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EFFECTIVE ENERGY INTEGRAL FUNCTIONALS FOR THIN FILMS IN THE ORLICZ–SOBOLEV SPACE SETTING

Abstract. We consider an elastic thin film as a bounded open subset ω of $\mathbb{R}^2$. First, the effective energy functional for the thin film ω is obtained, by Γ-convergence and $3D$-$2D$ dimension reduction techniques applied to the sequence of re-scaled total energy integral functionals of the elastic cylinders $\omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ as the thickness ε goes to 0. Then we prove the existence of minimizers of the film energy functional. These results are proved in the case when the energy density function for the elastic cylinders has the growth prescribed by an Orlicz convex function M. Here M is assumed to be non-power-growth-type and to satisfy the conditions Δ_2 and ∇_2 (that is equivalent to the reflexivity of Orlicz and Orlicz–Sobolev spaces generated by M). These results extend results of H. Le Dret and A. Raoult for the case $M(t) = |t|^p$ for some $p \in (1, \infty)$.

Introduction

The mathematical theory of nonlinear elasticity has a long history with major contributions from L. Euler, J. Bernoulli, A. Cauchy, G. Kirchhoff, A. E. Love, T. von Karman and many modern authors (see [4, 8, 16, 26]). One of main problems in this research is to understand relations between three-dimensional and two-dimensional theories for thin domains.

We consider an elastic thin film as a bounded open subset $\omega \subset \mathbb{R}^2$ with Lipschitz boundary. The set $\Omega_\varepsilon := \omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \subset \mathbb{R}^3$ for a small thickness ε is considered as an elastic cylinder approximate to the film ω. A three-dimensional deformation $U_\varepsilon : \Omega_\varepsilon \to \mathbb{R}^3$, defined on the thin cylinder Ω_ε, has elastic energy

$$\int_{\Omega_\varepsilon} W(DU_\varepsilon)\, dx$$

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and one seeks to understand the behavior as $\varepsilon \to 0$ of minimizers subject to appropriate boundary conditions. For solving this problem, there were investigated the limiting energies as $\varepsilon \to 0$ of the sequence of re-scaled elastic energies with different scales, for instance, of the energies

$$\frac{1}{\varepsilon^3} \int_{\Omega_\varepsilon} W(DU_\varepsilon)dx \quad \text{or} \quad \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} W(DU_\varepsilon)dx.$$

Here, the first re-scaled elastic energy agrees with the expression considered by G. Kirchhoff (see references in [8, 16, 26]) and the second was studied by H. Le Dret and A. Raoult in 1995 [25].

Let the energy density function $W : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ have the growth prescribed by an Orlicz convex function M. In the present paper, we investigate the above problem by the use of the second re-scaled elastic energy for the thin cylinder Ω_ε, assuming M is non-power-growth-type and satisfies the conditions Δ_2 and ∇_2 (that is equivalent to the reflexivity of Orlicz and Orlicz–Sobolev spaces generated by M).

Main results of the present paper (see Theorem 3.1 and Corollary 3.2) extend results established by H. Le Dret and A. Raoult in [25, Theorem 2, Theorem 8] (cf. [4, Theorem 12.2.1]) for the case of thin films in the reflexive Sobolev space setting with $M(t) = |t|^p$, for some $p \in (1, \infty)$.

Roughly speaking, in Theorem 3.1, the effective energy functional for the thin film ω is obtained, by Γ-convergence and $3D$-$2D$ dimension reduction techniques applied to the sequence of the re-scaled total energy integral functionals of the elastic cylinders Ω_ε as the thickness ε goes to 0. In Corollary 3.2, the existence of minimizers of the energy functional for the thin film is established by showing that some sequence of re-scaled minimizers weakly converges in an appropriate Orlicz–Sobolev space to a minimizer of the film energy functional.

Recall that various concrete examples of M with $M \in \Delta_2 \cap \nabla_2$ can be found in [24, Theorem 7.1, pp. 58–59] and [27, 28]. Furthermore, the assumption $M \in \Delta_2 \cap \nabla_2$ is indispensable in the regularity study of minimizers of multiple variational integrals with the M-growth on Orlicz–Sobolev spaces (see discussions and references for many other concrete examples in [12]).

In Section 4, we give the proofs of Theorem 3.1 and Corollary 3.2. Our proof scheme extends the proof scheme of H. Le Dret and A. Raoult [25]. For these proofs we apply also results: for Orlicz convex functions [21, Proposition 4], for Orlicz–Sobolev spaces [23, Theorem 5, Theorem 7] (cf. [11]), [18, Proposition 2.1], and for quasiconvex integral functionals and quasiconvexification in the Orlicz–Sobolev space setting [13].

1. Some terminology and notation

From now on, unless stated to the contrary, $M\colon \mathbb{R} \to [0,\infty)$ is assumed to be a non-power-growth-type Orlicz N-function (i.e., even convex function satisfying $\lim_{t\to 0} \frac{M(t)}{t} = 0$ and $\lim_{t\to+\infty} \frac{M(t)}{t} = +\infty$).

We assume $M \in \Delta_2 \cap \nabla_2$. Here, the condition $M \in \Delta_2$ means that $M(2t) \le c\, M(t)$ $(t \ge t_0)$, for some $t_0 \in [0,\infty)$ and $c \in (0,\infty)$. The condition $M \in \nabla_2$ means that $\exists\, l > 1, \exists\, t_* \in [0,\infty)$ such that $M(t) \le \frac{1}{2l} M(lt)$, for all $t \ge t_*$.

Let M^* be the complementary (conjugate) Orlicz N-function of M defined by $M^*(\tau) := \sup\{t\tau - M(t) : t \in \mathbb{R}\}$. It is known that the condition $M \in \nabla_2$ is equivalent to the condition $M^* \in \Delta_2$.

Denote by $|v|$ the Euclidean norm of v and by (u,v) the scalar product. Given an open bounded subset $G \subset \mathbb{R}^N$ with Lipschitz (e.g., C^2-smooth) boundary ∂G equipped with the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$. Denote by $L^M(G;\mathbb{R}^m)$ the Orlicz space of all (equivalent classes of) measurable functions $u\colon G \to \mathbb{R}^m$ equipped with the Luxemburg norm

$$\|u\|_{L^M(G;\mathbb{R}^m)} := \inf\{\lambda > 0 : \int_\Omega M(|u(x)|/\lambda)dx \le 1\}.$$

It is known that $M \in \Delta_2 \cap \nabla_2$ is equivalent to the reflexivity of $L^M(G;\mathbb{R}^m)$.

Recall that the Orlicz–Sobolev space $W^{1,M}(G;\mathbb{R}^3)$ is defined as the Banach space of $\mathbb{R}^3$-valued functions u of $L^M(G;\mathbb{R}^3)$ with the Sobolev–Schwartz distributional derivative $Du \in L^M(G;\mathbb{R}^{3\times N})$ equipped with the norm

$$\|u\|_{W^{1,M}(G;\mathbb{R}^3)} := \|u\|_{L^M(G;\mathbb{R}^3)} + \|Du\|_{L^M(G;\mathbb{R}^{3\times N})} < \infty.$$

The subspace $W_0^{1,M}(G;\mathbb{R}^3)$ is defined as the closure in $\|\cdot\|_{W^{1,M}(G;\mathbb{R}^3)}$-norm of the set $C_0^\infty(G;\mathbb{R}^3)$ of C^∞-smooth $\mathbb{R}^3$-valued functions with compact support in G. Since ∂G is Lipschitz and $M, M^* \in \Delta_2$, by [15, Theorems 2.1, 2.3], there exists the bounded linear trace operator

$$\mathrm{Tr} : W^{1,M}(G;\mathbb{R}^3) \to L^M(\partial G;\mathbb{R}^3)$$

such that: (i) $\mathrm{Tr}(u) = u_{|\partial G}$ $(\forall u \in C^\infty(\overline{G}))$ and (ii) $u \in W_0^{1,M}(G;\mathbb{R}^3)$ if and only if $\mathrm{Tr}(u) = 0$. So, for the simplicity of notation we will write "$u(x) = \varphi(x)$ on A" for $u \in W^{1,M}(G;\mathbb{R}^3)$ and $\varphi \in L^M(\partial G;\mathbb{R}^3)$ and $A \subset \partial G$ if $\mathrm{Tr}(u)(x) = \varphi(x)$ for almost every $x \in A$. Due to this reason, we also denote by "u on A" for "$\mathrm{Tr}(u)$ on A", etc.

