DEMONSTRATIO MATHEMATICA
Vol. XLVI No 3 2013

10.1515/dema-2013-0465

José Giménez, Lorena López, N. Merentes

THE NEMITSKIJ OPERATOR ON Lip^k-TYPE AND BV^k-TYPE SPACES

Abstract. In this paper, we discuss and present various results about acting and boundedness conditions of the autonomous Nemitskij operator on certain function spaces related to the space of all real valued Lipschitz (of bounded variation, absolutely continuous) functions defined on a compact interval of $\mathbb{R}$. We obtain a result concerning the integrability of products of the form $\psi \circ f \cdot f' \cdot f^{(k)}$ and a generalized version of the chain rule for functions a.e differentiable, in the sense of Lebesgue. As an application, we get a generalization of a theorem due to V. I. Burenkov for the case of functions of bounded Riesz-p-variation.

1. Introduction

Throughout this paper, we use the following notations: given two functions g and f, the expression $g \circ f$ stands for the composite function $g(f(t))$, whenever it is well-defined; $[a, b]$ denotes a compact interval in $\mathbb{R}$ (the field of all real numbers). Given two sets A and B, A^B denotes the set of all functions from B to A; if, in particular, $\mathbb{A}$ is a linear space, $\mathbb{A}^B$ denotes the linear space of all functions from B to $\mathbb{A}$. By λ we denote the Lebesgue measure on $\mathbb{R}$.

Let $\mathbb{X}$ be a subspace of $\mathbb{R}^{[a,b]}$. Given a function $g : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$, the Nemitskij (superposition or substitution, see e.g., [2, 16]) operator

$$S_g : \mathbb{X} \longrightarrow \mathbb{R}^{[a,b]},$$

generated by g, is defined as

$$S_g(f)(t) := g(t, f(t)), \qquad t \in [a, b].$$

2010 *Mathematics Subject Classification*: 47H30, 39B52.

Key words and phrases: Nemytskij operator, function of bounded variation, Lipschitz continuous function, absolutely continuous function.

This research has been partly supported by the Central Bank of Venezuela. We want to give thanks to the library staff of B.C.V. for compiling the references.

If the function g does not depend of the first variable; that is, $g : \mathbb{R} \to \mathbb{R}$, the mapping

$$S_g(f)(t) := (g \circ f)(t) = g\big(f(t)\big), \quad (t \in [a,b])$$

is known as the *autonomous Nemitskij operator.*

Given two normed spaces $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^{[a,b]}$ and a function $g : \mathbb{R} \to \mathbb{R}$, a primary objective of research in nonlinear analysis is to investigate under what conditions, on the generating function, the associated Nemitskij operator maps $\mathbb{X}$ into $\mathbb{Y}$. This problem is known as *the Superposition Operator Problem*; see, e.g., [2, 6]. Following [2], we will state the Superposition Operator Problem as the following set-theoretic identity:

$$\textsc{sop}\,(\mathbb{X}, \mathbb{Y}) := \{g : S_g(\mathbb{X}) \subset \mathbb{Y}\}$$

and one writes just $\textsc{sop}\,(\mathbb{X})$ if $\mathbb{X} = \mathbb{Y}$.

Recall that given two metric spaces $(\mathcal{M}, d_{\mathcal{M}})$ and $(\mathcal{N}, d_{\mathcal{N}})$, a function $F : \mathcal{M} \to \mathcal{N}$ is said to be Lipschitz continuous iff

$$L(F) := \sup\left\{\frac{d_{\mathcal{N}}(F(x), F(y))}{d_{\mathcal{M}}(x,y)} \;:\; x, y \in \mathcal{M},\; x \neq y\right\} < \infty.$$

The class of all Lipschitz continuos functions in $\mathcal{N}^{\mathcal{M}}$ is denoted as $Lip(\mathcal{M}, \mathcal{N})$. Accordingly, a function $F : \mathcal{M} \to \mathcal{N}$ is said to be locally Lipschitz, and one writes $F \in Lip_{loc}(\mathcal{M}, \mathcal{N})$, if it is Lipschitz continuous on every compact subset of $\mathcal{M}$.

If $\mathcal{M} = [a,b]$, $\mathcal{N} = \mathbb{R}$ (both equipped with the usual absolute value metric) we will simply use the notation $Lip\,[a,b]$. This is a linear space and the functional $\|f\| := |f(a)| + L(f)$ defines a norm with respect to which it is a Banach space.

In the autonomous case, it has been proved that the space of all functions that are locally Lipschitz on $\mathbb{R}$ is a set solution for the Superposition Operator Problem when $\mathbb{X} = C[a,b]$, $C^1[a,b]$, $Lip[a,b]$ or $AC[a,b]$; see, e.g., [3], [15] or [17].

Recently, J. Appell, Z. Jesús and O. Mejia, in [5], carried out the study of the action of the autonomous Nemitskij operator in various spaces of differentiable functions.

In this article, we present several results about acting and boundedness conditions of autonomous Nemitskij operators on certain function spaces of the kind that were considered in [5] but from a rather general point of view. We prove a result about the integrability of products of the form $g \circ f \cdot f' \cdot f^{(k)}$ $(k \in \mathbb{N})$ and a generalized version of the chain rule for functions λ-a.e differentiable; as a consequence, we also obtain a generalization of a theorem by V. I. Burenkov (on products of the form $(g \circ f) \cdot f^{(k)}$) to the case of functions of bounded Riesz p-variation (see Theorem 4.4 below).

2. Some function spaces

For the reader's convenience, in this section, we present a summary of some results related to the notions of function of bounded variation, absolutely continuous functions and the notion of function of bounded p-variation in the sense of Riesz.

Given an interval $[a,b] \subset \mathbb{R}$ and a function $f : [a,b] \to \mathbb{R}$, if $I = [c,d] \subset [a,b]$ we will use the following notations: $f[I] := f(d) - f(c)$ and $f_2[I] := \frac{f(d)-f(c)}{d-c}$. By $\mathfrak{I}[a,b]$ we will denote the family of all finite sequences $\{I_n := [a_n, b_n]\}_{n\geq 0}$ of non-overlapping closed intervals contained in $[a,b]$ and such that $|I_n| := b_n - a_n > 0$, $\forall n \geq 0$.

