covering of R_1 as well as $S(\beta)$ permits a translative covering of R_2. Consequently, $\rho_R(2) \geq 1$.

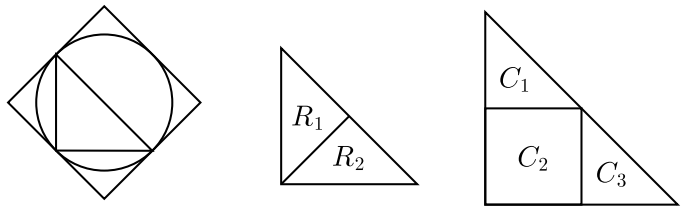

Fig. 6.

On the other hand, $S(0)$ can cover translatively only one vertex of $R(1+\epsilon)$, for any $\epsilon > 0$. This implies that two squares $S(0)$ do not permit a translative covering of $R(1+\epsilon)$, i.e., that $\rho_R(2) \leq 1$. ∎

Proof of $\rho_R(3) = \sqrt{2}$. Let C_1 and C_2 be right triangles of legs of length $\sqrt{2}/2$ and let C_3 be a square presented in the right-hand picture in Fig. 6. Any unit square permits a translative covering of C_i for $i \in \{1,2,3\}$. Hence, $\rho_R(3) \geq \sqrt{2}$.

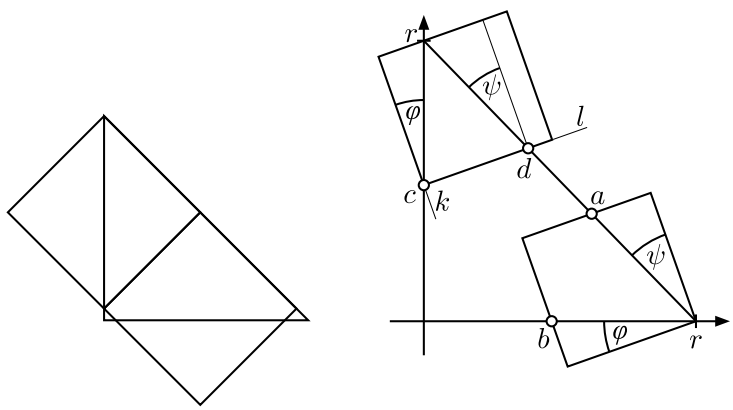

Fig. 7.

Obviously, in order to cover the hypotenuse of $R(\sqrt{2}+\epsilon)$, we need at least three squares $S(\pi/4)$. Moreover, no square $S(\pi/4)$ that covers a point of the hypotenuse, covers the left down vertex of $R(\sqrt{2}+\epsilon)$ (see Fig. 7, left). Consequently, $R(\sqrt{2}+\epsilon)$ cannot be covered translatively by three squares $S(\pi/4)$, for any $\epsilon > 0$, i.e., $\rho_R(3) \leq \sqrt{2}$. ∎

Put

$$\varphi = \arctan \nu \approx 0.3395,$$

where

$$\nu = \frac{1}{3}\left(-2 + \sqrt[3]{\frac{47+3\sqrt{249}}{2}} - 2\sqrt[3]{\frac{2}{47+3\sqrt{249}}}\right) \approx 0.3532$$

and let

$$r_0 = \frac{2\cos\varphi + \sin\varphi}{(\sin\varphi + \cos\varphi)\cos\varphi} \approx 1.84427.$$

The proof of $\rho_R(4) = r_0$ will be divided into two parts.

Proof of $\rho_R(4) \le r_0 \approx 1.844$. We show that four squares $S(\varphi)$ do not permit a translative covering of $R(r)$, for any $r > r_0$.

