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FIXED POINT ITERATIONS OF THREE ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS

Abstract. In this paper, we establish the strong convergence for a modified three-step iterative scheme with errors associated with three mappings in real Banach spaces. Moreover, our technique of proofs is of independent interest. Remark at the end simplifies many known results.

1. Introduction

Let E be a real normed space and K be a nonempty convex subset of E. Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 \text{ and } ||f^*|| = ||x||\},$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality map by j.

Let $T : D(T) \subset E \to E$ be a mapping with domain $D(T)$ in E.

Definition 1. The mapping T is said to be uniformly L-Lipschitzian, if there exists $L > 0$ such that, for all $x, y \in D(T)$

$$\|T^n x - T^n y\| \leq L \|x - y\| .$$

Definition 2. T is said to be nonexpansive if for all $x,\ y \in D(T)$, the following inequality holds:

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x,\ y \in D(T).$$

Definition 3. T is said to be asymptotically nonexpansive [8], if there exists a sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for all } x,\ y \in D(T),\ n \geq 0.$$

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Definition 4. T is said to be asymptotically pseudocontractive, if there exists a sequence $\{k_n\}_{n\geq 0} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n \|x-y\|^2, \text{ for all } x,\ y \in D(T),\ n \geq 0.$$

Remark 1. 1. It is easy to see that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian.

2. If T is asymptotically nonexpansive mapping then, for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\begin{aligned}\langle T^n x - T^n y, j(x-y)\rangle &\leq \|T^n x - T^n y\| \|x-y\| \\ &\leq k_n \|x-y\|^2,\ n \geq 0.\end{aligned}$$

Hence, every asymptotically nonexpansive mapping is asymptotically pseudocontractive.

3. Rhoades, in [14], showed that the class of asymptotically pseudocontractive mappings properly contains the class of asymptotically nonexpansive mappings.

4. The asymptotically pseudocontractive mappings were introduced by Schu [15].

Definition 5. Suppose that there exists a strictly increasing function $\psi : [0,\infty) \to [0,\infty)$, $\psi(0) = 0$. T is said to be ψ-asymptotically pseudocontractive, if there exists a sequence $\{k_n\}_{n\geq 0} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - p, j(x-p)\rangle \leq k_n ||x-p||^2 - \psi(||x-p||),\ \forall x, p \in K.$$

In recent years, Mann and Ishikawa iterative schemes [9, 11] have been studied extensively by many authors.

Let $T : K \to K$ be a mapping and K be a nonempty convex subset of E.

(a) The Mann iteration process is defined by the sequence $\{x_n\}$

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-b_n)\, x_n + b_n T x_n,\ n \geq 0, \end{cases} \tag{1.1}$$

where $\{b_n\}$ is a sequence in $[0,1]$.

(b) The sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-b_n)\, x_n + b_n T y_n, \\ y_n = (1-b'_n)\, x_n + b'_n T x_n,\ n \geq 0, \end{cases} \tag{1.2}$$

where $\{b_n\}, \{b'_n\}$ are sequences in $[0,1]$, is known as the Ishikawa iteration process.

In 1995, Liu [10] introduced iterative schemes with errors as follows:

(c) The sequence $\{x_n\}$ in K iteratively defined by:

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-b_n)x_n + b_n T y_n + u_n, \\ y_n = (1-b'_n)x_n + b'_n T x_n + v_n, \ n \geq 0, \end{cases} \tag{1.3}$$

where $\{b_n\}, \{b'_n\}$ are sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$ are sequences in K satisfying $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$, is known as Ishikawa iterative scheme with errors.

(d) The sequence $\{x_n\}$ iteratively defined by:

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-b_n)x_n + b_n T x_n + u_n, \ n \geq 0, \end{cases} \tag{1.4}$$

where $\{b_n\}$ is a sequence in $[0,1]$ and $\{u_n\}$ is a sequence in K satisfying $\sum_{n=1}^{\infty} \|u_n\| < \infty$, is known as Mann iterative scheme with errors.