Definition 2.1. A function $f : [a,b] \longrightarrow \mathbb{R}$ is said to be of bounded variation if the variation of f on $[a,b]$:

$$V(f) = V(f;[a,b]) := \sup\Big\{\sum |f[I_n]| : \{I_n\} \in \mathfrak{I}[a,b]\Big\} < \infty. \tag{2.1}$$

The variation of f on $[a,b]$ is denoted as $V(f;[a,b])$, or simply by $V(f)$, and it is the supremum of the sums (2.1). The class of all functions of bounded variation on $[a,b]$ is denoted as $BV[a,b]$ and it is a Banach space (algebra) if equipped with the norm:

$$\|f\|_{BV[a,b]} := |f(a)| + V(f;[a,b]).$$

The following result is well known:

Proposition 2.2. (C. Jordan [11]) *A function $f : [a,b] \longrightarrow \mathbb{R}$ is of bounded variation on $[a,b]$ if and only if it is the difference of two monotone functions.*

In particular, every function in $BV[a,b]$ has left limit $f(x-)$ at every point $x \in [a,b)$ and right limit $f(x+)$ at every point $x \in (a,b]$; also, by the celebrated Lebesgue's Theorem (see e.g. [13, Theorem 1.2.8]), every function in $BV[a,b]$ is λ-a.e. differentiable.

Recall that a function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is said to be *absolutely continuous* on $[a,b]$ if, given $\epsilon > 0$, there exists some $\delta > 0$ such that

$$\sum |f[I_n]| < \epsilon,$$

whenever $\{I_n = [a_n, b_n]\} \in \mathfrak{I}[a,b]$ is such that $\sum |a_n - b_n| < \delta$.

The class of all absolutely continuous functions on $[a,b]$, which is actually an algebra, is denoted as $AC[a,b]$.

Definition 2.3. (Luzin N property) A real-valued function defined on a finite interval $I \subset \mathbb{R}$ is said to satisfy the Luzin N property (or simply, N property) if it carries sets of λ-measure zero into sets of λ-measure zero.

It is easy to see that the property N is preserved under composition of functions. The class of all continuous functions that satisfy the property N on an interval $[a, b]$ will be denoted by $N[a, b]$. The fact that $N[a, b]$ is closed under pointwise multiplication is proved in [7, Lemma 2].

The following result is well known (cf. [13, Chapter 7]).

Proposition 2.4. *The following statements on a function $f : [a, b] \to \mathbb{R}$ are equivalent:*

(a) *f is absolutely continuous,*
(b) *$f \in BV[a, b] \cap C[a, b]$ and satisfies property N,*
(c) *f is λ-a.e. differentiable on $[a, b]$, $f' \in L_1[a, b]$ and*

$$f(x) = f(a) + \int_a^x f'(t)dt.$$

The equivalence (a)⇔(b) is known as the Banach–Zareckiĭ theorem (see, eg., [18]). The functional

$$\|f\|_{AC} := |f(a)| + \|f'\|_{L_1}$$

defines a norm on $AC[a, b]$; in fact, $\|f'\|_{L_1} = V(f; [a, b])$.

In 1910, F. Riesz introduced the concept of function of bounded p-variation ($1 < p < \infty$) as follows:

Definition 2.5. [19] Let $1 \leq p < \infty$. A function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be of bounded p-variation, in the sense of Riesz, if

$$V_p^R(f) = V_p^R(f; [a, b]) := \sup\Big\{\sum |f_2[I_n]|^p |I_n| : \{I_n\} \in \mathfrak{I}[a, b]\Big\} < \infty.$$

The class of all functions of bounded p-variation on $[a, b]$, in the sense of Riesz, is denoted by $RBV_p[a, b]$.

It readily follows from the definitions that $RBV_p[a, b] \subset C[a, b]$ and that, for all $f, g \in RBV_p[a, b]$,

$$V_p^R(fg) \leq \|f^p\|_\infty V_p^R(g) + \|g^p\|_\infty V_p^R(f);$$

in fact, the relation

$$||f||_p := |f(a)| + (V_p^R(f))^{\frac{1}{p}},$$

defines a norm in $RBV_p[a, b]$ respect to which it becomes a Banach algebra (see e.g., [20]).

Notice that $RBV_1[a, b] = BV[a, b]$; on the other hand, it is well known, that for $1 < p < \infty$, a function belongs to $RBV_p[a, b]$ if and only if, it is absolutely continuous and its derivative (which exists λ-a.e. in $[a, b]$) belongs to $L_p[a, b]$; in this case $V_p^R(f; [a, b]) = \|f'\|_{L_p}^p$ (this is the renowned Riesz's lemma, [19]). In particular, $Lip[a, b] \subset RBV_p[a, b]$. On the other hand, if

$f \in RBV_p[a,b]$ then a straightforward application of Jensen's inequality yields

$$\|f\|_{BV[a,b]} = \|f\|_{AC} \leq (1 + (b-a)^{\frac{p-1}{p}})\|f\|_{RBV_p[a,b]}. \tag{2.2}$$

Clearly, a continuously differentiable function is Lipschitz continuous and any Lipschitz continuous function is absolutely continuous. In fact, the following chain of strict inclusions holds (see e.g., [1], [5]):

$$C^1[a,b] \subset Lip[a,b] \subset RBVp[a,b] \subset AC[a,b] \subset BV[a,b] \quad (p > 1).$$

3. S_g on Lip^k-type spaces

In this section, we present several results about acting and boundedness conditions of Nemitskij operators defined on certain function spaces related to the spaces $Lip\,[a,b]$ and $BV\,[a,b]$. Throughout this section, it will be assumed that k is a positive integer and, as usual, the notation $f^{(k)}$ will stand for the k-th derivative of a function f.

Since we are going to consider higher order derivatives of composite functions, we begin this section by recalling a formula for the k-th derivative of the composition of two functions. For a few small values of k, such a formula, of course, is easy to deduce using the classical chain rule; thus for instance

$$(g \circ f)'''(t) = g'''(f(t))(f'(t))^3 + 3g''(f(t))f'(t)f''(t) + g'(f(t))f'''(t). \tag{3.1}$$

The formula, that we are going to present below, dates back to the year 1800, although it is now named after *Francesco Faà di Bruno*, which seems to have rediscovered it around 1855 (see, e.g., [10]). It was, of course, established assuming existence everywhere of the derivatives of the functions involved and therefore it holds if the functions considered are k-times continuously differentiable.