By [2, Proof of Theorem 3.9] and [20, Proof of Lemma 2.2], given a normed subspace $(X, \|\cdot\|_{W^{1,M}(G;\mathbb{R}^3)})$ and $\Lambda \in X^*$, there exist $h_0, h_1, \ldots, h_N \in$

$L^{M^*}(G;\mathbb{R}^3)$ such that

$$\Lambda(u) = \int_G (h_0, u)dx + \sum_{i=1}^{N} \int_G (h_i, \frac{\partial u}{\partial x_i})dx \quad (u \in X). \tag{1.1}$$

Conversely, every functional Λ defined by (1.1) in the case $h_0, h_1, \ldots, h_N \in L^{M^*}(G;\mathbb{R}^3)$, is an element of X^*.

2. Setup

Define $I := (-\frac{1}{2}, \frac{1}{2})$, $\Omega := \omega \times I$, $S^{\pm} := \omega \times \{\pm\frac{1}{2}\}$, $\Gamma := \partial\omega \times I$, and for each $\varepsilon > 0$, $S^{\pm}_{\varepsilon} := \omega \times \{\pm\frac{\varepsilon}{2}\}$, $\Gamma_{\varepsilon} := \partial\omega \times \varepsilon I$. Greek indexes will be used to distinguish the first two components of a vector, for instance (x_α) and (x_α, x_3), designates (x_1, x_2) and (x_1, x_2, x_3), respectively. We denote by $\mathbb{R}^{3\times 3}$ and $\mathbb{R}^{3\times 2}$ the vector spaces of respectively 3×3 and 3×2 real-valued matrices. Given $\bar{F} \in \mathbb{R}^{3\times 2}$ and $b \in \mathbb{R}^3$, denote by $(\bar{F}|b)$ the 3×3 matrix whose first two columns are those of $\bar{F}$ and the last column is b. By the analogous way, set $e_\alpha := (e_1|e_2) \in \mathbb{R}^{3\times 2}$ where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. Set $D_\alpha U := (\frac{\partial U}{\partial x_1}|\frac{\partial U}{\partial x_2})$, $D_3 U := \frac{\partial U}{\partial x_3}$, $DU := (D_\alpha U|D_3 U)$ for an $\mathbb{R}^3$-valued function U. Denote by C, $\widetilde{C}$ generic positive constants that may vary from line to line.

Let $W\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ be a continuous function satisfying the M-growth-type and coercivity conditions:

$$\frac{1}{C}(M(|F|) - 1) \leq W(F) \leq C(1 + M(|F|)) \quad (\forall F \in \mathbb{R}^{3\times 3}), \tag{2.1}$$

for some $C \in (0, \infty)$. Set

$$\widetilde{\Psi}_\varepsilon := \{U \in W^{1,M}(\Omega_\varepsilon;\mathbb{R}^3) : U(\tilde{x}) = \tilde{x} \text{ on } \Gamma_\varepsilon\}. \tag{2.2}$$

We consider the variational integral functional $\widetilde{J}_\varepsilon : \widetilde{\Psi}_\varepsilon \to \mathbb{R}$, where $\widetilde{J}_\varepsilon(U)$ (the re-scaled total energy of the elastic cylinder Ω_ε under a deformation $U : \Omega_\varepsilon \to \mathbb{R}^3$) is represented by the difference of the re-scaled bulk and surface energies:

$$\widetilde{J}_\varepsilon(U) := \frac{1}{\varepsilon}\Big(\int_{\Omega_\varepsilon} W(DU)d\tilde{x} - \int_{\Omega_\varepsilon} (f_\varepsilon, U)d\tilde{x}\Big) - \frac{1}{\varepsilon}\Big(\int_{S^+_\varepsilon} (g^+_\varepsilon, U)d\mathcal{H}^2 + \int_{S^-_\varepsilon} (g^-_\varepsilon, U)d\mathcal{H}^2\Big). \tag{2.3}$$

Here, $f_\varepsilon := f\left(\tilde{x}_\alpha, \frac{\tilde{x}_3}{\varepsilon}\right)$, $f \in L^{M^*}(\Omega;\mathbb{R}^3)$, $g^{\pm}_\varepsilon(\cdot, \pm\frac{\varepsilon}{2}) \in L^{M^*}(\omega;\mathbb{R}^3)$ and $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure in $\mathbb{R}^3$. We assume that $\frac{1}{\varepsilon}g^{\pm}_\varepsilon(\cdot, \pm\frac{\varepsilon}{2}) = g^{\pm}(\cdot, \pm\frac{1}{2})$.

Let $W_0 : \mathbb{R}^{3\times 2} \to \mathbb{R}$ be defined by

$$W_0(\bar{F}) = \inf_{z\in\mathbb{R}^3} W((\bar{F}|z)). \tag{2.4}$$

By [25, Proof of Proposition 1, p. 554], W_0 is continuous as the continuous function W satisfies the condition (2.1). Set

$$\overline{\Psi}_0 := \{\bar{u} \in W^{1,M}(\omega;\mathbb{R}^3) : \bar{u}(x_\alpha) = (x_\alpha, 0) \text{ on } \partial\omega\}. \tag{2.5}$$

Let $\bar{J}_0 : \overline{\Psi}_0 \to \mathbb{R}$ be defined by

$$\bar{J}_0(\bar{z}) := \int_\omega \mathcal{Q}W_0(D_\alpha \bar{z})\, dx_\alpha - \int_\omega (\mathcal{F}, \bar{z}) dx_\alpha \tag{2.6}$$

where $\mathcal{Q}W_0$ is the quasiconvex envelope of W_0 and

$$\mathcal{F}(x_\alpha) := \int_I f(x_\alpha, x_3)\, dx_3 + g^+\left(x_\alpha, \frac{1}{2}\right) + g^-\left(x_\alpha, -\frac{1}{2}\right). \tag{2.7}$$

Remind that the quasiconvex envelope $\mathcal{Q}g : \mathbb{R}^{m\times n} \to \mathbb{R}$ of a continuous function $g : \mathbb{R}^{m\times n} \to \mathbb{R}$ is defined (see [6, Definition 6.3], [9, Theorem 6.9]) by

$$\mathcal{Q}g(E) := \inf\left\{ \frac{1}{\operatorname{meas}(B)} \int_B g(E + D\varphi)\, dx : \varphi \in C_0^\infty(B;\mathbb{R}^m) \right\}, \tag{2.8}$$

for all $E \in \mathbb{R}^{m\times n}$ where B is the open unit ball of $\mathbb{R}^n$.

3. The formulation of main results

Let $\mathcal{Z}$ be the space of membrane deformations defined by

$$\mathcal{Z} = \{z \in W^{1,M}(\Omega;\mathbb{R}^3) : D_3 z = 0,\ z(x) = (x_\alpha, 0) \text{ on } \Gamma\}. \tag{3.1}$$

Observe that $\mathcal{Z}$ is canonically isomorphic to $\overline{\Psi}_0$ [29, Theorem 1.1.3/1]. Let $\bar{z}$ denote the element of $\overline{\Psi}_0$ that is associated with $z \in \mathcal{Z}$ through this isomorphism:

$$z(x_\alpha, x_3) = \bar{z}(x_\alpha) \text{ a.e.} \tag{3.2}$$

Since we want to identify the sequence convergence with the thickness of our domain tending to zero, for simplicity we assume this thickness parameter ε takes its values in a sequence $\varepsilon_n \to 0$.

