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# FOUR-DIMENSIONAL MATRIX TRANSFORMATION AND $A$-STATISTICAL FUZZY KOROVKIN TYPE APPROXIMATION

**Abstract.** In this paper, we prove a fuzzy Korovkin-type approximation theorem for fuzzy positive linear operators by using $A$-statistical convergence for four-dimensional summability matrices. Also, we obtain rates of $A$-statistical convergence of a double sequence of fuzzy positive linear operators for four-dimensional summability matrices.

## 1. Introduction

Anastassiou [3] first introduced the fuzzy analogue of the classical Korovkin theory (see also [1], [2], [4], [10]). Recently, some statistical fuzzy approximation theorems have been obtain by using the concept of statistical convergence (see, [5], [8]). In this paper, we prove a fuzzy Korovkin-type approximation theorem for fuzzy positive linear operators by using $A$-statistical convergence for four-dimensional summability matrices. Then, we construct an example such that our new approximation result works but its classical case does not work. Also we obtain rates of $A$-statistical convergence of a double sequence of fuzzy positive linear operators for four-dimensional summability matrices.

We now recall some basic definitions and notations used in the paper.

A fuzzy number is a function $\mu : \mathbb{R} \to [0,1]$, which is normal, convex, upper semi-continuous and the closure of the set $supp(\mu)$ is compact, where $supp(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$. The set of all fuzzy numbers are denoted by $\mathbb{R}_{\mathcal{F}}$. Let

$$[\mu]^0 = \overline{\{x \in \mathbb{R} : \mu(x) > 0\}} \text{ and } [\mu]^r = \{x \in \mathbb{R} : \mu(x) \geq r\}, \ (0 < r \leq 1).$$

Then, it is well-known [11] that, for each $r \in [0,1]$, the set $[\mu]^r$ is a closed

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and bounded interval of $\mathbb{R}$. For any $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, it is possible to define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ as follows:

$$[u \oplus v]^r = [u]^r + [v]^r \text{ and } [\lambda \odot u]^r = \lambda [u]^r, \ (0 \leq r \leq 1).$$

Now denote the interval $[u]^r$ by $\left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$ and $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$ for $r \in [0,1]$. Then, for $u, v \in \mathbb{R}_{\mathcal{F}}$, define

$$u \preceq v \Leftrightarrow u_-^{(r)} \leq v_-^{(r)} \text{ and } u_+^{(r)} \leq v_+^{(r)} \text{ for all } 0 \leq r \leq 1.$$

Define also the following metric $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$ by

$$D(u,v) = \sup_{r \in [0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\}.$$

Hence, $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space [18].

A double sequence $x = \{x_{m,n}\}$, $m, n \in \mathbb{N}$, is convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{m,n} - L| < \varepsilon$ whenever $m, n > N$. Then, $L$ is called the Pringsheim limit of $x$ and is denoted by $P - \lim_{m,n} x_{m,n} = L$ (see [16]). In this case, we say that $x = \{x_{m,n}\}$ is "$P$-convergent to $L$". Also, if there exists a positive number $M$ such that $|x_{m,n}| \leq M$ for all $(m,n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, then $x = \{x_{m,n}\}$ is said to be bounded. Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded. A double sequence $x = \{x_{m,n}\}$ is said to be non-increasing in Pringsheim's sense if, for all $(m,n) \in \mathbb{N}^2$, $x_{m+1,n+1} \leq x_{m,n}$.

Now let $A = [a_{j,k,m,n}]$, $j, k, m, n \in \mathbb{N}$, be a four-dimensional summability matrix. For a given double sequence $x = \{x_{m,n}\}$, the $A$-transform of $x$, denoted by $Ax := \{(Ax)_{j,k}\}$, is given by

$$(Ax)_{j,k} = \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n} x_{m,n}, \quad j, k \in \mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(j,k) \in \mathbb{N}^2$. In summability theory, a two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization for two dimensional matrix transformations is known as Silverman–Toeplitz conditions (see, for instance, [13]). In 1926, Robison [17] presented a four dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double $P$-convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices is known as Robison–Hamilton conditions, or briefly, $RH$-regularity (see, [12], [17]).