Proposition 3.1. *Let g and f be functions which posses derivatives up to order k, everywhere on an interval $I \subset \mathbb{R}$. Then*

$$\frac{d^k}{dt^k} g \circ f(t) = \sum \frac{k!}{n_1!n_2!\dots n_k!} g^{(i)}(f(t)) \prod_{j=1}^{k} \left(\frac{f^{(j)}(t)}{j!}\right)^{n_j}, \text{ for all } t \in I, \tag{3.2}$$

where the sum is taken over all different solutions in nonnegative integers $n_1, n_2, \dots, n_k$ of the equations (a) $n_1 + 2n_2 + \cdots + kn_k = k$ *and* (b) $i = n_1 + n_2 + \cdots + n_k$.

Remark 3.2. Note that in the formula (3.2), the terms (summands) that contain a k-th derivative either of g or f, as factors, correspond necessarily to the cases in which $i = k$ or $n_k \neq 0$. Since knowing explicitly such terms will be needed later on, we proceed now to compute them. Indeed, if $i = k$

in formula (3.2) then, combining equations (a) and (b), we must have

$$n_1 + 2n_2 + \cdots + kn_k = k = n_1 + n_2 + \cdots + n_k$$

which is possible if and only if $n_1 = k$ and $n_2 = n_3 = \cdots = n_k = 0$.

On the other hand, $n_k \neq 0$ if and only if $n_k = 1$ (cf. equation (a)) and $n_1 = n_2 = \cdots = n_{k-1} = 0$ (cf. equation (b)), and in this case $i = 1$.

We conclude that the summands that contain a k-th order derivative either of g or f in (3.2) are just $g^{(k)}(f(t)) \cdot (f'(t))^k$ and $g'(f(t)) \cdot f^{(k)}(t)$ (cf. (3.1)).

Since we are going to study the Superposition Operator Problem in the setting of some general normed spaces contained in $\mathbb{R}^{[a,b]}$, we introduce the following notation:

Definition 3.3. Let $\mathcal{P}$ be any generic property of real valued functions (injectivity, continuity, Lipschitz continuity etc.). By $\mathcal{P}[a,b]$ we will denote the set of all functions in $\mathbb{R}^{[a,b]}$ that satisfy property $\mathcal{P}$; in short,

$$\mathcal{P}[a,b] := \{f \in \mathbb{R}^{[a,b]} : f \in \mathcal{P}\}.$$

Also, if k is a nonnegative integer, we will write $\mathcal{P}^k[a,b]$ to denote the set

$$\mathcal{P}^k[a,b] := \{f \in \mathbb{R}^{[a,b]} : f \in C^k[a,b] \text{ and } f^{(k)} \in \mathcal{P}[a,b]\}.$$

Finally, if $I \subset \mathbb{R}$ is an interval, we denote as $P_{loc}(I)$ the set

$$\{f \in \mathbb{R}^{\mathbb{R}} : f|_{[a,b]} \in \mathcal{P}[a,b] \text{ for all } [a,b] \subset I\}.$$

Remark 3.4.

(a) We will use the traditional notation for any well known space such as $C[a,b]$, $Lip[a,b]$, $AC[a,b]$, etc, and their respective $\mathcal{P}^k$-counterparts.

(b) From the definitions, it readily follows that if $\mathcal{P}[a,b]$ is a linear space (resp. an algebra) and $\| \|_{\mathcal{P}}$ is a norm on it then $\mathcal{P}^k[a,b]$ is also a linear space (resp. algebra) and

$$\|u\| := \sum_{i=0}^{k} |u^{(i)}(a)| + \|u^{(k)}\|_{\mathcal{P}}$$

defines a norm in $\mathcal{P}^k[a,b]$. Moreover, if $\mathcal{P}[a,b]$ is a Banach space then $\mathcal{P}^k[a,b]$ is also a Banach space.

Remark 3.5. It was shown by J. G. Darboux in 1875 ([8]) that derivative functions satisfy the intermediate value property. Functions that satisfy the intermediate value property are now called Darboux functions. Notice that if $\mathcal{P}[a,b] \subseteq BV[a,b]$ then

$$\mathcal{P}^k[a,b] = \{f \in \mathbb{R}^{[a,b]} : f \text{ is } k\text{-times differentiable and } f^{(k)} \in \mathcal{P}[a,b]\};$$

that is, the requirement that $f^{(k)}$ be continuous is superfluous because it is satisfied automatically. Indeed, being a derivative, $f^{(k)}$ is a Darboux function and therefore it can only has discontinuities of the second kind, but this is not possible because any function of bounded variation can only have discontinuities of the first kind (by Proposition 2.2). Thus $f^{(k)}$ must be continuous on $[a, b]$. In particular $\mathcal{P}^k[a, b] \subset C^k[a, b]$.

Definition 3.6. We will say that a generic property $\mathcal{P}$, of real valued functions, is S-invariant if the composition (whenever it is well defined) of two functions that satisfy property $\mathcal{P}$ also satisfies property $\mathcal{P}$.

It is easy to check that the Luzin N property and Lipschitz continuity are S-invariants; on the other hand, it is well known that bounded variation and absolutely continuity are not S-invariant (see e.g., [13] and examples later on).

Definition 3.7. A space $\mathcal{P}^k[a, b]$ will be called a Lip^k-type space if $\mathcal{P}[a, b]$ satisfies the following properties:

(p1) $C^1[a, b] \subset \mathcal{P}[a, b]$,
(p2) $\mathcal{P}$ is S-invariant, and
(p3) $\mathcal{P}[a, b]$ is an algebra.

Remark 3.8. Notice that if $\mathcal{P}[a, b]$ is one of the spaces $N[a, b]$, $C^1[a, b]$ (see comments after Definition 2.3) or $Lip[a, b]$ then $\mathcal{P}^k[a, b]$ is a Lip^k-type space.

