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M. A. Pathan, O. A. Daman

LAPLACE TRANSFORMS OF THE LOGARITHMIC FUNCTIONS AND THEIR APPLICATIONS

Abstract. This paper deals with theorems and formulas using the technique of Laplace and Steiltjes transforms expressed in terms of interesting alternative logarithmic and related integral representations. The advantage of the proposed technique is illustrated by logarithms of integrals of importance in certain physical and statistical problems.

1. Introduction

The aim of this paper is to obtain some theorems and formulas for the evaluation of finite and infinite integrals for logarithmic and related functions using technique of Laplace transform. Basic properties of Laplace and Steiltjes transforms and Parseval type relations are explicitly used in combination with rules and theorems of operational calculus. Some of the integrals obtained here are related to stochastic calculus [6] and common mathematical objects, such as the logarithmic potential [3], logarithmic growth [2] and Whittaker functions [2, 3, 4, 6, 7] which are of importance in certain physical and statistical applications, in particular in energies, entropies [3, 5, 7 (22)], intermediate moment problem [2] and quantum electrodynamics. The advantage of the proposed technique is illustrated by the explicit computation of a number of different types of logarithmic integrals.

We recall here the definition of the Laplace transform

$$L\{f(t)\} = L[f(t); s] = \int_0^\infty e^{-st} f(t)\, dt. \tag{1}$$

Closely related to the Laplace transform is the generalized Stieltjes transform

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$$S_\rho\{f(t)\} = \int_0^\infty \frac{f(t)dt}{(a+t)^\rho} = G(a,\rho) \tag{2}$$

which, for $\rho = 1$, gives the Stieltjes transform

$$S\{f(t)\} = \int_0^\infty \frac{f(t)dt}{(a+t)} = G(a). \tag{3}$$

After a change of integration variable, (2) is transfomed into

$$\int_0^1 \frac{f(-a\ln x)}{x(1-\ln x)^\rho} dx = a^{\rho-1} G(a,\rho). \tag{4}$$

Special cases of (2) and (4), when $f(t) = t^{\lambda-1}e^{-t}$, are generalizations of gamma function given by Kobayashi [7]

$$\Gamma_\rho(\lambda, a) = \int_0^1 (-\ln x)^{\lambda-1}(a - \ln x)^{-\rho} dx, \quad \mathcal{R}(\lambda) > 0. \tag{5}$$

Kobayashi [7] applied this generalized gamma function integral in diffraction theory.

2. Theorems

In this section, we state and prove some theorems in the study of integral transforms, and briefly discuss some apparent, known and new special cases of these theorems. We will apply systematically the rules and theorems of the operational calculus assuming the existence of the Laplace transforms of the functions involved and the permissibility of performed mathematical operations.

Theorem 1. *If*

$$L\{f(t)\} = \varphi(s) \tag{6}$$

and

$$L\{h(t)\} = g(s) \tag{7}$$

then

$$\int_0^1 \left(\frac{-\ln x}{s}\right)^n g\left(\frac{-\ln x}{s}\right) f\left(\frac{-\ln x}{s}\right) dx$$

$$= s\int_s^\infty \varphi(t) h^n(t-s) dt \tag{8}$$

$$= s\int_0^\infty \varphi(s+x) h^n(x) dx \tag{9}$$

(10) $$\int_0^\infty L[\{h^n(t-a)H(t-a)\}; s]ds = \Gamma(n+1)\int_0^\infty \frac{h^n(t)}{(t+a)^{n+1}}dt$$

and

(11) $$\int_0^1 \frac{\phi(\frac{-\ln x}{u})}{(au-\ln x)}dx = \int_u^\infty L[\phi(t-a)H(t-a); x]dx$$

provided that, $f(t) \in L^2(0,\infty)$, $e^{-st}t^n g(t) \in L^2(0,\infty)$, $h(t) \in L^2(0,\infty)$ *and* $h^n(t)$ *denotes the* n^{th} *differential coefficient of* $h(t)$ *such that* $h'(0) = h''(0) = \cdots = h^{n-1}(0) = 0$. $H(t)$ *is the Heaviside's unit function and integrals in* (8) *to* (11) *are convergent.*