Theorem 3.1. *Let $\widetilde{J}_\varepsilon$ be defined in* (2.3) *and $\bar{J}_0$ be defined in* (2.6). *Assume $M \in \Delta_2 \cap \nabla_2$. Assume the continuous function $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfies the conditions* (2.1). *Let $\{U_\varepsilon\} \in \widetilde{\Psi}_\varepsilon$. For each $\varepsilon > 0$ and $\tilde{x} = (\tilde{x}_\alpha, \tilde{x}_3) \in \Omega_\varepsilon$ we associate $x = (x_\alpha, x_3) := \left(\tilde{x}_\alpha, \frac{1}{\varepsilon}\tilde{x}_3\right) \in \Omega$ and we set $z_\varepsilon(x_\alpha, x_3) := U_\varepsilon(\tilde{x}_\alpha, \tilde{x}_3)$.*

Then the sequence $\widetilde{J}_\varepsilon$ converges in $L^M(\Omega;\mathbb{R}^3)$-norm to $\bar{J}_0$ in the following sense:

(i) *(lower bound) if $z_\varepsilon \to z$ in $L^M(\Omega;\mathbb{R}^3)$-norm, $\|z_\varepsilon\|_{W^{1,M}(\Omega;\mathbb{R}^3)} < +\infty$ and $z \in \mathcal{Z}$ with $z(x_\alpha, x_3) = \bar{z}(x_\alpha)$ through the isomorphism* (3.2), *then*

$$\liminf_{\varepsilon\to 0} \widetilde{J}_\varepsilon(U_\varepsilon) \ge \bar{J}_0(\bar{z}),$$

(ii) *(attainment of lower bound) for every* $\bar{z} \in \overline{\Psi}_0$ *there exists a sequence* U_ε *such that* $z_\varepsilon \to z$ *in* $L^M(\Omega;\mathbb{R}^3)$*-norm, where* $\|z_\varepsilon\|_{W^{1,M}(\Omega;\mathbb{R}^3)} < +\infty$ *with* $z(x_\alpha, x_3) = \bar{z}(x_\alpha)$ *through the isomorphism* (3.2) *and*

$$\lim_{\varepsilon\to 0} \widetilde{J}_\varepsilon(U_\varepsilon) = \bar{J}_0(\bar{z}).$$

Consider the asymptotic behavior of $U_\varepsilon \in \widetilde{\Psi}_\varepsilon$ such that

$$\widetilde{J}_\varepsilon(U_\varepsilon) \le \inf_{U\in\widetilde{\Psi}_\varepsilon} \widetilde{J}_\varepsilon(U) + \gamma(\varepsilon), \tag{3.3}$$

where γ is a positive function such that $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Corollary 3.2. (The minimization problem) *Assume* $U_\varepsilon \in \widetilde{\Psi}_\varepsilon$ *satisfies* (3.3). *Let the functions* M, W *and* $z_\varepsilon, \bar{z}$ *be such as in Theorem* 3.1.
Then:

(i) *the sequence* z_ε *is relatively weakly compact in* $W^{1,M}(\Omega;\mathbb{R}^3)$*;*
(ii) *the set* $\mathcal{C}_{\text{film}}$ *of cluster points of the sequence* z_ε *in the weak topology is a non-empty subset of* $\mathcal{Z}$*;*
(iii) *any point* z_* *of* $\mathcal{C}_{\text{film}}$ *can be identified with* $\bar{z}_* \in \overline{\Psi}_0$ *by the* 3*D*-2*D* *dimension reduction isomorphism* (3.2) *and* $\bar{z}_*$ *is a solution of the minimization problem*

$$\inf_{\bar{u}\in\overline{\Psi}_0} \bar{J}_0(\bar{u}).$$

4. The proofs of Theorem 3.1 and Corollary 3.2

We will reformulate Theorem 3.1 and Corollary 3.2 by the use of the following equivalent functionals J_ε^1 and J_0 (see the re-formulation in Theorem 4.1 and Corollary 4.2). Define

$$u_{0,\varepsilon}(x) := (x_\alpha, \varepsilon x_3), \quad u_{0,0}(x) := (x_\alpha, 0). \tag{4.1}$$

Notice that after the change of variables as in Theorem 3.1 with the association

$$x = (x_\alpha, x_3) := \left(\tilde{x}_\alpha, \frac{1}{\varepsilon}\tilde{x}_3\right),\ u(x_\alpha, x_3) := U(\tilde{x}_\alpha, \tilde{x}_3), \tag{4.2}$$

the re-scaled energy $\widetilde{J}_\varepsilon(U)$ in (2.3) can be rewritten in the equivalent form

$$J_\varepsilon(u) = \int_\Omega W\left(D_\alpha u \Big| \frac{D_3 u}{\varepsilon}\right) dx - \int_\Omega (f, u)\, dx - \left(\int_{S^+} (g^+, u)\, d\mathcal{H}^2 + \int_{S^-} (g^-, u)\, d\mathcal{H}^2\right), \tag{4.3}$$

where u is an element of

$$\Psi_\varepsilon := \{u \in W^{1,M}(\Omega;\mathbb{R}^3) : u(x) = u_{0,\varepsilon}(x) \text{ on } \Gamma\}. \tag{4.4}$$

Observe that the re-scaled displacement $v = u - u_{0,\varepsilon}$ belongs to the set

$$V = W^{1,M}_{\Gamma}(\Omega;\mathbb{R}^3) := \{v \in W^{1,M}(\Omega;\mathbb{R}^3) : v(x) = 0 \text{ on } \Gamma\}$$

and

$$J_\varepsilon(v + u_{0,\varepsilon}) = \int_\Omega W(e_\alpha + D_\alpha v | e_3 + \frac{D_3 v}{\varepsilon}) dx \\ - \int_\Omega (f, u_{0,\varepsilon} + v)\, dx - \Big(\int_{S^+} (g^+, u_{0,\varepsilon} + v)\, d\mathcal{H}^2 + \int_{S^-} (g^-, u_{0,\varepsilon} + v)\, d\mathcal{H}^2 \Big).$$

Since the direct consideration of J_ε would imply the study involving the weak topology of the Orlicz–Sobolev space $W^{1,M}(\Omega;\mathbb{R}^3)$ which is non-metrizable on unbounded sets, then it is needed to extend J_ε to the functional $J^1_\varepsilon : L^M(\Omega;\mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ by

$$J^1_\varepsilon(v) = \begin{cases} J_\varepsilon(v + u_{0,\varepsilon}), & \text{if } v \in V, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.5}$$

Let $\mathcal{V}$ be the space of membrane displacements defined by

$$\mathcal{V} = \{v \in W^{1,M}(\Omega;\mathbb{R}^3) :\ D_3 v = 0,\ v(x) = 0 \text{ on } \Gamma\} \subset V. \tag{4.6}$$

Similarly as in (3.1)–(3.2), $\mathcal{V}$ is canonically isomorphic to $W^{1,M}_0(\omega;\mathbb{R}^3)$ [29, Theorem 1.1.3/1]. Let $\bar{v}$ denote the element of $W^{1,M}_0(\omega;\mathbb{R}^3)$ that is associated with $v \in \mathcal{V}$ through this isomorphism:

$$v(x_\alpha, x_3) = \bar{v}(x_\alpha) \text{ a.e.} \tag{4.7}$$

Define

$$J_0(v + u_{0,0}) = \int_\omega \mathcal{Q}W_0(e_\alpha + D_\alpha \bar{v})\ dx_\alpha - \int_\omega (\mathcal{F}, u_{0,0} + \bar{v}) dx_\alpha. \tag{4.8}$$

In this notion, we have for $U_\varepsilon \in \widetilde{\Psi}_\varepsilon$

$$\widetilde{J}_\varepsilon(U_\varepsilon) = J_\varepsilon(u_\varepsilon) = J_\varepsilon(v_\varepsilon + u_{0,\varepsilon}) = J^1_\varepsilon(v_\varepsilon),$$

where $u_\varepsilon \in \Psi_\varepsilon$, $v_\varepsilon \in V$ and

$$\bar{J}_0(\bar{z}) = J_0(v + u_{0,0}), \quad (v \in \mathcal{V},\ \bar{z} = \bar{v} + u_{0,0} \in \overline{\Psi}_0).$$

Recall [10], [6, Definition 7.1] that a sequence of functions I_ε from a metric space X to $\overline{R}$ is said to Γ-converge to I_0 for the topology of X if the following conditions are satisfied, for all $x \in X$:

$$\begin{cases} \forall x_\varepsilon \to x, & I_0(x) \le \liminf I_\varepsilon(x_\varepsilon), \\ \exists y_\varepsilon \to y, & I_\varepsilon(y_\varepsilon) \to I_0(y). \end{cases} \tag{4.9}$$

Theorem 4.1. *Let J^1_ε be defined in* (4.5) *and J_0 be defined in* (4.8). *Assume $M, M^* \in \Delta_2$. Assume the continuous function $W\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfies the conditions* (2.1).