Definition 3.9. Let $a, b \in \mathbb{R}$ be fixed. For any given pair of real numbers α, β with $\alpha < \beta$, we will denote by $f_{ab}^{\alpha\beta}$ the linear diffeomorphism $f_{ab}^{\alpha\beta} : [a, b] \to [\alpha, \beta]$ defined as

$$f_{ab}^{\alpha\beta}(x) := m_{ab}^{\alpha\beta}(x - a) + \alpha,$$

where $m_{ab}^{\alpha\beta} := \dfrac{\beta - \alpha}{b - a}$.

Notice that if $\alpha < \beta$ then, for all $[c, d] \subset \mathbb{R}$,

$$(f_{cd}^{\alpha\beta})^{-1} = f_{\alpha\beta}^{cd} \quad \text{and} \quad f_{\alpha\beta}^{cd} \circ f_{ab}^{\alpha\beta} = f_{ab}^{cd}. \tag{3.3}$$

Now, we are ready to present a space solution for the Superposition Operator Problem when the spaces considered are of Lip^k-type.

Theorem 3.10. *Let $\mathcal{P}$ be any generic property of functions in $\mathbb{R}^{[a,b]}$. Assume that $\mathcal{P}^k[a, b]$ is a Lip^k-type space. Then, $g \in \mathrm{SOP}(\mathcal{P}^k[a, b])$ if and only if $g \in \mathcal{P}^k_{loc}(\mathbb{R})$.*

Proof. Suppose, in the first place, that $g \in \mathcal{P}^k_{loc}(\mathbb{R})$.

If $f \in \mathcal{P}^k[a,b]$ then $g \circ f \in C^k[a,b]$, and property (p1) implies that each derivative $g^{(j)} \in \mathcal{P}^k_{loc}(\mathbb{R})$ and $f^{(j)}$ $j = 1, 2, \ldots, k-1$ is in $\mathcal{P}[a,b]$. On the other hand, by definition $g^{(k)} \in \mathcal{P}_{loc}(\mathbb{R})$ and $f^{(k)} \in \mathcal{P}[a,b]$. Hence, by properties (p2), (p3) and formula (3.2) $(g \circ f)^{(k)} \in \mathcal{P}[a,b]$ and therefore $g \circ f \in \mathcal{P}^k[a,b]$.

Conversely, assume that $S_g : \mathcal{P}^k[a,b] \to \mathcal{P}^k[a,b]$. We have to show that for all $[\alpha,\beta] \subset \mathbb{R}$: $g \in C^k[\alpha,\beta]$ and $g^k \in \mathcal{P}[\alpha,\beta]$.

Suppose $\alpha < \beta$. Since $f^{\alpha\beta}_{ab} \in \mathcal{P}^k[a,b]$, the hypothesis implies that $S_g(f^{\alpha\beta}_{ab}) \in \mathcal{P}^k[a,b]$. Hence $S_g(f^{\alpha\beta}_{ab}) \in C^k[a,b]$ and $[S_g(f^{\alpha\beta}_{ab})]^{(k)} \in \mathcal{P}[a,b]$. Thus, $g = (g \circ f^{\alpha\beta}_{ab}) \circ [f^{\alpha\beta}_{ab}]^{-1} \in C^k[\alpha,\beta]$. On the other hand,

$$\begin{aligned}
[S_g(f^{\alpha\beta}_{ab})]^{(k)} \in \mathcal{P}[a,b] &\Longrightarrow (g \circ f^{\alpha\beta}_{ab})^{(k)} \in \mathcal{P}[a,b] \\
&\Longrightarrow m^{\alpha\beta}_{ab}(g' \circ f^{\alpha\beta}_{ab})^{(k-1)} \in \mathcal{P}[a,b] \\
&\Longrightarrow [m^{\alpha\beta}_{ab}]^2(g^{(2)} \circ f^{\alpha\beta}_{ab})^{(k-2)} \in \mathcal{P}[a,b] \\
&\;\;\vdots \\
&\Longrightarrow [m^{\alpha\beta}_{ab}]^k(g^{(k)} \circ f^{\alpha\beta}_{ab}) \in \mathcal{P}[a,b].
\end{aligned}$$

Since $m^{\alpha\beta}_{ab} \neq 0$, we must have $g^{(k)} \circ f^{\alpha\beta}_{ab} \in \mathcal{P}[a,b]$. Therefore $g^{(k)}|_{[\alpha,\beta]} = (g^{(k)} \circ f^{\alpha\beta}_{ab}) \circ f^{ab}_{\alpha\beta} \in \mathcal{P}[\alpha,\beta]$ and consequently $g \in \mathcal{P}^k[\alpha,\beta]$. ■

4. S_g in BV^k-type spaces

The fact that the property of a function, in $\mathbb{R}^{[a,b]}$, of being of bounded variation or absolutely continuous is not S-invariant, entails necessarily that in the cases in which $\mathcal{P}[a,b] = BV[a,b], AC[a,b]$ or $RBV_p[a,b]$ the study of the action of the Nemitskij operator on $\mathcal{P}^k[a,b]$ requires an approach fundamentally distinct to the one we performed in the Lip^k-type spaces case. The first part of this section is devoted to discuss several aspects concerning the composition of functions in the referred spaces and its connection with the Superposition Operator Problem. As it will turn out in these cases, the intrinsic properties of the inner function (in the composition) will show to play a fundamental role, thus we begin by making a remark that involves the so called *linear composition operator with symbol* φ, C_φ. We recall that if D and E are given sets, X is a linear subspace of $\mathbb{R}^E$ and φ is a map from D to E, the operator $C_\varphi : X \to \mathbb{R}^D$ is defined by

$$C_\varphi(f) := f \circ \varphi.$$

Remark 4.1.