Recall that a four dimensional matrix $A = [a_{j,k,m,n}]$ is said to be $RH$-regular if it maps every bounded $P$-convergent sequence into a $P$-convergent sequence with the same $P$-limit. The Robison–Hamilton conditions state that a four dimensional matrix $A = [a_{j,k,m,n}]$ is $RH$-regular if and only if

$(i)$ $P - \lim_{j,k} a_{j,k,m,n} = 0$ for each $(m, n) \in \mathbb{N}^2$,

$(ii)$ $P - \lim_{j,k} \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} = 1$,

$(iii)$ $P - \lim_{j,k} \sum_{m\in\mathbb{N}} |a_{j,k,m,n}| = 0$ for each $n \in \mathbb{N}$,

$(iv)$ $P - \lim_{j,k} \sum_{n\in\mathbb{N}} |a_{j,k,m,n}| = 0$ for each $m \in \mathbb{N}$,

$(v)$ $\sum_{(m,n)\in\mathbb{N}^2} |a_{j,k,m,n}|$ is $P$-convergent for each $(j, k) \in \mathbb{N}^2$,

$(vi)$ there exist finite positive integers $A$ and $B$ such that $\sum_{m,n>B} |a_{j,k,m,n}| < A$ holds for every $(j, k) \in \mathbb{N}^2$.

Now let $A = [a_{j,k,m,n}]$ be a non-negative $RH$-regular summability matrix, and let $K \subset \mathbb{N}^2$. Then, a double sequence $\{x_{m,n}\}$ of fuzzy numbers is said to be $A$-statistically convergent to a fuzzy number $L \in \mathbb{R}_{\mathcal{F}}$ if, for every $\varepsilon > 0$,

$$P - \lim_{j,k} \sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \{(m, n) \in \mathbb{N}^2 : D(x_{m,n}, L) \geq \varepsilon\}.$$

In this case we write $st^2_{(A)} - \lim_{m,n} x_{m,n} = L$.

We should note that if we take $A = C(1; 1) := [c_{j,k,m,n}]$, the double Cesáro matrix, defined by

$$c_{j,k,m,n} = \begin{cases} \frac{1}{jk}, \text{ if } 1 \leq m \leq j \text{ and } 1 \leq n \leq k, \\ 0, \qquad\qquad \text{otherwise,} \end{cases}$$

then $C(1; 1)$-statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in [14], [15]. Finally, if we replace the matrix $A$ by the identity matrix for four-dimensional matrices, then $A$-statistical convergence reduces to the Pringsheim convergence [16].

## 2. $A$-statistical fuzzy Korovkin type approximation

Let us choose the real numbers $a; b; c; d$ so that $a < b, c < d$, and $U := [a; b] \times [c; d]$. Let $C(U)$ denote the space of all real valued continuous functions on $U$ endowed with the supremum norm

$$\|f\| = \sup_{(x,y)\in U} |f(x, y)|, \ (f \in C(U)).$$

Assume that $f : U \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy number valued function. Then $f$ is said to be fuzzy continuous at $x^0 := (x_0, y_0) \in U$ whenever $P - \lim_{m,n} x_{m,n} = x^0$, then $P-\lim D(f(x_{m,n}),f(x^0))=0$. If it is fuzzy continuous at every point $(x,y)\in U$, we say that $f$ is fuzzy continuous on $U$. The set of all fuzzy continuous functions on $U$ is denoted by $C_{\mathcal{F}}(U)$. Note that $C_{\mathcal{F}}(U)$ is a vector space. Now let $L:C_{\mathcal{F}}(U)\to C_{\mathcal{F}}(U)$ be an operator. Then $L$ is said to be fuzzy linear if, for every $\lambda_1,\lambda_2\in\mathbb{R}$ having the same sing and for every $f_1,f_2\in C_{\mathcal{F}}(U)$, and $(x,y)\in U$,

$$L(\lambda_1\odot f_1\oplus\lambda_2\odot f_2;x,y)=\lambda_1\odot L(f_1;x,y)\oplus\lambda_2\odot L(f_2;x,y)$$

holds. Also $L$ is called fuzzy positive linear operator if it is fuzzy linear and, the condition $L(f;x,y)\preceq L(g;x,y)$ is satisfied for any $f,g\in C_{\mathcal{F}}(U)$ and all $(x,y)\in U$ with $f(x,y)\preceq g(x,y)$. Also, if $f,g:U\to\mathbb{R}_{\mathcal{F}}$ are fuzzy number valued functions, then the distance between $f$ and $g$ is given by

$$D^*(f,g)=\sup_{(x,y)\in U}\sup_{r\in[0,1]}\max\left\{\left|f_-^{(r)}-g_-^{(r)}\right|,\left|f_+^{(r)}-g_+^{(r)}\right|\right\}$$

(see for details, [1], [2], [3], [4], [9], [10]). Throughout the paper we use the test functions given by

$$f_0(x,y)=1,\ \ f_1(x,y)=x,\ f_2(x,y)=y,\ \ f_3(x,y)=x^2+y^2.$$

**Theorem 2.1.** *Let $A=[a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix and let $\{L_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ be a double sequence of fuzzy positive linear operators from $C_{\mathcal{F}}(U)$ into itself. Assume that there exists a corresponding sequence $\{\widetilde{L}_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ of positive linear operators from $C(U)$ into itself with the property*

$$\{L_{m,n}(f;x,y)\}_\pm^{(r)}=\widetilde{L}_{m,n}\left(f_\pm^{(r)};x,y\right) \tag{2.1}$$