Let the vertices of $R(r)$ are $(0,0),\ (0,r),\ (r,0)$. No square $S(\varphi)$ can cover at the same time two vertices of $R(r)$. Let

$$a(x_a, y_a),\ b(x_b, 0),\ c(0, y_c),\ d(x_d, y_d),\ \psi = \frac{\pi}{4} - \varphi,$$

where (see Fig. 7, right)

$$x_a = r - \frac{1}{\sqrt{2}\cos\psi} = r - \frac{1}{\cos\varphi + \sin\varphi},\quad y_a = \frac{1}{\sqrt{2}\cos\psi} = \frac{1}{\cos\varphi + \sin\varphi},$$

$$x_b = r - \frac{1}{\cos\varphi},\ y_c = r - \frac{1}{\cos\varphi},$$

$$x_d = \frac{1}{\sqrt{2}\cos\psi} = \frac{1}{\cos\varphi + \sin\varphi},\quad y_d = r - \frac{1}{\sqrt{2}\cos\psi} = r - \frac{1}{\cos\varphi + \sin\varphi}.$$

A square $S(\varphi)$ that covers $(r,0)$ can cover neither $(x_b - \epsilon, 0)$ nor $(x_a - \epsilon, y_a + \epsilon)$, for any $\epsilon > 0$. Moreover, a square $S(\varphi)$ that covers $(0,r)$ can cover neither $(0, y_c - \epsilon)$ nor $(x_d + \epsilon, y_d - \epsilon)$, for any $\epsilon > 0$.

It is easy to verify that a square that covers $(0,0)$ can cover neither d nor a. Now, we show that a square $S(\varphi)$ that covers $(0,0)$ cannot cover at the same time both point b and c. It suffices to prove that the distance $d(b,l)$ between b and the straight line l going through points c and d is greater than 1. Since l is described by the equality

$$y = \tan\varphi \cdot x + y_c,$$

i.e.,

$$y\cos\varphi - x\sin\varphi - r\cos\varphi + 1 = 0,$$

we have

$$d(b,l) = \left| -\left(r - \frac{1}{\cos\varphi}\right)\sin\varphi - r\cos\varphi + 1 \right|.$$

Thus

$$d(b,l) = r(\sin\varphi + \cos\varphi) - \frac{\sin\varphi}{\cos\varphi} - 1.$$

Since $r > r_0$, it follows that

$$d(b,l) > \frac{2\cos\varphi + \sin\varphi}{\cos\varphi} - \frac{\sin\varphi}{\cos\varphi} - 1 = 1.$$

Moreover, $d(b,l) > 1$ implies that no square $S(\varphi)$ can cover at the same time both points b and d.

In the similar way, we show that no square $S(\varphi)$ can cover at the same time both points c and a. Let k be the straight line described by the equality

$$y = -\frac{1}{\tan\varphi} \cdot x + y_c,$$

i.e.,

$$y\sin\varphi + x\cos\varphi - \left(r - \frac{1}{\cos\varphi}\right)\sin\varphi = 0.$$

The distance between a and k is equal to

$$d(a,k) = \left|\frac{\sin\varphi}{\cos\varphi+\sin\varphi} + r\cos\varphi - \frac{\cos\varphi}{\cos\varphi+\sin\varphi} - r\sin\varphi + \frac{\sin\varphi}{\cos\varphi}\right|.$$

Since $r > r_0$, it follows that

$$d(a,k) > \frac{\sin\varphi}{\cos\varphi+\sin\varphi} + r_0(\cos\varphi-\sin\varphi) - \frac{\cos\varphi}{\cos\varphi+\sin\varphi} + \frac{\sin\varphi}{\cos\varphi} = 1.$$

This implies that four squares $S(\varphi)$ cannot cover translatively $R(r)$. Thus $\rho_R(4) \le r_0$. ∎

Remark 1. By the proof presented above, we deduce that four squares $S(\alpha)$, for $0 \le \alpha \le \frac{\pi}{4}$, do not permit a translative covering of $R(r)$, for any $r > f(\alpha)$, where

$$f(\alpha) = \frac{2\cos\alpha + \sin\alpha}{(\sin\alpha+\cos\alpha)\cos\alpha}.$$