Proof. Consider the Laplace transform

(12) $$L\{f(t)\} = L[f(t); s] = \int_0^\infty e^{-st}f(t)dt = \varphi(s).$$

Recall a well known property of the Laplace transform [4, p. 129] that is, if

(13) $$L\{h(t)\} = g(s)$$

then

(14) $$L\{h^n(t)\} = s^n g(s), h'(0) = h''(0) = \cdots = h^{n-1}(0) = 0$$

and

(15) $$L\{h^n(t-a)H(t-a)\} = e^{-as}s^n g(s),$$

where $H(t)$ is a Heaviside's unit function. To prove (8) and (9), we use (12) and (15) in the Parseval theorem and then, by changing the integration variable $x = e^{-st}$, we find

(16) $$\int_0^1 \left(-\frac{\ln x}{s}\right)^n g \quad \left(-\frac{\ln x}{s}\right) f\left(-\frac{\ln x}{s}\right) dx$$
$$= s\int_s^\infty \varphi(t)h^n(t-s)dt = s\int_0^\infty \varphi(s+x)h^n(x)dx,$$

where a is replaced by s. Now, we integrate both sides of (15) and use (14) to obtain

(17) $$\int_0^\infty L\{h^n(t-a)H(t-a)\}ds$$
$$= \int_0^\infty e^{-as}s^n \int_0^\infty e^{-st}h^n(t)dtds = \int_0^\infty h^n(t)\left[\int_0^\infty s^n e^{-as-st}ds\right]dt.$$

By evaluating the integral on the right hand side of (17), we obtain a Parseval relation for (13). Thus

$$\int_0^\infty L\{h^n(t-a)\}H(t-a)\}dt = \Gamma(n+1)\int_0^\infty \frac{h^n(t)}{(t+a)^{n+1}}dt. \tag{18}$$

For $n = 0$, (18) gives $\int_0^\infty L\{h(t-a)H(t-a)\}dt = \int_0^\infty \frac{h(t)}{(t+a)}dt$. For $n = 0$ and $a = 0$, (18) gives a known result [5, p. 110 (2.4)]

$$\int_0^\infty L\{h(t)\}ds = \int_0^\infty \frac{h(t)}{t}dt. \tag{19}$$

Now, set $h(t) = e^{-ut}\phi(t)$ and $n = 0$ in (19) and use shift property

$$L\{h(t)\} = L\{e^{-ut}\phi(t)\} = \int_0^\infty e^{-ut-st}\phi(t)dt = L[\phi(t); u+s], \tag{20}$$

to get another Parseval-type relation

$$\int_u^\infty L[\phi(t-a)H(t-a); x]dx = L\left[\frac{\phi(t)}{a+t}; u\right], \tag{21}$$

by making the change of variable $u + s = x$. On changing the integration variable $x = e^{-ut}$, we get (11). For $a = 0$, (21) gives a known result [5, p. 110 (2.8)]

$$\int_u^\infty L[\phi(t); x]dx = L\left[\frac{\phi(t)}{t}; u\right]. \tag{22}$$

If we take $h(t) = t^\lambda e^{-at}$ and use [4, p. 129]

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s), \quad n = 1, 2, ... \tag{23}$$

and binomial theorem in Theorem 1, we obtain

Theorem 2. *If* $L[f(t); s] = F(s)$ *then*

$$\int_0^1 x^{\frac{a}{s}} (\ln x)^n f\left(\frac{-\ln x}{s}\right) dx = s^{n+1}\frac{d^n}{ds^n}F(s+a) \tag{24}$$

and

$$\int_0^1 \left(-\frac{\ln x}{s}\right)^n \left(a - \frac{\ln x}{s}\right)^{-\lambda-1} f\left(-\frac{\ln x}{s}\right) dx = s\sum_{r=0}^n \frac{(-1)^{n-r}a^{n-r}}{\Gamma(\lambda-r+1)}\binom{n}{r}\phi(a; s) \tag{25}$$

where

$$\phi(a;s)=\int_0^\infty x^{\lambda-r}e^{-ax}F(s+x)dx \tag{26}$$

provided that Laplace transform of $|f(t)|$ *exists,* $\lambda>n-1,\mathcal{R}(s+a)>0$ *and the integrals in* (25) *and* (26) *are convergent.*

In the simplest case $f(t)=1$ and $F(s)=\frac{1}{s}$, we have immediately from (25) and [4, p. 294 (6)]

$$\int_0^1\left(-\frac{\ln x}{s}\right)^n\left(a-\frac{\ln x}{s}\right)^{-\lambda-1}dx = e^{\frac{as}{a}}\sum_{r=0}^{n}\frac{(-1)^{n-r}a^{n-(\lambda+r+1)/2}}{s^{(\lambda-r-1)/2}}\binom{n}{r}W_{k,m}(as), \tag{27}$$

where $k=(r-\lambda-1)/2$, $m=(\lambda-r)/2$ and $W_{k,m}(x)$ is Whittaker function [4].