Then the sequence J^1_ε Γ*-converges in* $L^M(\Omega;\mathbb{R}^3)$*-norm to some functional* $J^1_\infty : L^M(\Omega;\mathbb{R}^3) \to \mathbb{R}\cup\{+\infty\}$ *as* $\varepsilon \to 0$ *and moreover* $J^1_\infty(v) = J_0(v+u_{0,0})$*, for all* $v \in \mathcal{V}$*.*

Consider the asymptotic behavior of $u_\varepsilon \in \Psi_\varepsilon$ such that

$$J_\varepsilon(u_\varepsilon) \le \inf_{u\in\Psi_\varepsilon} J_\varepsilon(u) + \gamma(\varepsilon), \tag{4.10}$$

where γ is a positive function such that $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Corollary 4.2. (The minimization problem) *Let the functions* M *and* W *be such as in Theorem* 4.1*. Assume* $u_\varepsilon \in \Psi_\varepsilon$ *satisfies* (4.10)*.*

Then:

- (i) *the sequence* u_ε *is relatively weakly compact in* $W^{1,M}(\Omega;\mathbb{R}^3)$*;*
- (ii) *the set* $\mathcal{C}_{\text{film}}$ *of cluster points of the sequence* u_ε *in the weak topology is a non-empty subset of* $\mathcal{Z}$*;*
- (iii) *any point* u_* *of* $\mathcal{C}_{\text{film}}$ *can be identified with* $\bar{u}_* \in \overline{\Psi}_0$ *by the* 3*D*-2*D dimension reduction isomorphism* (3.2) *and* $\bar{u}_*$ *is a solution of the minimization problem*

$$\inf_{\bar{u}\in\overline{\Psi}_0} \bar{J}_0(\bar{u}).$$

We start the proofs of Theorem 4.1 and Corollary 4.2, with the following Lemmas 4.3–4.4.

We consider the following condition (4.11):

$$\begin{aligned} &\exists\, i(M) \in [1,\infty), \exists\, c \in (0,\infty), \exists\, a(M) \in (0,1] \quad \text{such that} \\ &M(at) \le c\, a^{i(M)} M(t) \quad (\forall t \ge 0, \forall a \le a(M)). \end{aligned} \tag{4.11}$$

The condition (4.11) is equivalent to the following condition (4.12):

$$\begin{aligned} &\exists\, i(M) \in [1,\infty), \exists\, c \in (0,\infty), \exists\, b(M) \in [1,\infty) \quad \text{such that} \\ &\frac{1}{c} b^{i(M)} M(s) \le M(bs) \quad (\forall s \ge 0, \forall b \ge b(M)). \end{aligned} \tag{4.12}$$

Recall [21] that the condition $M \in \Delta^q$ ($q \ge 1$) means the existence of $K > 0$ such that $K\lambda^q M(s) \ge M(\lambda s)$, for $s \ge 0$ and $\lambda \ge 1$. Furthermore, the condition $M \in \Delta^{*p}$ ($p \ge 1$) means the existence of $K > 0$ such that $K\lambda^p M(s) \le M(\lambda s)$, for $s \ge 0$ and $\lambda \ge 1$.

Note that the condition (4.12) with $i(M) = p$ is equivalent to the condition Δ^{*p} of M. In fact, if the condition (4.12) holds with $b(M) > 1$ then for $b \in [1, b(M)]$, we have

$$b^{i(M)} M(s) \le b(M)^{i(M)} M(s) \le b(M)^{i(M)} M(bs)$$

as $M(0) = 0$ and M is increasing on $[0,\infty)$. Therefore, the condition Δ^{*p} of M holds with $p = i(M)$ and $K = \min\{1/c, 1/b(M)^{i(M)}\}$.

The explanation given above shows that the following Lemma 4.3 is a reformulation of a part of [21, Proposition 4] (see the proof of the implications $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ of [21, Proposition 4]).

Lemma 4.3. *Assume the dual Orlicz N-function M^* satisfies the condition Δ_2^{glob}, i.e. $M^*(2\tau) \leq K\, M^*(\tau)$, for all $\tau \in [0,\infty)$ and for some $K \in (0,\infty)$.*

Then $M^* \in \Delta^q$ for some $q \in (1,\infty)$ and M satisfies the condition (4.11) *for $i(M) = \frac{q}{q-1} \in (1,\infty)$.*

Lemma 4.4. (Compactness) *Let M and W be such as in Theorem* 4.1. *Let $v_\varepsilon \in L^M(\Omega;\mathbb{R}^3)$ be a sequence such that*

$$\sup_{\varepsilon\in(0,1)} J_\varepsilon^1(v_\varepsilon) \leq d < +\infty. \tag{4.13}$$

Then there exists $\bar{d} > 0$ such that:

(i)

$$\sup_{\varepsilon\in(0,1)} \|v_\varepsilon\|_{W^{1,M}(\Omega;\mathbb{R}^3)} \leq \bar{d} < +\infty \tag{4.14}$$

and the sequence v_ε is relatively weakly compact;

(ii) *the set of cluster points of the sequence v_ε in the weak topology $\sigma(V,V^*)$ is a non-empty subset of $\mathcal{V}$.*

Proof. We divide the proof into Steps 4.1–4.6, where in Steps 4.2–4.5, we assume additionally $M^* \in \Delta_2^{glob}$.

Step 4.1. By (4.13) and (4.5) for J_ε^1, $v_\varepsilon \in V$, for all $\varepsilon > 0$. Denote $u_\varepsilon = v_\varepsilon + u_{0,\varepsilon}$. We claim that

$$\begin{aligned}\int_\Omega M\left(\left|\left(D_\alpha u_\varepsilon\Big|\frac{D_3 u_\varepsilon}{\varepsilon}\right)\right|\right)dx \leq C_1 + C_1((\|f\|_{L^{M^*}(\Omega;\mathbb{R}^3)} \\ + (\|g^+\|_{L^{M^*}(S^+;\mathbb{R}^3)} + \|g^-\|_{L^{M^*}(S^-;\mathbb{R}^3)})\|\operatorname{Tr}\|_{\mathcal{L}})\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})}),\end{aligned} \tag{4.15}$$

for some $C_1 \in (0,+\infty)$ and for all $\varepsilon \in (0,1)$. Here $\|\operatorname{Tr}\|_{\mathcal{L}} := N^+ + N^-$, where N^+ (resp., N^-) denotes the operator norm of the linear trace operator $\operatorname{Tr} : W^{1,M}(\Omega;\mathbb{R}^3) \to L^M(S^+;\mathbb{R}^3)$ (resp., $\operatorname{Tr} : W^{1,M}(\Omega;\mathbb{R}^3) \to L^M(S^-;\mathbb{R}^3)$).

For this, by the coercivity condition (2.1) together with (4.13), we infer that

$$\frac{1}{C}\bigg(\int_\Omega M(|(D_\alpha u_\varepsilon|\frac{D_3 u_\varepsilon}{\varepsilon})|)dx - |\Omega|\bigg) \tag{4.16}$$
$$\le d + \bigg|\int_\Omega (f, u_\varepsilon)\,dx\bigg| + \bigg|\int_{S^+}(g^+, u_\varepsilon)\,d\mathcal{H}^2\bigg| + \bigg|\int_{S^-}(g^-, u_\varepsilon)\,d\mathcal{H}^2\bigg|$$
$$= d + \bigg|\int_\Omega (f, u_\varepsilon)\,dx\bigg| + \bigg|\int_{S^+}(g^+, \mathrm{Tr}(u_\varepsilon))\,d\mathcal{H}^2\bigg| + \bigg|\int_{S^-}(g^-, \mathrm{Tr}(u_\varepsilon))\,d\mathcal{H}^2\bigg|.$$