Remark 2. The value φ was chosen so that

$$r_0 = f(\varphi) \le f(\alpha),$$

for any $0 \le \alpha \le \frac{\pi}{4}$. Indeed,

$$f'(\alpha) = \frac{\sin^3\alpha - \cos^3\alpha + 2\sin^2\alpha\cos\alpha + 2\sin\alpha\cos^2\alpha}{\left(\sin\alpha\cos\alpha+\cos^2\alpha\right)^2}.$$

Consequently, $f'(\alpha) = 0$ if and only if

$$\tan^3\alpha - 1 + 2\tan^2\alpha + 2\tan\alpha = 0.$$

Since $x = \nu \approx 0.3532$ is the only real root of

$$x^3 + 2x^2 + 2x - 1 = 0,$$

it is easy to check that $f(\alpha)$ is minimal at $\alpha = \varphi$.

Proof of $\rho_R(4) \ge r_0 \approx 1.844$. Let $S(\alpha_1)$, $S(\alpha_2)$, $S(\alpha_3)$ and $S(\alpha_4)$ be unit squares. We show that these squares permit a translative covering of $R(r_0)$. There are three cases to consider.

Case 1: at least three angles from among $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ are not greater than $\pi/4$.

There is no loss of generality in assuming that $\alpha_1 \le \alpha_2 \le \alpha_3 \le \frac{\pi}{4}$. Put $\alpha_1 = \beta$, $\alpha_2 = \alpha$ and $\alpha_3 = \gamma$.

We cover by $S(\alpha)$ the points:

$$(r_0, 0), \quad a\left(r_0 - \frac{1}{\sin\alpha + \cos\alpha}, \frac{1}{\sin\alpha + \cos\alpha}\right) \text{ and } b\left(r_0 - \frac{1}{\cos\alpha}, 0\right).$$

Moreover, we cover by $S(\beta)$ the part of $R(r_0)$ as in the left-hand picture in Fig. 8.

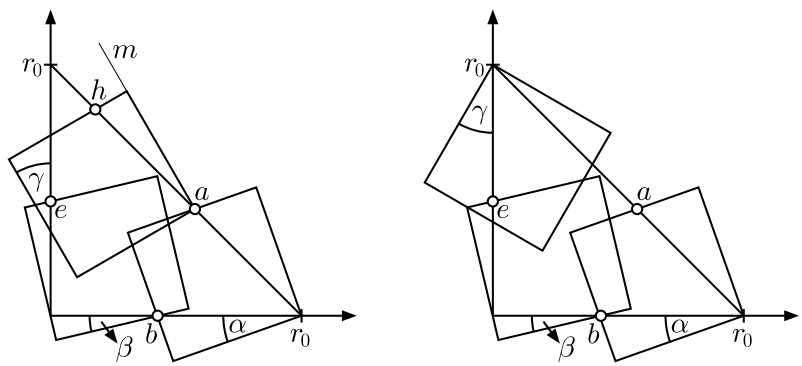

Fig. 8.

Now, we show that $S(\gamma)$ can cover at the same time both points a and $e(0, y_e)$, where

$$y_e = \frac{1 - x_b \sin\beta}{\cos\beta} = \frac{1 - \left(r_0 - \frac{1}{\cos\alpha}\right)\sin\beta}{\cos\beta}.$$

Observe that $y_a \le y_e$. Indeed, $\alpha \ge \beta$ implies that

$$y_a = \frac{1}{\sin\alpha + \cos\alpha} \le \frac{1}{\sin\beta + \cos\beta}$$

and that

$$y_e \ge \frac{1 - \left(r_0 - \frac{1}{\cos\beta}\right)\sin\beta}{\cos\beta}.$$

Moreover, the inequality

$$\frac{1}{\sin\beta + \cos\beta} \le \frac{1 - \left(r_0 - \frac{1}{\cos\beta}\right)\sin\beta}{\cos\beta}$$

is equivalent to

$$r_0 \le \frac{2\cos\beta + \sin\beta}{(\sin\beta + \cos\beta)\cos\beta} = f(\beta).$$

By Remark 2, we deduce that $y_a \le y_e$.