*for all $(x,y)\in U$, $r\in[0,1]$, $(m,n)\in\mathbb{N}^2$ and $f\in C_{\mathcal{F}}(U)$. Assume further that*

$$st^2_{(A)}-\lim_{m,n\to\infty}\left\|\widetilde{L}_{m,n}(f_i)-f_i\right\|=0\ \ \text{for each } i=0,1,2,3. \tag{2.2}$$

*Then, for all $f\in C_{\mathcal{F}}(U)$, we have*

$$st^2_{(A)}-\lim_{m,n\to\infty}D^*(L_{m,n}(f),f)=0.$$

**Proof.** Let $f\in C_{\mathcal{F}}(U)$, $(x,y)\in U$ and $r\in[0,1]$. By the hypothesis, since $f_\pm^{(r)}\in C(U)$, we can write, for every $\varepsilon>0$, that there exists a number $\delta>0$ such that $\left|f_\pm^{(r)}(u,v)-f_\pm^{(r)}(x,y)\right|<\varepsilon$ holds for every $(u,v)\in U$ satisfying $|u-x|<\delta$ and $|v-y|<\delta$. Then we immediately get for all $(u,v)\in U$, that

$$\left|f_\pm^{(r)}(u,v)-f_\pm^{(r)}(x,y)\right|\le\varepsilon+\frac{2M_\pm^{(r)}}{\delta^2}\left\{(u-x)^2+(v-y)^2\right\},$$

where $M_{\pm}^{(r)} := \|f_{\pm}^{(r)}\|$. Now, using the linearity and the positivity of the operators $\widetilde{L}_{m,n}$, we have, for each $(m,n) \in \mathbb{N}^2$, that

$$\begin{aligned}
&\left|\widetilde{L}_{m,n}(f_{\pm}^{(r)};x,y) - f_{\pm}^{(r)}(x,y)\right| \\
&\le \widetilde{L}_{m,n}\left(\left|f_{\pm}^{(r)}(u,v) - f_{\pm}^{(r)}(x,y)\right|;x,y\right) + M_{\pm}^{(r)}\left|\widetilde{L}_{m,n}(f_0;x,y) - f_0(x,y)\right| \\
&\le \widetilde{L}_{m,n}\left(\varepsilon + \frac{2M_{\pm}^{(r)}}{\delta^2}\left\{(u-x)^2 + (v-y)^2\right\};x,y\right) + M_{\pm}^{(r)}\left|\widetilde{L}_{m,n}(f_0;x,y) - f_0(x,y)\right| \\
&\le \varepsilon + \left(\varepsilon + M_{\pm}^{(r)}\right)\left|\widetilde{L}_{m,n}(f_0;x,y) - f_0(x,y)\right| + \frac{2M_{\pm}^{(r)}}{\delta^2}\widetilde{L}_{m,n}\left((u-x)^2 + (v-y)^2;x,y\right) \\
&\le \varepsilon + \left(\varepsilon + M_{\pm}^{(r)}\right)\left|\widetilde{L}_{m,n}(f_0;x,y) - f_0(x,y)\right| + \frac{2M_{\pm}^{(r)}}{\delta^2}\left\{\left|\widetilde{L}_{m,n}(f_3;x,y) - f_3(x,y)\right|\right. \\
&\quad + 2|x|\left|\widetilde{L}_{m,n}(f_1;x,y) - f_1(x,y)\right| + 2|y|\left|\widetilde{L}_{m,n}(f_2;x,y) - f_2(x,y)\right| \\
&\quad \left. + \left(x^2 + y^2\right)\left|\widetilde{L}_{m,n}(f_0;x,y) - f_0(x,y)\right|\right\} \\
&\le \varepsilon + \left(\varepsilon + M_{\pm}^{(r)} + \frac{2M_{\pm}^{(r)}}{\delta^2}\left(x^2 + y^2\right)\right)\left|\widetilde{L}_{m,n}(f_0;x,y) - f_0(x,y)\right| \\
&\quad + \frac{4M_{\pm}^{(r)}}{\delta^2}|x|\left|\widetilde{L}_{m,n}(f_1;x,y) - f_1(x,y)\right| + \frac{4M_{\pm}^{(r)}}{\delta^2}|y|\left|\widetilde{L}_{m,n}(f_2;x,y) - f_2(x,y)\right| \\
&\quad + \frac{2M_{\pm}^{(r)}}{\delta^2}\left|\widetilde{L}_{m,n}(f_3;x,y) - f_3(x,y)\right| \\
&\le \varepsilon + K_{\pm}^{(r)}(\varepsilon)\left\{\left|\widetilde{L}_{m,n}(f_0;x,y) - f_0(x,y)\right| + \left|\widetilde{L}_{m,n}(f_1;x,y) - f_1(x,y)\right|\right. \\
&\quad \left. + \left|\widetilde{L}_{m,n}(f_2;x,y) - f_2(x,y)\right| + \left|\widetilde{L}_{m,n}(f_3;x,y) - f_3(x,y)\right|\right\}
\end{aligned}$$