While it is clear that consideration of error terms in iterative schemes is an important part of the theory, it is also clear that the iterative schemes with errors introduced by Liu [10], as in (c) and (d) above, are not satisfactory. The errors can occur in a random way. The conditions imposed on the error terms in (c) and (d), which say that they tend to zero as n tends to infinity, are, therefore, unreasonable.

In 1998, Xu [16] introduced a more satisfactory error term in the following iterative schemes:

(e) The sequence $\{x_n\}$ iteratively defined by:

$$\begin{cases} x_0 \in K, \\ x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \ n \geq 0, \end{cases} \tag{1.5}$$

with $\{u_n\}$ a bounded sequence in K and $a_n + b_n + c_n = 1$, is known as Mann iterative scheme with errors.

(f) The sequence $\{x_n\}$ iteratively defined by:

$$\begin{cases} x_0 \in K, \\ x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \ n \geq 0, \end{cases} \tag{1.6}$$

with $\{u_n\}, \{v_n\}$ bounded sequences in K and $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$, is known as Ishikawa iterative scheme with errors.

Note that the error terms are now improved Mann and Ishikawa iterative schemes follow as special cases of the above schemes respectively.

In [15], Schu proved the following results:

Theorem 1. *Let K be a nonempty bounded closed convex subset of a Hilbert space H, $T : K \to K$ a completely continuous, uniformly L-Lipschitzian and asymptotically pseudocontractive mapping with sequence $\{k_n\} \subset [1, \infty)$; $q_n = 2k_n - 1$, $\forall n \in N$; $\sum(q_n^2 - 1) < \infty$; $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$; $\epsilon < \alpha_n < \beta_n \leq b$, $\forall n \in N$, and some $\epsilon > 0$ and some $b \in (0, L^{-2}[(1 + L^2)^{\frac{1}{2}} - 1])$. Let $x_1 \in K$, for all $n \in N$, define*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n.$$

Then $\{x_n\}$ converges to some fixed point of T.

Let

$$F(S) = \{x \in K : Sx = x\},$$
$$F(T) = \{x \in K : Tx = x\},$$
$$F(H) = \{x \in K : Hx = x\},$$

be the sets of fixed points of S, T and H. Also

$$F' = F(S) \cap F(T), \quad \text{and} \quad F = F(S) \cap F(T) \cap F(H).$$

In [13], E. U. Ofoedu proved the following results.

Theorem 2. *Let K be a nonempty closed convex subset of a real Banach space E, $T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n \geq 0} \subset [1, \infty)$, $\lim_{n \to \infty} k_n = 1$ such that $p \in F(T)$. Let $\{\alpha_n\}_{n \geq 0} \subset [0, 1]$ be such that $\sum_{n \geq 0} \alpha_n = \infty$, $\sum_{n \geq 0} \alpha_n^2 < \infty$ and $\sum_{n \geq 0} \alpha_n (k_n - 1) < \infty$. For arbitrary $x_0 \in K$, let $\{x_n\}_{n \geq 0}$ be iteratively defined by*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n, \ n \geq 0.$$

Suppose there exists a strictly increasing function $\psi : [0, \infty) \to [0, \infty)$, $\psi(0) = 0$ such that

$$\langle T^n x - p, j(x - p) \rangle \leq k_n ||x - p||^2 - \psi(||x - p||), \ \forall x \in K.$$

Then $\{x_n\}_{n \geq 0}$ converges strongly to p.

Many authors [1, 7] have studied the two mappings case of iterative schemes for different types of mappings.

Let K be a nonempty convex subset of a normed space E and $S, T : K \to K$ be two mappings.

Recently, Agarwal et al. [1] studied the following iteration process for a couple of quasi-contractive mappings.

(g) The sequence $\{x_n\}$ iteratively defined by:

$$\begin{cases} x_0 \in K, \\ x_{n+1} = a_n x_n + b_n S y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \ n \geq 0, \end{cases} \tag{1.7}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in K.

In [2], Chang et al. proved the following results.