(a) It readily follows from a result due to M. Josephy (see [12, Theorem 3]) that given a map $\varphi : [a,b] \to [c,d]$, the operator C_φ maps $BV[c,d]$ into $BV[a,b]$ if and only if there is a positive integer $\mathbf{n}$ such that $\varphi^{-1}[\alpha,\beta]$ can be expressed as a union of $\mathbf{n}$ subintervals of $[a,b]$, for all $[\alpha,\beta] \subseteq [c,d]$. The referred subintervals may be open or closed at either end and singletons are also allowed as degenerate closed intervals. The class of all functions that satisfy this property is denoted by $BV(\mathbf{n};[a,b])$. Josephy's result and the fact that both continuity and the Luzin N property are S-invariants imply that if $\varphi \in BV(\mathbf{n};[a,b]) \cap AC[a,b]$ then C_φ maps $AC[c,d]$ into $AC[a,b]$. The converse of this proposition is also true (see [9]).

(b) By the Fundamental Theorem of Algebra and Rolle's Theorem, if f is a polynomial of degree $\mathbf{n}$, then for all $[a,b] \subset \mathbb{R}$, $f \in AC[a,b] \cap BV(\mathbf{n};[a,b])$; also, every monotone absolutely continuous function $\varphi \in \mathbb{R}^{[a,b]}$ is in $BV(\mathbf{n};[a,b])$ for some $\mathbf{n} \in \mathbb{N}$. Indeed, in this case we have that C_φ maps $AC[\varphi([a,b])]$ into $AC[a,b]$ (see [14, page 97]) and the desired conclusion follows then from the mentioned result given in [9].

For convenience, now we state the next result as a single proposition. Its proof is based on three separate results of M. Josephy [12] (see also [4]), N. Merentes [15] and N. Merentes and S. Rivas [17].

Proposition 4.2. *Suppose $\mathcal{P}[a,b] = BV[a,b]$, $AC[a,b]$ or $RBV_p[a,b]$. Then $g \in \text{SOP}(\mathcal{P}[a,b])$ if and only if $g \in Lip_{loc}(\mathbb{R})$.*

In what follows we will observe some instances of a very remarkable phenomenon that often occurs in non-linear functional analysis: given two functions, say g and f, that satisfy a certain generic property $\mathcal{P}$ which is not S-invariant, the multiplication of $g \circ f$ by a derivative ($f^{(k)}$, $k \in \mathbb{N}$) of f improves the properties of the composition, whenever that derivative satisfies also property $\mathcal{P}$. Thus, the fact that the spaces $\mathcal{P}^k[a,b]$ are contained in the space $C^k[a,b]$ will allow us to rely on these kind of tools to deal with the Superposition Operator Problem in these cases. To begin with, we state a result due to V. I. Burenkov for the case $P[a,b] = BV[a,b]$, for the proof the reader is referred to [7, Theorem 5].

Theorem 4.3. (Burenkov) *Suppose that f has a derivative $f^{(k)}$ of order k everywhere on $[a,b]$. If $f^{(k)} \in BV[a,b]$ and if $g \in BV[c,d]$, $c := \min_{[a,b]} f$, $d := \max_{[a,b]} f$, then the function $g \circ f \cdot f^{(k)}$ is also of bounded variation on $[a,b]$; moreover,*

$$\|g \circ f \cdot f^{(k)}\|_{[a,b]} \leq (k+1)\|g\|_{[c,d]}\, \|f^{(k)}\|_{[a,b]}, \tag{4.1}$$

where $\|\cdot\|_{[a,b]}$ *is the norm*[1] *on* $BV[a,b]$ *defined as*

$$\|f\|_{[a,b]} := \sup_{[a,b]} |f(x)| + V(f;[a,b]).$$

In the case of absolutely continuous functions, we observe that not even the fact that the inner function is smooth enough ensures that the composition $g \circ f$ is absolutely continuous. For instance, if $g(u) := u^{\frac{1}{4}}$ and f is defined as $f(0) := 0$ and for $x > 0$, $f(x) := x^7(\sin\frac{1}{x^2} + 2)$, then $f \in C^2[0,1]$, $g \in AC[f([0.1])]$, but $g \circ f$ is not even of bounded variation. Notice, however, that the function $h(x) := g(f(x))\, f'(x)$ is absolutely continuous on $[0,1]$. Thus, in this case, multiplication by an absolutely continuous derivative of the inner function improves the properties of the composition $g \circ f$. In fact, as pointed out by V. I. Burenkov in ([7]), we have the following corollary of Theorem 4.3.

Proposition 4.4. *If f has an absolutely continuous k^{th}-derivative $f^{(k)}(x)$ on $[a,b]$ and if $g \in AC_{loc}(\mathbb{R})$, then the function $g \circ f \cdot f^{(k)}$ is also absolutely continuous on $[a,b]$ and inequality* (4.1) *holds.*

Proof. By Theorem [7, Theorem 4], the function $g \circ f \cdot f^{(k)} \in C[a,b] \cap BV[a,b]$. The result follows by the Banach–Zareckiĭ Theorem, due to the fact that the $N[a,b]$ is an algebra (see Remark 3.8). ■

Now we turn our attention to the RBV_p $(p \geq 1)$ case. By the Riesz's Lemma, similar considerations as those discussed above about compositions apply also in this case. Now, we will be particularly interested in determining whether a product of the form $g \circ f \cdot f'$ (or $|g|^p \circ f \cdot f'$) is integrable. Notice that it is not enough that g be integrable and f be absolutely continuous to guarantee that the product $g \circ f \cdot f'$ be integrable; for instance, let $g(0) := f(0) := 0$ and for $x > 0$ let $g(x) := 1/\sqrt{x}$ and $f(x) := x^6\big(\sin\frac{1}{x^3} + 2\big)$, then $f \in AC[0,1]$, g is integrable in $f([0,1])$, but $g \circ f \cdot f'$ is not integrable in $[0,1]$. In that respect, the following proposition is known (see, e.g., [14, Theorem 3.54]):

Proposition 4.5. (Change of variables) *Let $g : [c,d] \to \mathbb{R}$ be an integrable function and let $f : [a,b] \to [c,d]$ be a function differentiable λ-a.e. in $[a,b]$. Then $g \circ f \cdot f'$ is integrable and*

$$\int_{f(\alpha)}^{f(\beta)} g(t)\,dt = \int_{\alpha}^{\beta} g(f(x))f'(x)\,dx$$

[1] Notice that $\|f\|_{BV[a,b]} \leq \|f\|_{[a,b]} \leq 2\|f\|_{BV[a,b]}$.

holds for all $\alpha, \beta \in [a, b]$ if and only if the function $G \circ f \in AC[a, b]$, where

$$G(z) := \int_c^z g(t)\, dt, \quad z \in [c, d]. \tag{4.2}$$

Of course, the condition "$G \circ f \in AC[a, b]$" brings us back to the same situations discussed above. However, also in this case, it turns out that multiplication by a derivative of f, which is in BV, improves the (integrability) properties of a product of the form $g \circ f \cdot f'$, whenever g is an integrable function. Indeed, now we present a proposition which is, somehow, a version of Theorem 4.3 when the outer function in the composition is an integrable function.