where $K_{\pm}^{(r)}(\varepsilon) := \max\left\{\varepsilon + M_{\pm}^{(r)} + \frac{2M_{\pm}^{(r)}}{\delta^2}\left(A^2 + B^2\right), \frac{4M_{\pm}^{(r)}}{\delta^2}A, \frac{4M_{\pm}^{(r)}}{\delta^2}B, \frac{2M_{\pm}^{(r)}}{\delta^2}\right\}$, $A := \max\{|a|, |b|\}$, $B := \max\{|c|, |d|\}$. Also taking supremum over $(x,y) \in U$, the above inequality implies that

$$\begin{aligned}
\text{(2.3)} \quad &\left\|\widetilde{L}_{m,n}(f_{\pm}^{(r)}) - f_{\pm}^{(r)}\right\| \\
&\le \varepsilon + K_{\pm}^{(r)}(\varepsilon)\left\{\left\|\widetilde{L}_{m,n}(f_0) - f_0\right\| + \left\|\widetilde{L}_{m,n}(f_1) - f_1\right\|\right. \\
&\qquad \left. + \left\|\widetilde{L}_{m,n}(f_2) - f_2\right\| + \left\|\widetilde{L}_{m,n}(f_3) - f_3\right\|\right\}.
\end{aligned}$$

Now, it follows from (2.1) that

$$\begin{aligned} D^*\left(L_{m,n}(f), f\right) &= \sup_{(x,y)\in U} D\left(L_{m,n}(f;x,y), f(x,y)\right) \\ &= \sup_{(x,y)\in U} \sup_{r\in[0,1]} \max\Big\{\big|\widetilde{L}_{m,n}\big(f_-^{(r)};x,y\big) - f_-^{(r)}(x,y)\big|, \\ &\qquad\qquad \big|\widetilde{L}_{m,n}\big(f_+^{(r)};x,y\big) - f_+^{(r)}(x,y)\big|\Big\} \\ &= \sup_{r\in[0,1]} \max\Big\{\big\|\widetilde{L}_{m,n}\big(f_-^{(r)}\big) - f_-^{(r)}\big\|, \big\|\widetilde{L}_{m,n}\big(f_+^{(r)}\big) - f_+^{(r)}\big\|\Big\}. \end{aligned}$$

Combining the above equality with (2.3), we have

$$\begin{aligned} D^*\left(L_{m,n}(f), f\right) \leq \varepsilon + K(\varepsilon)\Big\{&\big\|\widetilde{L}_{m,n}(f_0) - f_0\big\| + \big\|\widetilde{L}_{m,n}(f_1) - f_1\big\| \\ &+ \big\|\widetilde{L}_{m,n}(f_2) - f_2\big\| + \big\|\widetilde{L}_{m,n}(f_3) - f_3\big\|\Big\} \end{aligned} \tag{2.4}$$

where $K(\varepsilon) := \sup_{r\in[0,1]} \max\big\{K_-^{(r)}(\varepsilon), K_+^{(r)}(\varepsilon)\big\}$.

Now, for a given $r > 0$, choose $\varepsilon > 0$ such that $0 < \varepsilon < r$, and also define the following sets:

$$\begin{aligned} G &:= \big\{(m,n) \in \mathbb{N}^2 : D^*\left(L_{m,n}(f), f\right) \geq r\big\}, \\ G_0 &:= \left\{(m,n) \in \mathbb{N}^2 : \big\|\widetilde{L}_{m,n}(f_0) - f_0\big\| \geq \frac{r-\varepsilon}{4K(\varepsilon)}\right\}, \\ G_1 &:= \left\{(m,n) \in \mathbb{N}^2 : \big\|\widetilde{L}_{m,n}(f_1) - f_1\big\| \geq \frac{r-\varepsilon}{4K(\varepsilon)}\right\}, \\ G_2 &:= \left\{(m,n) \in \mathbb{N}^2 : \big\|\widetilde{L}_{m,n}(f_2) - f_2\big\| \geq \frac{r-\varepsilon}{4K(\varepsilon)}\right\}, \\ G_3 &:= \left\{(m,n) \in \mathbb{N}^2 : \big\|\widetilde{L}_{m,n}(f_3) - f_3\big\| \geq \frac{r-\varepsilon}{4K(\varepsilon)}\right\}. \end{aligned}$$