Theorem 3. *Let K be a nonempty closed convex subset of a real Banach space E, S,$T : K \to K$ be two uniformly L-Lipschitzian ψ-asymptotically pseudocontractive mappings with sequence $\{k_n\}_{n\geq 0} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ and $p \in F'$. Let $\{\alpha_n\}_{n\geq 0}, \{\beta_n\}_{n\geq 0} \subset [0,1]$ be two sequences such that $\sum_{n\geq 0} \alpha_n = \infty$, $\sum_{n\geq 0} \alpha_n^2 < \infty$, $\sum_{n\geq 0} \beta_n < \infty$ and $\sum_{n\geq 0} \alpha_n (k_n - 1) < \infty$. For arbitrary $x_0 \in K$, let $\{x_n\}_{n\geq 0}$ be a sequence iteratively defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\, x_n + \alpha_n S^n y_n, \\ y_n &= (1-\beta_n)\, x_n + \beta_n T^n x_n, \ n \geq 0. \end{aligned} \tag{1.8}$$

Then $\{x_n\}_{n\geq 0}$ converges strongly to p.

Remark 2. (i) One can see that, with $\sum_{n\geq 0} \alpha_n = \infty$, the conditions $\sum_{n\geq 0} \alpha_n^2 < \infty$, and $\sum_{n\geq 0} \alpha_n (k_n - 1) < \infty$ are not always true. Let us take $\alpha_n = \frac{1}{\sqrt{n}}$ and $k_n = 1 + \frac{1}{\sqrt{n}}$, then obviously $\sum \alpha_n = \infty$, but $\sum \alpha_n^2 = \infty = \sum \alpha_n (k_n - 1)$. Hence, the results of Ofoedu [13] and Chang et al. [2] are to be improved.

(ii) The same argument can be applied on the results of [4–6].

These facts motivated us to introduce and analyze a class of three-step iterative scheme for three ψ-asymptotically pseudocontractive mappings. This scheme is defined as follows:

Let K be a nonempty closed convex subset of a real Banach space E, $S, T, H : K \to K$ be three ψ-asymptotically pseudocontractive mappings at $p \in F$ with common ψ.

(h) The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in K, \\ x_{n+1} = a_n x_n + b_n S^n y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T^n z_n + c'_n v_n, \\ z_n = a''_n x_n + b''_n H^n x_n + c''_n w_n, \ n \geq 0, \end{cases} \tag{1.9}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{a''_n\}$, $\{b''_n\}$, $\{c''_n\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n$ and $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in K.

It can be easily seen that,

(i) for $S = T = H$ and $c_n = 0 = c'_n = c''_n$, (1.9) reduces to [17]

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-b_n)\, x_n + b_n S^n y_n, \\ y_n = (1-b'_n)\, x_n + b'_n T^n z_n, \\ z_n = (1-b''_n)\, x_n + b''_n H^n x_n, \ n \geq 0, \end{cases} \tag{1.10}$$

where $\{b_n\}, \{b'_n\}, \{b''_n\}$ are sequences in $[0,1]$,

(j) for $H = I$ and $c_n = 0 = c'_n = b''_n$, (1.9) reduces to

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-b_n)\, x_n + b_n S^n y_n, \\ y_n = (1-b'_n)\, x_n + b'_n T^n x_n, \ n \geq 0, \end{cases} \tag{1.11}$$

where $\{b_n\}, \{b'_n\}$ are sequences in $[0,1]$,

(k) for $H = T = I$ and $c_n = 0 = c'_n = b'_n = b''_n$, (1.9) reduces to

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-b_n)\, x_n + b_n S^n x_n, \ n \geq 0, \end{cases} \tag{1.12}$$

where $\{b_n\}$ is a sequence in $[0,1]$.

In this paper, we establish the strong convergence for a modified three-step iterative scheme with errors associated with three mappings in real Banach spaces.

2. Main results

The following lemmas are now well known.