By the generalized Hölder inequality (see, e.g., [31, Theorems 13.13, 13.11], [24, 34]), we deduce that

$$\frac{1}{C}\bigg(\int_\Omega M(|(D_\alpha u_\varepsilon|\frac{D_3 u_\varepsilon}{\varepsilon})|)dx - |\Omega|\bigg) \tag{4.17}$$
$$\begin{aligned}&\le d + 2\|f\|_{L^{M^*}(\Omega;\mathbb{R}^3)}\|u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)}\\ &\quad + 2(\|g^+\|_{L^{M^*}(S^+;\mathbb{R}^3)}\|\mathrm{Tr}(u_\varepsilon)\|_{L^M(S^+;\mathbb{R}^3)}\\ &\quad + \|g^-\|_{L^{M^*}(S^-;\mathbb{R}^3)}\|\mathrm{Tr}(u_\varepsilon)\|_{L^M(S^-;\mathbb{R}^3)})\\ &\le d + 2\|f\|_{L^{M^*}(\Omega;\mathbb{R}^3)}\|u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} + 2(\|g^+\|_{L^{M^*}(S^+;\mathbb{R}^3)}\\ &\quad + \|g^-\|_{L^{M^*}(S^-;\mathbb{R}^3)})\|\mathrm{Tr}\|_{\mathcal{L}}(\|u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)}\\ &\quad + \|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})}).\end{aligned}$$

By the $W^{1,M}_\Gamma$-generalization (see [23, Theorem 5, Theorem 7] together with [11, Theorem 3.9], [19, Lemma 4.14], [18, Proposition 2.1]) for the Poincaré–Sobolev-type inequality (see [30, Theorem 3.6.4]), there exists $\widetilde{C} \in (0,\infty)$ such that

$$\begin{aligned}\|u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} &\le \widetilde{C}\bigg(\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})} + \int_\Gamma |u_\varepsilon|d\mathcal{H}^2\bigg)\\ &= \widetilde{C}\bigg(\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})} + \int_\Gamma |u_{0,\varepsilon}|d\mathcal{H}^2\bigg)\\ &\le \widetilde{C}(\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})} + \mathcal{H}^2(\Gamma)\sup_{x\in\Omega}|x|) < \infty,\end{aligned} \tag{4.18}$$

for all $\varepsilon \in (0,1)$.

Then (4.17)–(4.18) imply (4.15).

STEP 4.2. By the additional assumption $M^* \in \Delta_2^{glob}$, we may apply Lemma 4.3, and so M satisfies the condition (4.11) for some $i(M) \in (1,\infty)$.

We claim that

$$\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})} \le C_2 < \infty, \quad (\forall \varepsilon \in (0,1)), \tag{4.19}$$
$$\|u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} \le C_3 < \infty, \quad (\forall \varepsilon \in (0,1)), \tag{4.20}$$

$$\int_\Omega M(|(D_\alpha u_\varepsilon|\frac{D_3 u_\varepsilon}{\varepsilon})|)dx \le C_4 < \infty, \quad (\forall \varepsilon \in (0,1)), \tag{4.21}$$

for some C_2, C_3, C_4.

For this, by (4.15), we infer that

$$\frac{1}{1 + \|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})}} \int_\Omega M(|Du_\varepsilon|)dx \le C_5 < \infty, \quad (\forall \varepsilon \in (0,1)), \tag{4.22}$$

for some C_5.

Consider the case when $\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})}/2 \ge a(M)^{-1} > 0$, where $a(M) \in (0,\infty)$ in (4.11). Since

$$0 < \frac{\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})}}{2} < \|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})},$$

by the definition of the Luxemburg norm and by (4.11), we deduce that

$$\begin{aligned} 1 &< \int_\Omega M\left(\frac{|Du_\varepsilon|}{\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})}/2}\right)dx \\ &\le \left(\frac{2}{\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})}}\right)^{i(M)} \int_\Omega M(|Du_\varepsilon|)dx, \quad (\forall \varepsilon \in (0,1)). \end{aligned} \tag{4.23}$$

Therefore, (4.22) and (4.23) imply that

$$A(\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})}) \le C_5 < \infty \tag{4.24}$$

whenever $\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})} \ge 2\,a(M)^{-1}$. Here

$$A(s) := \frac{s^{i(M)}}{2^{i(M)}(1+s)}.$$

Since $i(M) > 1$, $A(s) \to +\infty$ as $s \to +\infty$, and so there exists $C_6 \in (0,\infty)$ such that $A(s) > C_5$ $(\forall s > C_6)$. Hence, (4.24) implies the claim (4.19): $\|Du_\varepsilon\|_{L^M(\Omega;\mathbb{R}^{3\times 3})} \le C_2 := \min\{C_6, 2\,a(M)^{-1}\}$, $(\forall \varepsilon \in (0,1))$. By (4.18) and (4.15), we deduce the claims (4.20) and (4.21).

Step 4.3. Obviously,

$$C_7 := \sup_{\varepsilon \in (0,1)} \|u_{0,\varepsilon}\|_{W^{1,M}(\Omega;\mathbb{R}^3)} < +\infty.$$

Therefore, (4.19)–(4.20) imply (4.14):

$$\sup_{\varepsilon \in (0,1)} \|v_\varepsilon\|_{W^{1,M}(\Omega;\mathbb{R}^3)} \le \bar{d} := C_2 + C_3 + C_7 < \infty. \tag{4.25}$$

Step 4.4. We claim that

$$\lim_{\varepsilon \to 0} \|D_3 u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} = 0. \tag{4.26}$$

For this, by the convexity of M and $M(0) = 0$,

$$M(t) = M\left(\frac{\varepsilon^{-1}t}{\varepsilon^{-1}}\right) \leq \frac{1}{\varepsilon^{-1}} M(\varepsilon^{-1}t), \quad (\forall \varepsilon \in (0,1)).$$

Since $|(z_\alpha|\varepsilon^{-1}z_3)| \geq \varepsilon^{-1}|z_3|$, we deduce, by (4.21) that

$$\begin{aligned} 0 &\leq \int_\Omega M(|D_3u_\varepsilon|)dx \leq \varepsilon \int_\Omega M(\varepsilon^{-1}|D_3u_\varepsilon|)dx \\ &\leq \varepsilon \int_\Omega M(|(D_\alpha u_\varepsilon|\varepsilon^{-1}D_3u_\varepsilon)|)dx \leq \varepsilon \cdot C_4 < \infty, \quad (\forall \varepsilon \in (0,1)). \end{aligned}$$

Hence

$$\lim_{\varepsilon \to 0} \int_\Omega M(|D_3u_\varepsilon|)dx = 0. \tag{4.27}$$

It is known (see, e.g., [24], [31]) that (4.27) implies (4.26) as $M \in \Delta_2$.

Step 4.5. It is known (see, e.g., [20, Theorems 1.1, 3.3]) that $W^{1,M}(\Omega;\mathbb{R}^3)$ is a separable reflexive Banach space as $M, M^* \in \Delta_2$. By the reflexivity and separability of the closed subspace $V = W^{1,M}_\Gamma(\Omega;\mathbb{R}^3)$ of $W^{1,M}(\Omega;\mathbb{R}^3)$, the Alaoglu–Bourbaki theorem together with [22, Theorem V.7.6] imply that any closed ball of V equipped with the weak topology is compact and metrizable. Therefore, (4.14) implies the existence of some cluster point of the sequence v_ε in the weak topology of V.

Now, let v be a cluster point in the weak topology $\sigma(V, V^*)$. Analogously, (4.19)–(4.20) imply that there exist $u \in W^{1,M}(\Omega;\mathbb{R}^3)$ and a subsequence (not relabeled) of the sequence u_ε such that u_ε converges weakly to u in $W^{1,M}(\Omega;\mathbb{R}^3)$. Then it is easy to check by the representation (1.1) that $v_\varepsilon = u_\varepsilon - u_{0,\varepsilon}$ converges weakly to $u - u_{0,0}$ in $W^{1,M}(\Omega;\mathbb{R}^3)$. Therefore, $u - u_{0,0} = v$ and D_3u_ε converges to D_3u in the weak topology $\sigma(L^M(\Omega;\mathbb{R}^3), L^{M^*}(\Omega;\mathbb{R}^3))$. By (4.26) and the generalized Hölder inequality [31, Theorems 13.13, 13.11], for every $y \in L^{M^*}(\Omega;\mathbb{R}^3)$, we deduce that

$$\begin{aligned} \left|\int_\Omega (y, D_3u)dx\right| &= \lim_{\varepsilon \to 0}\left|\int_\Omega (y, D_3u_\varepsilon)dx\right| \\ &\leq \lim_{\varepsilon \to 0} 2\|y\|_{L^{M^*}(\Omega;\mathbb{R}^3)}\|D_3u_\varepsilon\|_{L^M(\Omega;\mathbb{R}^3)} = 0. \end{aligned}$$

Therefore, $\int_\Omega (y, D_3u)dx = 0$, for every $y \in L^{M^*}(\Omega;\mathbb{R}^3)$, and so $D_3u = 0$ a.e. Since $D_3u_{0,0} = 0$, $D_3v = 0$ follows, and so $v \in \mathcal{V}$.