Then inequality (2.4) gives

$$G \subset G_0 \cup G_1 \cup G_2 \cup G_3$$

which guarantees that, for each $(j,k) \in \mathbb{N}^2$

$$\begin{aligned} \sum_{(m,n)\in G} a_{j,k,m,n} \leq \sum_{(m,n)\in G_0} a_{j,k,m,n} &+ \sum_{(m,n)\in G_1} a_{j,k,m,n} \\ + \sum_{(m,n)\in G_2} a_{j,k,m,n} &+ \sum_{(m,n)\in G_3} a_{j,k,m,n}. \end{aligned} \tag{2.5}$$

If we take the limit as $j,k \to \infty$ on the both sides of inequality (2.5) and use the hypothesis (2.2), we immediately see that

$$\lim_{j,k} \sum_{(m,n)\in G} a_{j,k,m,n} = 0$$

whence the result. ∎

If $A = I$, the identity matrix, then we obtain the following new fuzzy Korovkin theorem in Pringsheim's sense.

**Theorem 2.2.** *Let $\{L_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ be a double sequence of fuzzy positive linear operators from $C_{\mathcal{F}}(U)$ into itself. Assume that there exists a corresponding sequence $\{\widetilde{L}_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ of positive linear operators from $C(U)$ into itself with the property (2.1). Assume further that*

$$P-\lim_{m,n\to\infty} \left\| \widetilde{L}_{m,n}(f_i) - f_i \right\| = 0 \quad \text{for each } i = 0,1,2,3.$$

*Then, for all $f \in C_{\mathcal{F}}(U)$, we have*

$$P-\lim_{m,n\to\infty} D^*\left(L_{m,n}(f), f\right) = 0.$$

We will now show that our result Theorem 2.1 is stronger than its classical (Theorem 2.2) version.

**Example 2.3.** Take $A = C(1,1) := [c_{j,k,m,n}]$, the double Cesáro matrix, and define the double sequence $\{u_{m,n}\}$ by

$$u_{m,n} = \begin{cases} \sqrt{mn}, & \text{if } m \text{ and } n \text{ are square,} \\ 0, & \text{otherwise.} \end{cases}$$

We observe that, $st^{(2)}_{C(1,1)} - \lim_{m,n\to\infty} u_{m,n} = 0$. But $\{u_{m,n}\}$ is neither $P$-convergent nor bounded. Then consider the fuzzy Bernstein-type polynomials as follows:

$$B^{(\mathcal{F})}_{m,n}(f;x,y) = (1+u_{m,n}) \odot \bigoplus_{s=0}^{m} \odot \bigoplus_{t=0}^{n} \binom{m}{s}\binom{n}{t} x^s y^t (1-x)^{m-s}(1-y)^{n-t} \odot f\left(\frac{s}{m}, \frac{t}{n}\right), \tag{2.6}$$

where $f \in C_{\mathcal{F}}(U)$, $(x,y) \in U$, $(m,n) \in \mathbb{N}^2$. In this case, we write

$$\begin{aligned} \left\{B^{(\mathcal{F})}_{m,n}(f;x,y)\right\}^{(r)}_{\pm} &= \widetilde{B}_{m,n}\left(f^{(r)}_{\pm};x,y\right) \\ &= (1+u_{m,n}) \sum_{s=0}^{m}\sum_{t=0}^{n} \binom{m}{s}\binom{n}{t} x^s y^t (1-x)^{m-s}(1-y)^{n-t} f^{(r)}_{\pm}\left(\frac{s}{m}, \frac{t}{n}\right), \end{aligned}$$

where $f_{\pm}^{(r)} \in C(U)$. Then, we get

$$\begin{aligned}
\widetilde{B}_{m,n}(f_0;x,y) &= (1+u_{m,n})\, f_0(x,y),\\
\widetilde{B}_{m,n}(f_1;x,y) &= (1+u_{m,n})\, f_1(x,y),\\
\widetilde{B}_{m,n}(f_2;x,y) &= (1+u_{m,n})\, f_2(x,y),\\
\widetilde{B}_{m,n}(f_3;x,y) &= (1+u_{m,n})\left(f_3(x,y) + \frac{x-x^2}{m} + \frac{y-y^2}{n}\right).
\end{aligned}$$

So we conclude that

$$st^2_{C(1,1)} - \lim_{m,n\to\infty} \left\|\widetilde{B}_{m,n}(f_i) - f_i\right\| = 0 \quad \text{for each } i = 0,1,2,3.$$

By Theorem 2.1, we obtain for all $f \in C_{\mathcal{F}}(U)$, that

$$st^2_{C(1,1)} - \lim_{m,n\to\infty} D^*\left(B^{(\mathcal{F})}_{m,n}(f), f\right) = 0.$$

However, since the sequence $\{u_{m,n}\}$ is not convergent (in the Pringsheim's sense), we conclude that Theorem 2.2 do not work for the operators $\left\{B^{(\mathcal{F})}_{m,n}(f;x,y)\right\}$ in (2.6) while our Theorem 2.1 still works.