Lemma 1. [18] *Let $J : E \to 2^{E^*}$ be the normalized duality mapping. Then for any $x, y \in E$, we have*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

Lemma 2. [12] *Let $\{\theta_n\}$ be a sequence of nonnegative real numbers, $\{\lambda_n\}$ be a real sequence satisfying*

$$0 \leq \lambda_n \leq 1, \ \sum_{n=0}^{\infty} \lambda_n = \infty.$$

Suppose there exists a strictly increasing function $\psi : [0, \infty) \to [0, \infty)$ *with* $\psi(0) = 0$. *If there exists a positive integer* n_0 *such that*

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \psi(\theta_{n+1}) + \sigma_n,$$

for all $n \geq n_0$, *with* $\sigma_n \geq 0$, $\forall n \in \mathbb{N}$, *and* $\sigma_n = o(\lambda_n)$, *then* $\lim_{n\to\infty} \theta_n = 0$.

Theorem 4. *Let* K *be a nonempty closed convex subset of a real Banach space* E, $S, T, H : K \to K$ *be three* ψ*-asymptotically pseudocontractive mappings at* $p \in F$ *with common* ψ *and having* $S(K)$ *and* $T(K)$ *bounded with sequence* $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$. *Further, let* S *be uniformly continuous and* $\{a_n\}_{n\geq 0}, \{b_n\}_{n\geq 0}, \{c_n\}_{n\geq 0}, \{a'_n\}_{n\geq 0}, \{b'_n\}_{n\geq 0}, \{c'_n\}_{n\geq 0}, \{a''_n\}_{n\geq 0}, \{b''_n\}_{n\geq 0}, \{c''_n\}_{n\geq 0}$ *be real sequences in* $[0, 1]$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} b_n = 0 = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c'_n$;
(ii) $\sum_{n\geq 0} b_n = \infty$;
(iii) $c_n = o(b_n)$.

For arbitrary $x_0 \in K$ *let* $\{x_n\}$ *be iteratively defined by* (1.9), *then* $\{x_n\}_{n\geq 0}$ *converges strongly to a common fixed point of* S, T *and* H.

Proof. Since p is a common fixed point of S, T and H, then the set F is nonempty.

Because $S(K)$ and $T(K)$ are bounded, we set

$$(2.1) \quad M_1 = \|x_0 - p\| + \sup_{n\geq 0} \|S^n y_n - p\| + \sup_{n\geq 0} \|T^n z_n - p\| + \sup_{n\geq 0} \|u_n - p\| + \sup_{n\geq 0} \|v_n - p\| + \sup_{n\geq 0} \|w_n - p\|.$$

Obviously $M_1 < \infty$.

It is clear that $||x_0 - p|| \leq M_1$. Let $||x_n - p|| \leq M_1$. Next, we will prove that $||x_{n+1} - p|| \leq M_1$.

By the condition (iii) which implies $c_n = t_n b_n$; $t_n \to 0$ as $n \to \infty$, consider

$$\begin{aligned}
\|x_{n+1} - p\| &= \|a_n x_n + b_n S^n y_n + c_n u_n - p\| \\
&= \|a_n(x_n - p) + b_n(S^n y_n - p) + c_n(u_n - p)\| \\
&\leq (1 - b_n) \|x_n - p\| + b_n \|S^n y_n - p\| + c_n \|u_n - p\| \\
&\leq (1 - b_n) M_1 + b_n \|S^n y_n - p\| + c_n \|u_n - p\| \\
&= (1 - b_n) \big[\|x_0 - p\| + \sup_{n\geq 0} \|S^n y_n - p\| + \sup_{n\geq 0} \|T^n z_n - p\| \\
&\quad + \sup_{n\geq 0} \|u_n - p\| + \sup_{n\geq 0} \|v_n - p\| + \sup_{n\geq 0} \|w_n - p\| \big] \\
&\quad + b_n \|S^n y_n - p\| + c_n \|u_n - p\|
\end{aligned}$$