Theorem 4.6. *Suppose that $g \in L_{1,loc}(\mathbb{R})$ and that $f^{(k)} \in BV[a, b]$. Then $g \circ f \cdot f' \cdot f^{(k)} \in L_1[a, b]$; moreover,*

$$\|g \circ f \cdot f' \cdot f^{(k)}\|_{L_1[a,b]} \le k \|f^{(k)}\|_{[a,b]}\, \|g\|_{L_1(f([a,b]))}. \tag{4.3}$$

Proof. By Remark 3.5, $f^{(k)}$ is continuous on [a,b], so, we can express the set $S = \{x \in (a, b) : f^{(k)}(x) \neq 0\}$ as a countable union of component intervals, say $S = \cup_{i=1}^N (a_i, b_i)$, where $a_i < b_i$ and $N \in \mathbb{N}$ or $N = \infty$. Now, since $f^{(k)} \neq 0$ on S, given $i \in \{1, \ldots, N\}$ there is a nonnegative integer $m_i \le k$ such that $[a_i, b_i]$ can be decomposed into m_i intervals, $[a_i^1, b_i^1]$, $[a_i^2, b_i^2]$, …, $[a_i^{m_i}, b_i^{m_i}]$, on which f is monotone (this follows from finitely many successive applications of Rolle's theorem).

By Remark 4.1, the monotonicity of f on $[a_i^j, b_i^j]$ implies that if G is the function defined by (4.2) then $G \circ f \in AC[a, b]$; consequently, $g \circ f \cdot f'$ is integrable on this interval (so, a fortiori, measurable on each $[a_i^j, b_i^j]$) and, since f' does not change sign on $[a_i^j, b_i^j]$, we must have

$$\int_{a_i^j}^{b_i^j} |g(f(t))|\, |f'(t)|\, dt = \pm \int_{a_i^j}^{b_i^j} |g(f(t))|\, f'(t)\, dt = \int_{\langle f(a_i^j), f(b_i^j)\rangle} |g(x)|\, dx$$

where the notation $\langle \alpha, \beta \rangle$ stands for $[\alpha, \beta]$ if $\alpha < \beta$ or $[\beta, \alpha]$ otherwise.

Notice that from the last considerations, it readily follows that $g \circ f \cdot f' \cdot f^{(k)}$ is a measurable function on $[a, b]$.

Now, since $|f^{(k)}|$ is continuous, the (*generalized*) *mean value theorem for integrals* implies that, on each $[a_i^j, b_i^j]$, there is a point c_i^j such that

$$\int_{a_i^j}^{b_i^j} |g(f(t))\, f'(t)|\, |f^{(k)}(t)|\, dt = |f^{(k)}(c_i^j)| \int_{\langle f(a_i^j), f(b_i^j)\rangle} |g(x)|\, dx.$$

Now we just need to examine two cases:
(a) If $f^{(k)}(x) \neq 0$, on all of $[a,b]$ then $N = 1$, and

$$\int_a^b \left|g(f(t))\, f'(t)\, f^{(k)}(t)\right| dt = \sum_{j=1}^{m_1} \left|f^{(k)}(c_1^j)\right| \int_{\langle f(a_1^j), f(b_1^j)\rangle} \left|g(x)\right| dx$$

$$\leq k\|f^{(k)}\|_\infty \,\|g\|_{L_1(f([a,b]))}.$$

(b) If $f^{(k)}(x) = 0$, anywhere on $[a,b]$ then, for all $i \in \{1,\ldots,N\}$, either $f^{(k)}(a_i) = 0$ or $f^{(k)}(b_i) = 0$; say $f^{(k)}(a_i) = 0$. Then

$$\int_a^b \left|g(f(t))\, f'(t)\, f^{(k)}(t)\right| dt = \sum_{i=1}^{N}\sum_{j=1}^{m_i} \left|f^{(k)}(c_i^j)\right| \int_{\langle f(a_i^j), f(b_i^j)\rangle} \left|g(x)\right| dx$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{m_i} \left|f^{(k)}(c_i^j) - f^{(k)}(a_i)\right| \int_{\langle f(a_i^j), f(b_i^j)\rangle} \left|g(x)\right| dx$$

$$\leq \|g\|_{L_1(f([a,b]))} \sum_{i=1}^{N} V\big(f^{(k)}; [a_i, b_i]\big)$$

$$\leq \|g\|_{L_1(f([a,b]))}\, V\big(f^{(k)}; [a,b]\big).$$

We conclude that

$$\int_a^b \left|g(f(t))\, f'(t)\, f^{(k)}(t)\right| dt \leq \big(k\|f^{(k)}\|_\infty + V\big(f^{(k)}; [a,b]\big)\|g\|_{L_1(f([a,b]))}$$

$$\leq k\|f^{(k)}\|_{[a,b]}\, \|g\|_{L_1(f([a,b]))},$$

and the proof is complete. ■

At this point, let us recall the following fact (see, e.g., [14, Theorem 3.44]):

Suppose that g, f are functions defined on intervals and that $g \circ f$ is well defined. If g, f and $g \circ f$ are λ-a.e differentiable functions and g satisfy the property N then,

$$(g \circ f)'(x) = g'(f(x))f'(x) \quad \textit{for} \quad \lambda\text{-a.e. } x, \tag{4.4}$$

where $g'(f(x))f'(x)$ is interpreted to be zero whenever $f^{(k)}(x) = 0$.