Step 4.6. Now consider the general assumption $M, M^* \in \Delta_2$. By [24, (4.5) in p. 24], there exists some Orlicz N-function $N_1 \in \Delta_2^{glob}$ such that

$$N_1(\tau) = M^*(\tau), \quad (\forall \tau \geq \tau_0),$$

for some $\tau_0 \in (0, \infty)$. Let $M_1 := N_1^*$. By known results of the theory of N-functions and Orlicz spaces [24, 31, 33], we deduce the following assertions: $(M^*)^* = M$, $M_1^* = (N_1^*)^* = N_1 \in \Delta_2^{glob}$, $L_{M^*} = L_{N_1}$ and $L_M = L_{(M^*)^*} \cong (L_{M^*})^* = (L_{N_1})^* \cong L_{N_1^*}$ with equivalent norms, $L_M = L_{N_1^*} = L_{M_1}$ and $M_1 = N_1^* \in \Delta_2$ and $(L_M)^* = (L_{M_1})^* \cong L_{M^*} = L_{M_1^*}$ with equivalent norms.

So, $M_1 \in \Delta_2$, $M_1^* \in \Delta_2^{glob}$, $W_0^{1,M}(\Omega;\mathbb{R}^3) = W_0^{1,M_1}(\Omega;\mathbb{R}^3)$, $W^{1,M}(\Omega;\mathbb{R}^3) = W^{1,M_1}(\Omega;\mathbb{R}^3)$ with equivalent norms.

Furthermore, we deduce that the continuous function $W\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfying the conditions (2.1) with respect to M, satisfies the conditions (2.1) with respect to M_1:

$$\frac{1}{C'}(M_1(|F|) - 1) \le W(F) \le C'(1 + M_1(|F|)), \quad (\forall F \in \mathbb{R}^{3\times 3}),$$

for some $C' \in (0, \infty)$.

Therefore, we can apply the results of Steps 4.1–4.5 with respect to M_1 in place of M. Then by the above assertions for relations between M, M^* and M_1, M_1^*, we deduce all assertions of Lemma 4.4 with respect to M under the general assumption $M, M^* \in \Delta_2$. ∎

Corollary 4.5. *If $v \in L^M(\Omega;\mathbb{R}^3)$, but $v \notin \mathcal{V}$ then $J^1_\infty(v) = +\infty$.*

Proof. This follows from Lemma 4.4 by the same argument as in [25, Proof of Corollary 4, p. 555] ∎

Lemma 4.6. (The lower bound) *For all $v \in \mathcal{V}$ with $\bar{v}$ from the isomorphism* (4.7)*, we have that*

$$J^1_\infty(v) \ge \int_\omega \mathcal{Q}W_0(e_\alpha + D_\alpha \bar{v})\, dx_\alpha - \int_\omega (\mathcal{F}, u_{0,0} + \bar{v}) dx_\alpha.$$

Proof. Fix $v \in \mathcal{V}$. Using $D_3 v = 0$, we infer that $\overline{C} := \sup_{\varepsilon\in(0,1)} J_\varepsilon(v + u_{0,\varepsilon}) < +\infty$. Then, by the definition of Γ-convergence (4.9), $J^1_\infty(v) \le \overline{C} < +\infty$. By (4.9), there exists a sequence $v_\varepsilon \in V$ such that $v_\varepsilon \to v$ in $L^M(\Omega;\mathbb{R}^3)$-norm and $J^1_\varepsilon(v_\varepsilon) \to J^1_\infty(v)$.

Then, by Lemma 4.4, there exists some subsequence (not relabeled) such that v_ε converges to v in the weak topology $\sigma(V, V^*)$. Hence, it is easy to check by the representation (1.1) and the isomorphism (4.7) and by the Fubini theorem that

$$\int_\Omega (f, u_{0,\varepsilon} + v_\varepsilon)\, dx + \int_{S^+} (g^+, u_{0,\varepsilon} + v_\varepsilon) d\mathcal{H}^2 + \int_{S^-} (g^-, u_{0,\varepsilon} + v_\varepsilon) d\mathcal{H}^2 \quad \to \quad \int_\omega (\mathcal{F}, u_{0,0} + \bar{v}) dx_\alpha \tag{4.28}$$

where $\mathcal{F}$ is from (2.7).

Define $u_\varepsilon := v_\varepsilon + u_{0,\varepsilon}$. By (2.4) and (2.8), we infer that

$$\int_\Omega W(D_\alpha u_\varepsilon | \varepsilon^{-1} D_3 u_\varepsilon) dx \geq \int_\Omega W_0(D_\alpha u_\varepsilon) dx \geq \int_\Omega \mathcal{Q}W_0(D_\alpha u_\varepsilon) dx. \tag{4.29}$$

Define $Z : \mathbb{R}^{3\times 3} \to \mathbb{R}$ by

$$Z(z_\alpha | z_3) := \mathcal{Q}W_0(z_\alpha).$$

By the same arguments as in [25, p. 556] and in [25, Proof of Proposition 1, p. 554], we deduce that Z is quasiconvex (i.e., $Z = \mathcal{Q}Z$) and W_0, Z satisfy the conditions (cf. (2.1)):

$$\begin{aligned} &\frac{1}{C}(M(|\bar{F}|) - 1) \leq W_0(\bar{F}) \leq C(1 + M(|\bar{F}|)), \\ &-\frac{1}{C} \leq Z(F) \leq C(1 + M(|F|)), \end{aligned} \tag{4.30}$$

for all $\bar{F} \in \mathbb{R}^{3\times 2}$ and all $F \in \mathbb{R}^{3\times 3}$ and for some $C \in (0, \infty)$. Define $G : W^{1,M}(\Omega; \mathbb{R}^3) \to \mathbb{R}$ by

$$G(u) := \int_\Omega Z(Du)\, dx = \int_\Omega \mathcal{Q}W_0(D_\alpha u)\, dx. \tag{4.31}$$

Then by the $W^{1,M}$-generalization [13, Theorem 3.1] of the Acerbi–Fusco weak lower semi-continuity $W^{1,p}$-theorem [1], G is sequentially weakly lower semi-continuous on $W^{1,M}(\Omega; \mathbb{R}^3)$. Since u_ε converges weakly to $u := v + u_{0,0}$ in $W^{1,M}(\Omega; \mathbb{R}^3)$, (4.29) implies that

$$\begin{aligned} \liminf_{\varepsilon\to 0} \int_\Omega W(D_\alpha u_\varepsilon | \varepsilon^{-1} D_3 u_\varepsilon) dx &\geq \liminf_{\varepsilon \to 0} G(u_\varepsilon) \geq G(u) \\ &= \int_\Omega \mathcal{Q}W_0(e_\alpha + D_\alpha v) dx = \int_\omega \mathcal{Q}W_0(e_\alpha + D_\alpha \bar{v}) dx_\alpha. \end{aligned} \tag{4.32}$$

Hence, by (4.32) and (4.28), the statement of Lemma 4.6 follows. ∎

Lemma 4.7. (The upper bound) *For all $v \in \mathcal{V}$ with $\bar{v}$ from the isomorphism* (4.7)*, we have that*

$$J^1_\infty(v) \leq \int_\omega \mathcal{Q}W_0(e_\alpha + D_\alpha \bar{v})\ dx_\alpha - \int_\omega (\mathcal{F}, u_{0,0} + \bar{v}) dx_\alpha.$$

Proof. We divide the proof into Steps 4.7–4.11.