Let m be the straight line described by the equality

$$y - y_a = -\frac{1}{\tan\gamma} \cdot (x - x_a), \quad \text{i.e.,}$$

$$y \sin\gamma + x \cos\gamma - y_a \sin\gamma - x_a \cos\gamma = 0.$$

The distance between e and m is equal to

$$d(e,m) = |y_e \sin\gamma - y_a \sin\gamma - x_a \cos\gamma| = (y_a - y_e)\sin\gamma + x_a \cos\gamma.$$

Since $y_a - y_e \leq 0$, it follows that

$$d(e,m) \leq x_a \cos\gamma \leq x_a \cos\alpha = \big(r_0 - \frac{1}{\sin\alpha + \cos\alpha}\big)\cos\alpha.$$

By $r_0 \leq f(\alpha)$ (see Remark 2), we obtain

$$d(m,e) \leq \frac{2\cos\alpha + \sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha} = 1.$$

It is easy to see that the part of $R(r_0)$, not covered by $S(\alpha)$, $S(\beta)$ and $S(\gamma)$, can be covered translatively by $S(\alpha_4)$ (the distance between the points $(0, r_0)$ and h in the left-hand picture in Fig. 8 is not greater than $r_0\sqrt{2} - 2 < \sqrt{2}/2$).

Case 2: at least three angles from among $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ are not smaller than $\pi/4$. Obviously, if $\alpha \geq \pi/4$ then $\pi/2 - \alpha \leq \pi/4$. We proceed in a similar way as in Case 1. In that case, we start with covering of the left side of $R(r_0)$ instead of the bottom.

Case 3: exactly two angles from among $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ are not greater than $\pi/4$.

We can assume that $\alpha_1 \leq \alpha_2 \leq \frac{\pi}{4} < \alpha_3 \leq \alpha_4$.

Subcase 3A: $\alpha_2 + \alpha_3 \leq \pi/2$. Put $\alpha_1 = \beta$, $\alpha_2 = \alpha$ and $\gamma = \frac{\pi}{2} - \alpha_3$. Obviously, $\gamma \geq \alpha$. We cover by $S(\alpha)$ and $S(\beta)$ the part of $R(r_0)$ as in the right-hand picture in Fig. 8.

Observe that $S(\alpha_3)$ can cover at the same time two points $(0, r_0)$ and e. The reason is that

$$\frac{1}{\cos\gamma} + y_e \geq \frac{1}{\cos\beta} + \frac{1 - \big(r_0 - \frac{1}{\cos\beta}\big)\sin\beta}{\cos\beta},$$

and that the inequality

$$\frac{2}{\cos\beta} - \frac{r_0 \sin\beta}{\cos\beta} + \frac{\sin\beta}{\cos^2\beta} \geq r_0$$

is equivalent to

$$r_0 \leq \frac{2\cos\beta + \sin\beta}{(\sin\beta + \cos\beta)\cos\beta} = f(\beta).$$

Consequently, by Remark 2, we conclude that $\frac{1}{\cos\gamma} + y_e \geq r_0$.

We cover by $S(\alpha_3)$ the part of $R(r_0)$ as in the right-hand picture in Fig. 8. It is easy to see that the part of $R(r_0)$ not covered by $S(\alpha_1)$, $S(\alpha_2)$ and $S(\alpha_3)$ can be covered translatively by $S(\alpha_4)$.

Subcase 3B: $\alpha_2 + \alpha_3 > \pi/2$. Put $\beta = \frac{\pi}{2} - \alpha_4$, $\alpha = \frac{\pi}{2} - \alpha_3$ and $\gamma = \alpha_2$. Obviously, $\beta \leq \alpha < \gamma$. We proceed in a similar way as in Subcase 3A. In that case we cover by $S(\alpha_3)$ and $S(\alpha_4)$ the left side of $R(r_0)$. ∎

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