Thus, although the required (sufficient) conditions are rather mild, one still needs to be careful, since, the composition of λ-a.e differentiable functions need not be λ-a.e differentiable (see e.g. [7, §4]). Now, we present a result that somehow shows the extent to which multiplication by an absolutely continuous derivative of the inner function improves the differentiability properties of the composition $g \circ f$, whenever $g \in AC[a,b]$. This fact might have some interest in itself.

Lemma 4.7. *Suppose $g \in AC(\mathbb{R})$ and let $f^k \in AC[a,b]$. Then for λ-a.e. $x \in [a,b]$:*

$$(g \circ f \cdot f^{(k)})'(x) = g'(f(x))f'(x)f^{(k)}(x) + g(f(x))\, f^{(k+1)}(x), \tag{4.5}$$

where $g'(f(x))f'(x)$ is interpreted to be zero whenever $f'(x) = 0$ (even if g is not differentiable at $f(x)$).

Proof. By Proposition 4.4, $g \circ f \cdot f^k \in AC[a,b]$.

Let $S, N, \{(a_i,b_i)\}_{i=1}^N$ and $\{(a_i^j, b_i^j)\}_{j=1}^{m_i}$ be as in the proof of Theorem 4.6. Then, the monotonicity of f on $[a_i^j, b_i^j]$ implies that $g \circ f|_{(a_i^j, b_i^j)} \in AC[a_i^j, b_i^j]$ and hence (4.4) and (4.5) holds for λ-a.e. x in S. The desired conclusion follows if we define $(g \circ f \cdot f^{(k)})'(x) \equiv 0$ on the set $[a,b] \setminus S$. ∎

Now, we present a version of Theorem 4.3 in the RBV_p case.

Theorem 4.8. *Let $1<p<\infty$. If $f^{(k)} \in RBV_p[a,b]$ and if $g \in RBV_{p,loc}(\mathbb{R})$, then the function $g \circ f \cdot f^{(k)}$ is in $RBV_p[a,b]$; moreover, there are constants M_1 and M_2, that depend on g, such that*

$$\begin{aligned} &V_p^R(g \circ f \cdot f^{(k)}; [a,b]) \\ &\quad \le M_1\big(\||f'|^{p-1}|f^{(k)}|^{p-1}\|_\infty \, \|f^{(k)}\|_{RBV_p[a,b]}\big) + M_2\|f^{(k+1)}\|^p_{L_p[a,b]}. \end{aligned} \tag{4.6}$$

Proof. By Riesz Lemma $f^{(k)} \in AC[a,b]$, $f^{(k+1)} \in L_p[a,b]$, $g \in AC(\mathbb{R})$ and $g' \in L_{p,loc}(\mathbb{R})$. Thus, by Theorem 4.3, $g \circ f \cdot f^{(k)} \in AC[a,b]$ and, by Lemma 4.7, for λ-a.e. $x \in [a,b]$:

$$\begin{aligned} &\left|\frac{d}{dx}[g(f(x))\, f^{(k)}(x)]\right|^p \\ &\qquad = \big|g'(f(x))f'(x)f^{(k)}(x) + g(f(x))\, f^{(k+1)}(x)\big|^p \\ &\qquad \le 2^p\big(\big|g'(f(x))f'(x)f^{(k)}(x)\big|^p + \big|g(f(x))\, f^{(k+1)}(x)\big|^p\big) \\ &\qquad \le 2^p\big(\big||g'|^p(f(x))\, f'(x)f^{(k)}(x))\big|\cdot\big\||f'|^{p-1}\, |f^{(k)}|^{p-1}\big\|_\infty \\ &\qquad\quad + \||g|^p \circ f\|_\infty \, |f^{(k+1)}|^p(x)\big). \end{aligned} \tag{4.7}$$

Hence, by Theorem 4.6, $\frac{d}{dx}[g(f(x))\, f^{(k)}(x)]$ belongs to $L_p[a,b]$ and another application of Riesz Lemma implies that $g \circ f \cdot f^{(k)} \in RBV_p[a,b]$. Finally, from estimates (4.7), (4.3) and the fact that $\|f^{(k)}\|_{BV[a,b]} = \|f^{(k)}\|_{AC[a,b]}$ we have

$$\left\| \frac{d}{dx}[g \circ f \cdot f^{(k)}] \right\|^p_{L_p[a,b]}$$
$$\leq 2^p \big(\| |f'|^{p-1} |f^{(k)}|^{p-1} \|_\infty \, 2k \, \|f^{(k)}\|_{AC[a,b]} \, \|(g')\|^p_{L_p[a,b]}$$
$$+ \| |g|^p \circ f \|_\infty \, \|f^{(k+1)}\|^p_{L_p[a,b]} \big)$$

which, by vitue of (2.2), yields (4.6). ■

Our next proposition shows that a result similar to Theorem 3.10 is possible for the spaces $BV[a,b]$, $RBV_p[a,b]$ or $AC[a,b]$; in this case, condition (p2) of Definition 3.7 needs to be replaced by the following condition:

(p2') For $g \in \mathcal{P}_{loc}(\mathbb{R})$ and $f \in \mathcal{P}^k[a,b]$: the function $g \circ f \cdot f^{(j)} \in \mathcal{P}[a,b]$, $j = 1, \ldots, k$.

Thus, to be consistent with Definition 3.7, any space $P^k[a,b]$ that satisfies properties (p1), (p2') and (p3) shall be called a BV^k-type space.

Theorem 4.9. *Let $\mathcal{P}^k[a,b]$ be a BV^k-type space and let $g \in \mathbb{R}^{\mathbb{R}}$. Then $g \in \mathrm{SOP}(\mathcal{P}^k[a,b])$ if and only if $g \in \mathcal{P}^k_{loc}(\mathbb{R})$.*

Proof. Suppose that $g \in \mathcal{P}^k_{loc}(\mathbb{R})$.

If $f \in \mathcal{P}^k[a,b]$, then $g \circ f \in C^k[a,b]$ and since each derivative $g^{(j)}$, $f^{(j)}$, $j = 1, 2, \ldots, k-1$, is of class C^1, it is Lipschitz continuous and hence every summand of (3.2) of the form

$$c_i \, g^{(i)}(f(t)) \prod_{j=1}^{k-1} \left(\frac{f^{(j)}(t)}{j!} \right)^{n_j}, \qquad i = 1, \ldots k-1,$$

is in $\mathcal{P}[a,b]$ (by Proposition 4.2).