Step 4.7. Let $\bar{v} \in W^{1,M}_0(\omega; \mathbb{R}^3)$ from the isomorphism (4.7) for $v \in \mathcal{V}$ and set $\bar{u} := \bar{v} + u_{0,0}$. We claim that

$$J^1_\infty(v) \leq G_1(\bar{v}), \quad (\forall v \in \mathcal{V}) \tag{4.33}$$

where $G_1 : W_0^{1,M}(\omega;\mathbb{R}) \to \mathbb{R}$ is defined by

$$G_1(\bar{v}) := \int_\omega W_0(e_\alpha + D_\alpha\bar{v})\,dx_\alpha - \int_\omega(\mathcal{F}, u_{0,0} + \bar{v})\,dx_\alpha \tag{4.34}$$
$$= \int_\omega W_0(D_\alpha\bar{u})\,dx_\alpha - \int_\omega(\mathcal{F}, u_{0,0} + \bar{v})\,dx_\alpha.$$

For this, for every $w \in W_0^{1,M}(\omega;\mathbb{R}^3)$, define

$$v_\varepsilon(x) := \bar{v}(x_\alpha) + \varepsilon x_3 w(x_\alpha).$$

It is easy to check that $v_\varepsilon \to v$ in $W^{1,M}(\Omega;\mathbb{R}^3)$-norm.

Under the conditions (2.1), the continuous function W generates the continuous superposition operator $\Lambda_W : L^M(\Omega;\mathbb{R}^{3\times3}) \to L^1(\Omega;\mathbb{R})$ defined by $\Lambda_W(\tilde{u})(x) := W(\tilde{u}(x))$ (see, e.g., [3, Theorem 3], [32, Theorem 3.2]). Hence the functional $\tilde{u} \mapsto \int_\Omega \Lambda_W(\tilde{u})dx$ is continuous on $L^M(\Omega;\mathbb{R}^{3\times3})$. Therefore,

$$\int_\Omega W(e_\alpha + D_\alpha v_\varepsilon|e_3 + \frac{D_3 v_\varepsilon}{\varepsilon})dx = \int_\Omega W(D_\alpha(\bar{u} + \varepsilon x_3 w)|e_3 + w)dx \to$$
$$\int_\Omega W(D_\alpha\bar{u}|e_3 + w)dx = \int_\omega W(D_\alpha\bar{u}|e_3 + w)dx_\alpha.$$

Hence as in (4.28), we deduce that

$$J_\varepsilon^1(v_\varepsilon) \to \int_\omega W(D_\alpha\bar{u}|e_3 + w)dx_\alpha - \int_\omega(\mathcal{F}, u_{0,0} + \bar{v})dx_\alpha.$$

By (4.9) we infer that

$$J_\infty^1(v) \le \inf_{w\in W_0^{1,M}(\omega;\mathbb{R}^3)} \int_\omega W(D_\alpha\bar{u}|e_3 + w)dx_\alpha - \int_\omega(\mathcal{F}, u_{0,0} + \bar{v})dx_\alpha.$$

Analogously, we deduce that the functional $\tilde{u} \mapsto \int_\omega \Lambda_W(\tilde{u})dx_\alpha$ is $L^M(\omega;\mathbb{R}^{3\times3})$ -norm-continuous. It is known (see [11, Lemma 2.1], [2]) that the set $C_0^\infty(\omega;\mathbb{R}^3) \subset W_0^{1,M}(\omega;\mathbb{R}^3)$ is $L^M(\omega;\mathbb{R}^3)$-norm-dense in $L^M(\omega;\mathbb{R}^3)$. Hence, we infer that

$$\inf_{w\in W_0^{1,M}(\omega;\mathbb{R}^3)} \int_\omega W(D_\alpha\bar{u}|e_3 + w)dx_\alpha \tag{4.35}$$
$$= \inf_{w\in C_0^\infty(\omega;\mathbb{R}^3)} \int_\omega W(D_\alpha\bar{u}|e_3 + w)dx_\alpha$$
$$= \inf_{w\in L^M(\omega;\mathbb{R}^3)} \int_\omega W(D_\alpha\bar{u}|e_3 + w)dx_\alpha.$$

Since W is continuous function, the function $g : \omega \times \mathbb{R}^3 \to \mathbb{R}$, defined by $g(x,z) := W(D_\alpha\bar{u}(x)|e_3 + z)$, is a Carathéodory function. This means that there exists $\omega_0 \subset \omega$ such that $\mathcal{L}^2(\omega \setminus \omega_0) = 0$ and $g(x,\cdot)$ is continuous for any $x \in \omega_0$ and $g(\cdot,z)$ is measurable for any $z \in \mathbb{R}^3$. Fix $\delta \in (0,\infty)$.

The set $M_\delta := \{(x,z) \in \omega_0 \times \mathbb{R}^3 : g(x,z) \le W_0(D_\alpha \bar{u}(x)) + \delta\}$ belongs to $\mathcal{A}_{\mathcal{L}^2}(\omega_0) \times \mathcal{B}(\mathbb{R}^3)$ (where $\mathcal{A}_{\mathcal{L}^2}(\omega_0)$ is the σ-algebra of Lebesgue-measurable subsets of ω_0 and $\mathcal{B}(\mathbb{R}^3)$ is the σ-algebra of Borel subsets of $\mathbb{R}^3$). Observe that $M_\delta(x) := \{z \in M_\delta : (x,z) \in M_\delta\} \neq \emptyset$. By the Measurable Selection Theorem (see, e.g. [7, 17]), there exists a measurable function w_δ such that $w_\delta(x) \in M_\delta(x)$ $(x \in \omega_0)$. Hence,

$$W(D_\alpha \bar{u}(x) | e_3 + w_\delta(x)) \le \delta + W_0(D_\alpha \bar{u}(x)),$$

for almost every $x \in \omega$. By the coercivity condition (2.1), we infer that $w_\delta \in L^M(\omega; \mathbb{R}^3)$. Therefore,

$$\begin{aligned} \inf_{w \in L^M(\omega;\mathbb{R}^3)} \int_\omega W(D_\alpha \bar{u} | e_3 + w) dx_\alpha &\le \int_\omega W(D_\alpha \bar{u} | e_3 + w_\delta) dx_\alpha \\ &\le \int_\omega W_0(D_\alpha \bar{u}) dx_\alpha + \delta |\omega| \quad (\forall \delta \in (0, +\infty)). \end{aligned}$$

So

$$\inf_{w \in L^M(\omega;\mathbb{R}^3)} \int_\omega W(D_\alpha \bar{u} | e_3 + w) dx_\alpha \le \int_\omega W_0(D_\alpha \bar{u}) dx_\alpha. \tag{4.36}$$

By (4.7), (4.35) and (4.36), we infer (4.33).

Step 4.8. Define $\tilde{G}_1 : L^M(\Omega; \mathbb{R}^3) \to \mathbb{R} \cup \{\infty\}$ by $\tilde{G}_1(v) = G_1(\bar{v})$ if $v \in \mathcal{V}$ and $\bar{v} \in W_0^{1,M}(\omega; \mathbb{R}^3)$ from the isomorphism (4.7), $\tilde{G}_1(v) = +\infty$ otherwise. Then by Corollary 4.5, (4.33) implies

$$J^1_\infty(v) \le \tilde{G}_1(v), \quad (\forall v \in L^M(\Omega; \mathbb{R}^3)). \tag{4.37}$$

Let $\Pi(\tilde{G}_1)$ denote the lower semicontinuous envelope of $\tilde{G}_1$ on $L^M(\Omega; \mathbb{R}^3)$. Since J^1_∞ is lower semicontinuous on $L^M(\Omega; \mathbb{R}^3)$ as the Γ-limit in (4.9) (see, e.g., [6, Remark 7.3/(i)]), (4.37) implies

$$J^1_\infty(v) \le \Pi(\tilde{G}_1)(v). \tag{4.38}$$

Step 4.9. By the $W_0^{1,M}(\omega; \mathbb{R}^3)$-generalization of Dacorogna's $W_0^{1,p}(\omega; \mathbb{R}^3)$-relaxation theorem [9, Theorem 9.1] in the case $M \in \Delta_2$, we deduce, under the condition (4.30) for W_0, that the sequential weak lower semicontinuous envelope $\Pi(G_1)$ of G_1 on $W_0^{1,M}(\omega; \mathbb{R}^3)$ is calculated by

$$\Pi(G_1)(\bar{v}) = \int_\omega \mathcal{Q}W_0(e_\alpha + D_\alpha \bar{v})\, dx_\alpha - \int_\omega (\mathcal{F}, u_{0,0} + \bar{v}) dx_\alpha. \tag{4.39}$$

Step 4.10. We claim that G_1 is coercive in the sense:

$$G_1(\bar{v}) \to +\infty \text{ as } \|\bar{v}\|_{W_0^{1,M}(\omega;\mathbb{R}^3)} \to +\infty. \tag{4.40}$$

For this, observe that $\lambda = \int_\Omega M^*(|f(x_\alpha, x_3)|) dx < \infty$ as $f \in L^{M^*}(\Omega; \mathbb{R}^3)$ and $M^* \in \Delta_2$ (see [24, 34]). By the Fubini theorem and the Jensen inequality

[24, p. 62], we infer that

$$\infty > \lambda = \int_\omega \left(\int_I M^*(|f(x_\alpha, x_3)|)\, dx_3 \right) dx_\alpha \geq \int_\omega M^* \left(\int_I |f(x_\alpha, x_3)|\, dx_3 \right) dx_\alpha.$$

Hence, $\int_I |f(\cdot, x_3)| dx_3 \in L^{M^*}(\omega; \mathbb{R}^3)$ that implies $\mathcal{F} \in L^{M^*}(\omega; \mathbb{R}^3)$.