## 3. $A$-statistical fuzzy rates

Various ways of defining rates of convergence in the $A$-statistical sense for two-dimensional summability matrices were introduced in [7]. In a similar way, we obtain fuzzy approximation theorems based on $A$-statistical rates for four-dimensional summability matrices.

**Definition 3.1.** Let $A = [a_{j,k,m,n}]$ be a non-negative $RH$-regular summability matrix and let $\{\alpha_{m,n}\}$ be a non-increasing double sequence of positive real numbers. A double sequence $x = \{x_{m,n}\}$ of fuzzy numbers is $A$-statistically convergent to a fuzzy number $L$ with the rate of $o(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$P - \lim_{j,k\to\infty} \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \left\{(m,n) \in \mathbb{N}^2 : D(x_{m,n}, L) \geq \varepsilon\right\}.$$

In this case, we write

$$D(x_{m,n}, L) = st^2_{(A)} - o(\alpha_{m,n}) \quad \text{as} \quad m,n \to \infty.$$

**Definition 3.2.** Let $A = [a_{j,k,m,n}]$ and $\{\alpha_{m,n}\}$ be the same as in Definition 3.1. Then, a double sequence $x = \{x_{m,n}\}$ of fuzzy numbers is $A$-statistically convergent to a fuzzy number $L$ with the rate of $o_{m,n}(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$P - \lim_{j,k\to\infty} \sum_{(m,n)\in M(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$M(\varepsilon) := \{(m,n) \in \mathbb{N}^2 : D(x_{m,n}, L) \geq \varepsilon\ \alpha_{m,n}\}.$$

In this case, we write

$$D(x_{m,n}, L) = st^2_{(A)} - o_{m,n}(\alpha_{m,n}) \text{ as } m,n \to \infty.$$

Note that the rate of convergence given by Definition 3.1 is more controlled by the entries of the summability matrices rather than the terms of the sequence $x = \{x_{m,n}\}$. However, according to the statistical rate given by Definition 3.2, the rate is mainly controlled by the terms of the fuzzy sequence $x = \{x_{m,n}\}$.

Also, we can give the corresponding $A$-statistical rates of real sequence $\{x_{m,n}\}$.

**Definition 3.3.** [6] Let $A = [a_{j,k,m,n}]$ be a non-negative $RH$-regular summability matrix and let $\{\alpha_{m,n}\}$ be a non-increasing double sequence of positive real numbers. A double sequence $x = \{x_{m,n}\}$ is $A$-statistically convergent to a number $L$ with the rate of $o(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$P - \lim_{j,k\to\infty} \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\}.$$

In this case, we write

$$x_{m,n} - L = st^2_{(A)} - o(\alpha_{m,n}) \text{ as } m,n \to \infty.$$

**Definition 3.4.** [6] Let $A = [a_{j,k,m,n}]$ and $\{\alpha_{m,n}\}$ be the same as in Definition 3.3. Then, a double sequence $x = \{x_{m,n}\}$ is $A$-statistically convergent to a number $L$ with the rate of $o_{m,n}(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$P - \lim_{j,k\to\infty} \sum_{(m,n)\in M(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$M(\varepsilon) := \{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\ \alpha_{m,n}\}.$$

In this case, we write

$$x_{m,n} - L = st^2_{(A)} - o_{m,n}(\alpha_{m,n}) \text{ as } m,n \to \infty.$$

Then we have the following.

**Theorem 3.5.** *Let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix and let $\{L_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ be a double sequence of fuzzy positive linear operators from $C_{\mathcal{F}}(U)$ into itself. Assume that there exists a corresponding sequence $\{\widetilde{L}_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ of positive linear operators from $C(U)$ into itself with the property (2.1). Assume further that $\{\alpha_{i,m,n}\}_{(m,n)\in\mathbb{N}^2}$, $i = 0,1,2,3$ are non-ingreasing sequences of positive real numbers. If, for each $i = 0,1,2,3$*

$$\left\|\widetilde{L}_{m,n}(f_i) - f_i\right\| = st^2_{(A)} - o(\alpha_{i,m,n}) \quad as \quad m,n\to\infty \tag{3.1}$$

*then, for all $f \in C_{\mathcal{F}}(U)$, we have*

$$D^*\left(L_{m,n}(f), f\right) = st^2_{(A)} - o(\gamma_{m,n}) \quad as \quad m,n\to\infty \tag{3.2}$$

*where $\gamma_{m,n} := \max\limits_{0\le i\le 3}\{\alpha_{i,m,n}\}$ for every $(m,n)\in\mathbb{N}^2$.*