$$
\begin{aligned}
&\le \|x_0 - p\| + \big((1-b_n)\sup_{n\ge 0}\|S^n y_n - p\| + b_n\|S^n y_n - p\|\big) \\
&\quad + \sup_{n\ge 0}\|T^n z_n - p\| + \big((1-b_n)\sup_{n\ge 0}\|u_n - p\| + c_n\|u_n - p\|\big) \\
&\quad + \sup_{n\ge 0}\|v_n - p\| + \sup_{n\ge 0}\|w_n - p\| \\
&\le \|x_0 - p\| + \big((1-b_n)\sup_{n\ge 0}\|S^n y_n - p\| + b_n\sup_{n\ge 0}\|S^n y_n - p\|\big) \\
&\quad + \sup_{n\ge 0}\|T^n z_n - p\| + \big((1-b_n)\sup_{n\ge 0}\|u_n - p\| + b_n\sup_{n\ge 0}\|u_n - p\|\big) \\
&\quad + \sup_{n\ge 0}\|v_n - p\| + \sup_{n\ge 0}\|w_n - p\| \\
&= \|x_0 - p\| + \sup_{n\ge 0}\|S^n y_n - p\| + \sup_{n\ge 0}\|T^n z_n - p\| + \sup_{n\ge 0}\|u_n - p\| \\
&\quad + \sup_{n\ge 0}\|v_n - p\| + \sup_{n\ge 0}\|w_n - p\| = M_1.
\end{aligned}
$$

So, from the above discussion, we can conclude that the sequence $\{x_n - p\}_{n\ge 0}$ is bounded. Let $M_2 = \sup_{n\ge 0}||x_n - p||$.

Denote $M = M_1 + M_2$. Obviously $M < \infty$.

Consider

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|a_n x_n + b_n S^n y_n + c_n u_n - p\|^2 \\
&= \|a_n(x_n - p) + b_n(S^n y_n - p) + c_n(u_n - p)\|^2 \\
&\le a_n\|x_n - p\|^2 + b_n\|S^n y_n - p\|^2 + c_n\|u_n - p\|^2 \\
&\le \|x_n - p\|^2 + M^2 b_n + M^2 c_n,
\end{aligned} \tag{2.2}
$$

where the first inequality holds by the convexity of $\|.\|^2$.

Now from Lemma 1, for all $n \ge 0$, we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= ||a_n x_n + b_n S^n y_n + c_n u_n - p||^2 \\
&= ||a_n(x_n - p) + b_n(S^n y_n - p) + c_n(u_n - p)||^2 \\
&\le (1-b_n)^2||x_n - p||^2 + 2b_n\langle S^n y_n - p, j(x_{n+1} - p)\rangle \\
&\quad + 2c_n\langle u_n - p, j(x_{n+1} - p)\rangle \\
&\le (1-b_n)^2||x_n - p||^2 + 2b_n\langle S^n x_{n+1} - p,\ j(x_{n+1} - p)\rangle \\
&\quad + 2b_n\langle S^n y_n - S^n x_{n+1}, j(x_{n+1} - p)\rangle + 2M^2 c_n \\
&\le (1-b_n)^2||x_n - p||^2 + 2b_n k_n||x_{n+1} - p||^2 \\
&\quad - 2b_n\psi(\|x_{n+1} - p\|) \\
&\quad + 2b_n\|S^n y_n - S^n x_{n+1}\|\,\|x_{n+1} - p\| + 2M^2 c_n
\end{aligned} \tag{2.3}
$$

$$\le (1-b_n)^2||x_n-p||^2 + 2b_nk_n||x_{n+1}-p||^2 - 2b_n\psi(\|x_{n+1}-p\|) + 2b_nd_n + 2M^2c_n,$$

where

$$d_n = M\,\|S^n y_n - S^n x_{n+1}\|\,. \tag{2.4}$$

From (1.9) we have

$$\begin{aligned}(2.5)\quad &\|y_n - x_{n+1}\| \\ &= \left\|b_n'(T^n z_n - x_n) + c_n'(v_n - x_n) + b_n(x_n - S^n y_n) - c_n(u_n - x_n)\right\| \\ &\le b_n'\,\|T^n z_n - x_n\| + c_n'\,\|v_n - x_n\| + b_n\,\|x_n - S^n y_n\| + c_n\,\|u_n - x_n\| \\ &\le 2M(b_n + c_n + b_n' + c_n').\end{aligned}$$