To complete the proof of the sufficiency of the condition it remains only to verify that each summand of (3.2) that contains a k^{th}-order derivative is also in $\mathcal{P}[a,b]$. Now, by definition $g^{(k)} \in \mathcal{P}[a,b]$ and $f^{(k)} \in \mathcal{P}[a,b]$. Hence, by property (p2') and the fact that $\mathcal{P}[a,b]$ is an algebra, both summands $g^{(k)} \circ f \cdot (f')^k$ and $g' \circ f \cdot f^{(k)}$ are in $\mathcal{P}[a,b]$. Thus, by Remark 3.2, every summand in formula (3.2) for $(g \circ f)^{(k)}$ is in $\mathcal{P}[a,b]$ and therefore $g \circ f \in \mathcal{P}^k[a,b]$.

To prove the necessity of the condition suppose that $\alpha < \beta$. Then, proceeding as in the proof of Theorem 3.10, one has

$$g^{(k)} \circ f^{\alpha\beta}_{ab} \cdot (m^{\alpha\beta}_{ab})^k = (S_g \circ f^{\alpha\beta}_{ab})^{(k)} \in \mathcal{P}[a,b]$$

to conclude, since $m^{\alpha\beta}_{ab} \neq 0$, that $g^{(k)} \circ f^{\alpha\beta}_{ab} \in \mathcal{P}[a,b]$. The assertion then follows by noticing that $g^{(k)}|_{[\alpha,\beta]} = (g^{(k)} \circ f^{\alpha\beta}_{ab}) \circ f^{ab}_{\alpha\beta} \in \mathcal{P}[\alpha,\beta]$. ■

Corollary 4.10. *Let $\mathcal{P}[a,b]$ be as in the statement of Theorem* 4.9. *If $g \in \mathcal{P}^k_{loc}(\mathbb{R})$ then the operator $S_g : \mathcal{P}[a,b] \to \mathcal{P}[a,b]$ is bounded with respect to norm $\|f\|_{k,[a,b]} := \sum_{i=1}^k \|f^{(j)}\|_\infty + \|f\|_{\mathcal{P}[a,b]}$.*

Proof. Indeed, this follows by a straightforward application of Theorem 4.9 and the norm estimate (4.1) (for the cases BV and AC), or inequality (4.6), and formula (3.2). ∎

References

[1] J. Appell, *Some counterexamples for your calculus course*, Analysis 31 (2010), 1001–1012.

[2] J. Appell, P. P. Zabrejko, *Nonlinear Superposition Operator*, Cambridge University Press, New York, 1990.

[3] J. Appell, N. Guanda, N. Merentes, J. L. Sánchez, *Some boundedness and continuity properties of nonlinear composition operators: a survey*, Commun. Appl. Anal. (2010), to appear.

[4] J. Appell, P. P. Zabrejko, *Remarks on the superposition operator problem in various function spaces*, Complex Var. Elliptic Equ. 55(8) (2010), 727–737.

[5] J. Appell, Z. Jesús, O. Mejía, *Some remarks on nonlinear composition operators in spaces of differentiable functions*, Boll. Unione Mat. Ital. Serie IX, 4(3) (2011), 321–336.

[6] G. Bourdaud, M. Lanza de Cristoforis, W. Sickel, *Superposition operators and functions of bounded p-variation*, Rev. Mat. Iberoamericana 22(2) (2006), 455–487.

[7] V. I. Burenkov, *On integration by parts and a problem on composition of absolutely continuous functions which arises in this connection*, Theory of functions and its applications, Trudy Mat. Inst. Steklov. 134 (1975), 38–46.

[8] G. Darboux, *Memoire sur les fonctions discontinues*, Ann. Sci. Scuola Norm. Sup. 4 (1875), 57–112.

[9] G. M. Fihtengol'c, *On absolutely continuous functions*, Mat. Sb. 31 (1924), 286–295.

[10] W. Johnson, *The curious history of Faá di Bruno's formula*, Amer. Math. Monthly 109(3) (2002), 217–234.

[11] C. Jordan, *Sur la série de Fourier*, C. R. Math. Acad. Sci. Paris 2 (1881), 228–230.

[12] M. Josephy, *Composing functions of bounded variation*, Proc. Amer. Math. Soc. 83(2) (1981), 354–356.

[13] R. Kannan, C. K. Krueger, *Advanced Analysis on the Real Line*, Springer, New York, 1996.

[14] G. Leoni, *A first course in Sobolev spaces*, Amer. Math. Soc., GSM 105, Providence, RI, (2009).

[15] N. Merentes, *On the composition operator in AC[a, b]*, Collect. Math. 42(1) (1991), 121–127.

[16] N. Merentes, S. Rivas, *El Operador de Composición en Espacios de Funciones con algún tipo de Variación Acotada*, IX Escuela Venezolana de Matemáticas, Facultad de Ciencias-ULA, Mérida-Venezuela, 1996.

[17] N. Merentes, S. Rivas, *On characterization of the Lipschitzian composition operators between spaces of functions of bounded p-variation*, Czechoslovak Math. J. 45(120) (1995), 627–637.

[18] I. P. Natanson, *Theory of Functions of a Real Variable*, vol. I, rev. ed., Ungar, New York, (1961).
[19] F. Riesz, *Untersuchugen ĺuber Systeme Integrierberer Funktionen*, Math. Ann. 69 (1910), 449–497.
[20] F. Szigeti, *Composition of Sobolev functions and applications*, Notas de Matematicas, Univ. Los Andes, Venezuela 86 (1987), 1–25.

José Giménez, Lorena López
UNIVERSIDAD DE LOS ANDES
FACULTAD DE CIENCIAS
DPTO. DE MATEMÁTICAS
MÉRIDA-VENEZUELA
E-mails: jgimenez@ula.ve
lomalopez@ula.ve

Nelson Merentes
UNIVERSIDAD CENTRAL DE VENEZUELA
ESCUELA DE MATEMÁTICAS
CARACAS-VENEZUELA
E-mail: nmerucv@gmail.com

Received September 20, 2011.