**Proof.** Let $f \in C_{\mathcal{F}}(U)$, $(x,y)\in U$ and $r\in[0,1]$. Then, we immediately see from Theorem 2.1's proof that, for every $\varepsilon > 0$,

$$\begin{aligned} D^*\left(L_{m,n}(f), f\right) \le \varepsilon + K(\varepsilon)\Big\{&\left\|\widetilde{L}_{m,n}(f_0) - f_0\right\| + \left\|\widetilde{L}_{m,n}(f_1) - f_1\right\| \\ &+ \left\|\widetilde{L}_{m,n}(f_2) - f_2\right\| + \left\|\widetilde{L}_{m,n}(f_3) - f_3\right\|\Big\} \end{aligned} \tag{3.3}$$

where $K(\varepsilon) := \sup\limits_{r\in[0,1]} \max\left\{K_-^{(r)}(\varepsilon), K_+^{(r)}(\varepsilon)\right\}$.

Now, for a given $r > 0$, choose $\varepsilon > 0$ such that $0 < \varepsilon < r$, and also define the following sets:

$$\begin{aligned} G &:= \left\{(m,n)\in\mathbb{N}^2 : D^*\left(L_{m,n}(f), f\right) \ge r\right\}, \\ G_0 &:= \left\{(m,n)\in\mathbb{N}^2 : \left\|\widetilde{L}_{m,n}(f_0) - f_0\right\| \ge \frac{r-\varepsilon}{4K(\varepsilon)}\right\}, \\ G_1 &:= \left\{(m,n)\in\mathbb{N}^2 : \left\|\widetilde{L}_{m,n}(f_1) - f_1\right\| \ge \frac{r-\varepsilon}{4K(\varepsilon)}\right\}, \\ G_2 &:= \left\{(m,n)\in\mathbb{N}^2 : \left\|\widetilde{L}_{m,n}(f_2) - f_2\right\| \ge \frac{r-\varepsilon}{4K(\varepsilon)}\right\}, \\ G_3 &:= \left\{(m,n)\in\mathbb{N}^2 : \left\|\widetilde{L}_{m,n}(f_3) - f_3\right\| \ge \frac{r-\varepsilon}{4K(\varepsilon)}\right\}. \end{aligned}$$

Then inequality (3.3) gives

$$G \subset G_0 \cup G_1 \cup G_2 \cup G_3$$

which guarantees that, for each $(j,k)\in\mathbb{N}^2$

$$\sum_{(m,n)\in G} a_{j,k,m,n} \le \sum_{i=0}^{3}\Big(\sum_{(m,n)\in G_i} a_{j,k,m,n}\Big).$$

Also, by the definition of $(\gamma_{m,n})_{(m,n)\in\mathbb{N}^2}$, we have

$$(3.4)\qquad \frac{1}{\gamma_{j,k}}\sum_{(m,n)\in G} a_{j,k,m,n} \le \sum_{i=0}^{3}\left(\frac{1}{\alpha_{i,j,k}}\sum_{(m,n)\in G_i} a_{j,k,m,n}\right).$$

If we take the limit as $j,k\to\infty$ on both sides of inequality (3.4) and use the hypothesis (3.1), we immediately see that

$$P-\lim_{j,k\to\infty}\frac{1}{\gamma_{j,k}}\sum_{(m,n)\in G} a_{j,k,m,n},$$

which gives (3.2). So, the proof is completed. ■

We also give the next result.

**Theorem 3.6.** *Let $A=[a_{j,k,m,n}]$, $\{\alpha_{i,m,n}\}_{(m,n)\in\mathbb{N}^2}$ $(i=0,1,2,3)$, $\{\gamma_{m,n}\}_{(m,n)\in\mathbb{N}^2}$, $\{L_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ and $\{\widetilde{L}_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ be the same as in Theorem* 3.5 *with the property* (2.1). *If, for each $i=0,1,2,3$*

$$(3.5)\qquad \left\|\widetilde{L}_{m,n}(f_i)-f_i\right\| = st^2_{(A)}-o_{m,n}(\alpha_{i,m,n}) \quad \text{as}\quad m,n\to\infty$$

*then, for all $f\in C_{\mathcal{F}}(U)$, we have*

$$(3.6)\qquad D^*\left(L_{m,n}(f),f\right) = st^2_{(A)}-o_{m,n}(\gamma_{m,n}) \quad \text{as}\quad m,n\to\infty.$$