Evidently, if we set $f(t)=1$ in (24), we get

$$\int_0^1 x^{a/s}(\ln x)^n dx=(-1)^n n!\left(\frac{s+a}{s}\right)^{-n-1}. \tag{28}$$

Another example is

$$L[f(t)]=L\left[\frac{1}{1+e^{-t}}\right]=\frac{1}{2}\left[\psi\left(\frac{s+1}{s}\right)-\psi\left(\frac{s}{2}\right)\right],\quad \mathcal{R}(s)>0$$

which leads to

$$\int_0^1\frac{x^{a/s}(\ln x)^n}{1+x^{1/s}}=\frac{s^{n+1}}{2^{n+1}}\left[\psi^{(n)}\left(\frac{a+s+1}{2}\right)-\psi^{(n)}\left(\frac{a+s}{2}\right)\right], \tag{29}$$

$n=1,2,\ldots,$ $\mathcal{R}(s)>0$, where $\psi(\zeta)$ is a psi-function and $\psi^{(n)}(\zeta)$ means the n^{th} derivatives of psi-function [4]. On the other hand, the special cases of (24) and (25), for $n=0$ and $a=0$, yield known results [1, p. 241, equations (23), (26) and (28)].

Theorem 3. *Let* $\alpha>0,\beta>0,$ *then*

$$\int_0^1\frac{f(\frac{-\ln x}{\alpha+\beta})}{\sqrt{\ln(1/x)}}dx=\sqrt{\pi\alpha(\alpha+\beta)}L[g(\theta,t);\alpha] \tag{30}$$

where

$$g(\theta,t)=\int_0^t\frac{e^{-\beta\theta}f(\theta)}{\pi\sqrt{\theta(t-\theta)}}d\theta. \tag{31}$$

Proof. Let $g(\theta, t)$ be defined by (31). Then

$$\begin{aligned} L[g(\theta,t);\alpha] &= \int_0^\infty e^{-\alpha t}\left[\int_0^t \frac{e^{-\beta\theta} f(\theta)}{\pi\sqrt{\theta(t-\theta)}} d\theta\right] dt \\ &= \int_0^\infty \frac{e^{-\beta\theta}}{\pi\sqrt{\theta}} f(\theta)\left[\int_\theta^\infty \frac{e^{-\alpha t}}{\sqrt{t-\theta}} dt\right] d\theta \\ &= \frac{1}{\pi}\int_0^\infty \frac{e^{-(\alpha+\beta)\theta}}{\sqrt{\theta}} f(\theta)\left[\int_0^\infty \frac{e^{-\alpha s}}{\sqrt{s}} ds\right] d\theta \end{aligned}$$

where in the inner integral, we have changed the variable of integration by setting $t - \theta = s$. It follows from

(32) $$\int_0^\infty \frac{e^{-\tau t}}{\sqrt{t}} dt = \sqrt{\frac{\pi}{\tau}}, \quad \mathcal{R}(\tau) > 0$$

that $L[g(\theta,t);\alpha] = \frac{1}{\sqrt{\pi\alpha}} \int_0^\infty \frac{e^{-(\alpha+\beta)\theta}}{\sqrt{\theta}} f(\theta) d\theta$. The uniqueness of Laplace transforms and the substitution $e^{-(\alpha+\beta)\theta} = x$ implies the required result. ∎

It will be shown that, if we set $f(\theta) = 1$ in the integral (31) and use (32), then Theorem 3 reduces to the following P. Levy's Arc-Sine Law for occupation time of $(0, \infty)$ [6, p. 273, Art 4.11].