From the conditions (i), (iii) and (2.5), we obtain

$$\lim_{n\to\infty}\|y_n - x_{n+1}\| = 0,$$

and the uniform continuity of S leads to

$$\lim_{n\to\infty}\|S^n y_n - S^n x_{n+1}\| = 0,$$

thus, we have

$$\lim_{n\to\infty} d_n = 0. \tag{2.6}$$

Substituting (2.2) in (2.3), and with the help of condition (iii), we get

$$\begin{aligned}(2.7)\quad \|x_{n+1}-p\|^2 &\le [(1-b_n)^2 + 2b_nk_n]||x_n-p||^2 - 2b_n\psi(\|x_{n+1}-p\|) \\ &\quad + 2b_n[M^2\,(k_n(b_n+c_n)+t_n) + d_n] \\ &= [1 + b_n^2 + 2b_n(k_n-1)]||x_n-p||^2 - 2b_n\psi(\|x_{n+1}-p\|) \\ &\quad + 2b_n[M^2\,(k_n(b_n+c_n)+t_n) + d_n] \\ &\le ||x_n-p||^2 - 2b_n\psi(\|x_{n+1}-p\|) \\ &\quad + b_n[M^2\,(2(k_n-1)+b_n) \\ &\quad + 2(M^2\,(k_n(b_n+c_n)+t_n) + d_n)].\end{aligned}$$

Denote

$$\theta_n = ||x_n - p||, \qquad \lambda_n = 2b_n,$$
$$\sigma_n = b_n[M^2\,(2(k_n-1)+b_n) + 2(M^2\,(k_n(b_n+c_n)+t_n) + d_n)].$$

Condition (i) assures the existence of a rank $n_0 \in \mathbb{N}$ such that $\lambda_n = 2b_n \le 1$, for all $n \ge n_0$. Now with the help of (i), (ii), (iii), (2.6) and Lemma 2, we obtain from (2.7) that

$$\lim_{n\to\infty}||x_n - p|| = 0,$$

completing the proof. ■

Corollary 1. *Let K be a nonempty closed convex subset of a real Banach space E, $S, T, H : K \to K$ be three ψ-asymptotically pseudocontractive mappings at $p \in F$ with common ψ and having $S(K)$ and $T(K)$ bounded with sequence $\{k_n\}_{n\geq 0} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$. Further, let S be uniformly L-Lipschitzian and $\{a_n\}_{n\geq 0}, \{b_n\}_{n\geq 0}$, $\{c_n\}_{n\geq 0}$, $\{a'_n\}_{n\geq 0}$, $\{b'_n\}_{n\geq 0}$, $\{c'_n\}_{n\geq 0}$, $\{a''_n\}_{n\geq 0}$, $\{b''_n\}_{n\geq 0}$, $\{c''_n\}_{n\geq 0}$ be real sequences in $[0,1]$ satisfying the following conditions:*

(i) $\lim\limits_{n\to\infty} b_n = 0 = \lim\limits_{n\to\infty} b'_n = \lim\limits_{n\to\infty} c'_n$;
(ii) $\sum_{n\geq 0} b_n = \infty$;
(iii) $c_n = o(b_n)$.

For arbitrary $x_0 \in K$, let $\{x_n\}$ be iteratively defined by (1.9), then $\{x_n\}_{n\geq 0}$ converges strongly to a common fixed point of S, T and H.

Remark 3. 1. For $H = I$ and $c_n = 0 = c'_n = b''_n$, we recapture the results of Chang et al. [2].

2. For $H = T = I$ and $c_n = 0 = c'_n = b'_n = b''_n$, we recapture the results of Schu [15] in real Banach spaces.

3. For $H = T = I$ and $c_n = 0 = c'_n = b'_n = b''_n$, we recapture the modified version of the results of Ofoedu [13].

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