**Proof.** By (3.3), it is clear that, for any $\varepsilon>0$,

$$(3.7)\qquad \begin{aligned} &D^*\left(L_{m,n}(f),f\right)\\ &\le \varepsilon\gamma_{m,n}+B(\varepsilon)\left\{\left\|\widetilde{L}_{m,n}(f_0)-f_0\right\|+\left\|\widetilde{L}_{m,n}(f_1)-f_1\right\|\right.\\ &\qquad\left.+\left\|\widetilde{L}_{m,n}(f_2)-f_2\right\|+\left\|\widetilde{L}_{m,n}(f_3)-f_3\right\|\right\}\end{aligned}$$

holds for some $B(\varepsilon)>0$. Now, as in the proof of Theorem 3.5, for a given $\varepsilon'>0$, choosing $\varepsilon>0$ such that $\varepsilon<\varepsilon'$. Now we define the following sets:

$$E := \left\{(m,n)\in\mathbb{N}^2 : D^*\left(L_{m,n}(f),f\right)\ge \varepsilon'\gamma_{m,n}\right\},$$

$$E_0 := \left\{(m,n)\in\mathbb{N}^2 : \left\|\widetilde{L}_{m,n}(f_0)-f_0\right\|\ge\left(\frac{\varepsilon'-\varepsilon}{4B(\varepsilon)}\right)\alpha_{0,m,n}\right\},$$

$$E_1 := \left\{(m,n)\in\mathbb{N}^2 : \left\|\widetilde{L}_{m,n}(f_1)-f_1\right\|\ge\left(\frac{\varepsilon'-\varepsilon}{4B(\varepsilon)}\right)\alpha_{1,m,n}\right\},$$

$$E_2 := \left\{(m,n)\in\mathbb{N}^2 : \left\|\widetilde{L}_{m,n}(f_2)-f_2\right\|\ge\left(\frac{\varepsilon'-\varepsilon}{4B(\varepsilon)}\right)\alpha_{2,m,n}\right\},$$

$$E_3 := \left\{(m,n)\in\mathbb{N}^2 : \left\|\widetilde{L}_{m,n}(f_3)-f_3\right\|\ge\left(\frac{\varepsilon'-\varepsilon}{4B(\varepsilon)}\right)\alpha_{3,m,n}\right\}.$$

In this case, we claim that

$$E \subset E_0 \cup E_1 \cup E_2 \cup E_3. \tag{3.8}$$

Indeed, otherwise, there would be an element $(m,n) \in E$ but $(m,n) \notin E_0 \cup E_1 \cup E_2 \cup E_3$. So, we get

$$(m,n) \notin E_0 \Rightarrow \left\| \widetilde{L}_{m,n}(f_0) - f_0 \right\| < \left( \frac{\varepsilon' - \varepsilon}{4B(\varepsilon)} \right) \alpha_{0,m,n},$$

$$(m,n) \notin E_1 \Rightarrow \left\| \widetilde{L}_{m,n}(f_1) - f_1 \right\| < \left( \frac{\varepsilon' - \varepsilon}{4B(\varepsilon)} \right) \alpha_{1,m,n},$$

$$(m,n) \notin E_2 \Rightarrow \left\| \widetilde{L}_{m,n}(f_2) - f_2 \right\| < \left( \frac{\varepsilon' - \varepsilon}{4B(\varepsilon)} \right) \alpha_{2,m,n},$$

$$(m,n) \notin E_3 \Rightarrow \left\| \widetilde{L}_{m,n}(f_3) - f_3 \right\| < \left( \frac{\varepsilon' - \varepsilon}{4B(\varepsilon)} \right) \alpha_{3,m,n}.$$

By the definition of $\{\gamma_{m,n}\}_{(m,n)\in\mathbb{N}^2}$, we immediately see that

$$B(\varepsilon) \sum_{k=0}^{3} \left\| \widetilde{L}_{m,n}(f_k) - f_k \right\| < (\varepsilon' - \varepsilon)\, \gamma_{m,n}. \tag{3.9}$$

Since $(m,n) \in E$, we have $D^*(L_{m,n}(f), f) \geq \varepsilon' \gamma_{m,n}$, and hence, by (3.7),

$$B(\varepsilon) \sum_{k=0}^{3} \left\| \widetilde{L}_{m,n}(f_k) - f_k \right\| \geq (\varepsilon' - \varepsilon)\, \gamma_{m,n},$$

which contradicts with (3.9). So, our claim (3.8) holds true. Now, it follows from (3.8) that

$$\sum_{(m,n)\in E} a_{j,k,m,n} \leq \sum_{i=0}^{3} \Big( \sum_{(m,n)\in E_i} a_{j,k,m,n} \Big). \tag{3.10}$$

Letting $j,k \to \infty$ in (3.10) and using (3.5), we observe that

$$P - \lim_{j,k\to\infty} \sum_{(m,n)\in E} a_{j,k,m,n},$$

which means (3.6). The proof is completed. ■

**Remark 3.7.** If $\alpha_{i,m,n} \equiv 1$ for each $i = 0,1,2,3$, then Theorem 3.6 reduced to Theorem 2.1. Also, if $A = I$, the identity matrix, $\alpha_{i,m,n} \equiv 1$ for each $i = 0,1,2,3$, then Theorem 3.6 reduced to Theorem 2.2.

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