Let $\alpha > 0$, $\beta > 0$. Then

(33) $$L[h(\theta,t);\alpha] = \frac{1}{\sqrt{\alpha(\alpha+\beta)}}$$

where

(34) $$h(\theta,t) = \frac{1}{\pi}\int_0^t \frac{e^{-\beta\theta}}{\sqrt{\theta(t-\theta)}} d\theta.$$

Assuming the existence of the Laplace transforms of $f(t)$, we consider

$$L\{f(t)\} = F(s)$$

and then using the rules of Laplace transform

$$L\{f(t+a)\} = e^{as}[F(s) - \int_0^a e^{-su} f(u) du], \quad a \geq 0$$

and

(35) $$L\{e^{-bt} f(t+a)\} = e^{a(s+b)}[F(s+b) - \int_0^a e^{-(s+b)u} f(u) du],$$

we get the following theorem.

Theorem 4. *If* $L\{f(t)\} = F(s)$, *then*

$$(36)\qquad \int_0^1 e^{(b\ln x)/s} f\left(a - \frac{\ln x}{s}\right) dx = s e^{a(s+b)} \left[F(s+b) - \int_0^a e^{-(s+b)u} f(u) du \right]$$

provided that $a \geq 0, \mathcal{R}(s) > 0$, *Laplace transform of* $|f(t)|$ *exist and integrals in* (36) *are convergent.*

Equation (36) is a generalization of the result [1, p. 239 (10)] which follows for $b = 0$.

Theorem 5. *If*

$$S_\rho\{g(t)\} = G(a;\rho)$$

then

$$(37)\qquad \int_0^1 \frac{x^{as-1} g(-a\ln x)}{(1-\ln x)^\rho} dx = e^{-s} a^{\rho-1} G(a;\rho).$$

Proof. The above theorem can be proved easily if the definition integral (2) is applied in the form

$$S_\rho\{e^{-st} g(t)\} = \int_0^\infty \frac{e^{-st} g(t)}{(a+t)^\rho} dt$$

which, after a change of integration variable and mere integration by parts together with (36), leads to

$$\begin{aligned}\int_0^1 \frac{x^{as-1} g(-a\ln x)}{(1-\ln x)^\rho} dx &= a^{\rho-1}[(e^{-s}-1)G(a;\rho) + s\int_0^\infty e^{-st} G(a;\rho) dt] \\ &= a^{\rho-1}[(e^{-s}-1)G(a;\rho) + G(a;\rho)].\end{aligned}$$

Particularly, for $s = 0$, (37) reduces to [1, p. 250, equation 81(b)]. ■

3. Applications

A number of applications of the formulas for the evaluation of finite and infinite logarithmic integrals, using the operational calculus technique of Section 2, can be given. We list some of them.

As an example of (36), we take $f(t) = 1/\sqrt{t}$ so that $L\{1/\sqrt{t}\} = \sqrt{\frac{\pi}{s}} = F(s)$ and (36) gives

$$(38)\qquad \int_0^1 e^{(b\ln x)/s} \frac{dx}{\sqrt{a - \ln x/s}} = s e^{(s+b)} \left[\sqrt{\frac{\pi}{s+b}} (1 - erf\sqrt{a(s+b)}) \right]$$

where $er f\xi$ is the error function [4]. For $b = 0$, (38) becomes a known result [1, p. 240 (14)].

Put $\alpha = as$ and $\beta = b/s$ in (36) and use $1 - erf\sqrt{\alpha} = erfc\sqrt{\alpha}$, where $\alpha > 0$, to get

$$\int_0^1 e^{\beta \ln x} \frac{dx}{\sqrt{\alpha - \ln x}} = e^{\alpha(1+\beta)} \sqrt{\frac{\pi}{1+\beta}} erfc\sqrt{\alpha(1+\beta)} \tag{39}$$

because $erf\xi$ then becomes the complementary error function $erfc\xi$ (see [4]).

In the second case, we start with $f(t) = \ln t$ and we use (36) to get

$$\int_0^1 e^{(b\ln x)/s} \ln\left(a - \frac{\ln x}{s}\right) dx = \frac{se^{a(s+b)}}{s+b} \left[e^{-a(s+b)} \ln a - Ei(-as - ab)\right], \tag{40}$$

where τ is the Euler's constant [4] and $\mathrm{Ei}(\xi)$ is the exponential integral [4].