Assume that $G_1(\bar{v}) \leq d_1 < \infty$. By (4.30) for W_0 and by the generalized Hölder inequality [31, Theorems 13.13, 13.11], we deduce that

$$\frac{1}{C} \int_\omega (M(|e_\alpha + D_\alpha \bar{v}|) - 1) dx_\alpha \leq d_1 + \left| \int_\omega (\mathcal{F}, u_{0,0} + \bar{v}) dx_\alpha \right|$$
$$\leq d_1 + 2\|\mathcal{F}\|_{L^{M^*}(\omega;\mathbb{R}^3)} \|u_{0,0}\|_{L^M(\omega;\mathbb{R}^3)} + 2\|\mathcal{F}\|_{L^{M^*}(\omega;\mathbb{R}^3)} \|\bar{v}\|_{L^M(\omega;\mathbb{R}^3)}.$$

Observe that the Poincaré-type inequality of [19, Corollary 5.8] implies that $\|\bar{v}\|_{L^M(\omega;\mathbb{R}^3)} \leq \widetilde{C} \|D_\alpha \bar{v}\|_{L^M(\omega;\mathbb{R}^{3\times 2})}$ ($\bar{v} \in W_0^{1,M}(\omega; \mathbb{R}^3)$).

The convexity of M implies that $M(|e_\alpha + D_\alpha \bar{v}|) \geq 2M(\frac{1}{2}|D_\alpha \bar{v}|) - M(\sqrt{2})$.

First, we assume additionally $M^* \in \Delta_2^{glob}$. Then we may apply Lemma 4.3, and so M satisfies the condition (4.11), for some $i(M) \in (1, \infty)$. By the arguments analogous to the arguments from (4.17) up to (4.25) and the end of Step 4.2 in the proof of Lemma 4.4, we infer that

$$\|\bar{v}\|_{W_0^{1,M}(\omega;\mathbb{R}^3)} \leq h(d_1) < \infty \text{ whenever } G_1(\bar{v}) \leq d_1 < \infty, \tag{4.41}$$

for some function $h : (0, \infty) \to (0, \infty)$. This is equivalent to (4.40).

Now consider the general assumption $M, M^* \in \Delta_2$. Let M_1 be the Orlicz N-function defined in Step 4.6 such that $M_1 \in \Delta_2$ and $M_1^* \in \Delta_2^{glob}$. By all arguments and assertions obtained in Step 4.6 for relations between M, M^* and M_1, M_1^*, we can apply the coerciveness property (4.40) proved before with respect to M_1 in place of M, and then we deduce the coerciveness property (4.40) with respect to M under the general assumption $M, M^* \in \Delta_2$.

Step 4.11. Since $W_0^{1,M}(\omega; \mathbb{R}^3)$ is a reflexive separable Banach space and G_1 satisfies (4.40), then (see, e.g., [14, Proposition 3.16])

$$\Pi(G_1)(\bar{v}) = \min\{ \liminf G_1(\bar{v}_\varepsilon) : \text{ a sequence } \bar{v}_\varepsilon \text{ converges weakly to } \bar{v} \text{ in } W_0^{1,M}(\omega; \mathbb{R}^3)\}.$$

Then, using the isomorphism (4.7) and the representation (1.1), further arguments which are analogous to the arguments in [25, Proof of Lemma 5, pages 555–556] imply that

$$\Pi(\widetilde{G}_1)(v) = \Pi(G_1)(\bar{v}), \quad (v \in \mathcal{V}). \tag{4.42}$$

Hence, (4.38)–(4.39) and (4.42) imply the upper bound in Lemma 4.7. ∎

Proof of Theorem 4.1. The assertions of Theorem 4.1 follow from Corollary 4.5 for the case $v \notin \mathcal{V}$ and from Lemmas 4.6 and 4.7 for the case $v \in \mathcal{V}$. ■

Proof of Corollary 4.2. Observe that $v_\varepsilon := u_\varepsilon - u_{0,\varepsilon}$ belongs to V. By (4.10),

$$J_\varepsilon(u_\varepsilon) = J^1_\varepsilon(v_\varepsilon) \le \inf_{v\in V} J^1_\varepsilon(v) + \gamma(\varepsilon). \tag{4.43}$$

It is easy to check that

$$J^1_\varepsilon(0) = J_\varepsilon(u_{0,\varepsilon}) = \int_\Omega W(e_\alpha|e_3)dx + \\ + \left(- \int_\Omega (f, u_{0,\varepsilon})dx - \int_{S^+} (g^+, u_{0,\varepsilon})d\mathcal{H}^2 + \int_{S^-} (g^-, u_{0,\varepsilon})d\mathcal{H}^2 \right) \le C < +\infty,$$

for some C and for all $\varepsilon \in (0,1)$. Hence, (4.43) implies that $\sup_{\varepsilon\in(0,1)} J^1_\varepsilon(v_\varepsilon) < +\infty$. Therefore, by Lemma 4.4, the sequence v_ε is relatively weakly compact and the set of cluster points of the sequence v_ε in the weak topology $\sigma(V, V^*)$ is a non-empty subset of $\mathcal{V}$.

By (4.43) and (4.5),

$$J^1_\varepsilon(v_\varepsilon) \le \inf_{v\in V} J^1_\varepsilon(v) + \gamma(\varepsilon) = \inf_{v\in L^M(\Omega;\mathbb{R}^3)} J^1_\varepsilon(v) + \gamma(\varepsilon). \tag{4.44}$$

Fix $\tilde v \in L^M(\Omega;\mathbb{R}^3)$. By Theorem 4.1 and (4.9), there exists a sequence $\tilde v_\varepsilon \in L^M(\Omega;\mathbb{R}^3)$ such that $\tilde v_\varepsilon \to \tilde v$ in $L^M(\Omega;\mathbb{R}^3)$-norm and $J^1_\varepsilon(\tilde v_\varepsilon) \to J^1_\infty(\tilde v)$. Therefore, applying Theorem 4.1, (4.9), (4.44) and the assumption $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$, we infer that

$$\begin{aligned} J_0(v_* + u_{0,0}) = J^1_\infty(v_*) &\le \liminf_{\varepsilon\to 0} J^1_\varepsilon(v_\varepsilon) \le \liminf_{\varepsilon\to 0}(J^1_\varepsilon(\tilde v_\varepsilon) + \gamma(\varepsilon)) \\ &= J^1_\infty(\tilde v) = J_0(\tilde v + u_{0,0}). \end{aligned}$$

Using the isomorphism (4.7) and the representation (1.1), we re-write the statements obtained above for v_ε and v_*. By this way, we deduce all statements of Corollary 4.2. ■

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Let us inform that we have recently obtained results in the setting of the Orlicz–Sobolev space $W^{1,M}$ that extend other known results for thin films in the case $M(t) = |t|^p$ for some $p \in (1,\infty)$. In particular, our results extend results obtained by G. Friesecke, R. D. James and S. Müller in 2002 [16] for rigid thin films, and by G. Bouchitté, I. Fonseca and M. L. Mascarenhas in 2004 [5] for thin films with bending moment. Their proofs require other techniques and we will discuss these issues in our forthcoming papers.

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