Using the Stieltjes transform $S_\rho\{e^{-t}\} = e^a \Gamma(1-\rho, a)$, $\mathcal{R}(\rho) > 0$ where $\Gamma(*.*)$ is an incomplete gamma function in Theorem 5, we get

$$\begin{aligned} \int_0^1 \frac{x^{as+a-1} dx}{(1-\ln x)^\rho} &= a^{\rho-1} S_\rho\{e^{-t(s+1)}\} \\ &= [a(s+1)]^{\rho-1} e^{a(s+1)} \Gamma(1-\rho, a(s+1)), \quad a > 0,\ \rho > 0. \end{aligned} \tag{41}$$

For $s = 0$, we get

$$\int_0^1 \frac{x^{a-1} dx}{(1-\ln x)^\rho} = a^{\rho-1} e^a \Gamma(1-\rho, a), \quad a > 0, \rho > 0 \tag{42}$$

which is a correct form of the result [1, p. 252, 97(b)]. (42) becomes a known result [1, p. 252, 97(a)] when $\rho = 1$.

Next, we will turn our attention to the case when $g(t) = \sqrt{t}e^{-st}$ in (37). Thus, we have from [8, p. 233 (14.30)]

$$\int_0^1 \frac{\ln(x^{-1/s}) dx}{(as - \ln x)} = \sqrt{\pi a} e^{as} \Gamma(-\frac{1}{2}, as). \tag{43}$$

Formulas (41) to (42) show that a more general case can be considered by using $g(t) = t^{\lambda-1}e^{-st}$ in Theorem 5. By defining a generalization of gamma function (see Kobayashi [7])

$$\Gamma_{\rho,s}(\lambda, a) = \int_0^\infty t^{\lambda-1} e^{-st} (t+a)^{-\rho} dt, \quad \mathcal{R}(\lambda), \mathcal{R}(s) > 0 \tag{44}$$

and changing the integration variable, we have

$$\Gamma_{\rho,s}(\lambda, a) = s^{-\lambda} \int_0^1 \left(a - \frac{\ln x}{s}\right)^{-\rho} (-\ln x)^{\lambda-1} dx, \quad \mathcal{R}(\lambda), \mathcal{R}(s) > 0 \tag{45}$$

which, for $s = 1$, reduces to the known generalization of gamma function (5) given by Kobayashi [7]. Note that $\Gamma_{\rho,1}(\lambda, a) = \Gamma_\rho(\lambda, a)$. In view of the result [4, p. 294 (6)], it is more natural to work with (37) and (45) to obtain

$$\Gamma_{\rho,s}(\lambda, a) = \Gamma(\lambda)(sa)^{(\rho-\lambda-1)/2}e^{as/2}W_{k,m}(a, s), \tag{46}$$

where $k = \frac{1-\lambda-\rho}{2}$, $m = \frac{\lambda-\rho}{2}$ and $\mathcal{R}(\lambda) > 0$. Thus, we have provided an integral representation for $\Gamma_{\rho,s}(\lambda, a)$ in terms of Whittaker functions.

We now use Theorem 2 and result (24) to obtain a generalization of the result of Apelbalt [1, p. 238, (4)] involving the nth derivatives of psi-function

$$\int_0^1 \frac{x^{b/s}(\ln x)^n}{1 - x^{a/s}}dx = -\left(\frac{s}{a}\right)^{n+1}\varphi^{(n)}\left(\frac{s+b}{a}\right). \tag{47}$$

For $b = 0$, (47) gives a corrected form of the result [1, p. 238 (4)]

$$\int_0^1 \frac{(\ln x)^n}{1 - x^\alpha}dx = -\frac{1}{(\alpha)^{n+1}}\varphi^{(n)}\left(\frac{1}{\alpha}\right) \tag{48}$$

where

$$\alpha = a/s.$$

To prove (47), notice that for $n = 1, 2, \ldots,$

$$L\{f(t)\} = L\left\{\frac{t^n}{1 - e^{-at}}\right\} = (-a)^{-n-1}\varphi^{(n)}\left(\frac{s}{a}\right) = F(s) \tag{49}$$

which is a well-known result (see [4] and [1, p. 238 (3)]). From the operational relations (24) and (49), result (47) follows.

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O. A. Daman
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BOTSWANA
P/BAG 0022, GABORONE, BOTSWANA
E-mail: damanoa@mopip.ub.bw

M. A. Pathan
CENTRE FOR MATHEMATICAL SCIENCES
ARUNAPURAM P.O., PALA
KERALA - 686574, INDIA
E-mail: mapathan@gmail.com

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