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ON EXPONENTIAL DECAY FOR LINEAR POROUS-THERMO-ELASTICITY SYSTEM

Abstract. We study the problem of exponential decaying for solutions of porous-thermoelasticity system, when time $t \rightarrow \infty$. For sufficiently small values of the parameter of intensity of elasticity-porosity interactions the exponential decaying is established.

In the considerations we apply the idea of compact decoupling for the system of equations. The exponential decaying property is proved first for the corresponding decoupled system, which is simpler to handle, then the property is derived for the original system.

1. Introduction

In this paper we continue research, which we have begun in [7], on decaying of solutions as $t \rightarrow \infty$ for the thermoelasticity system of viscoporous media.

We consider the following system:

$$\begin{aligned}
 (1.1) \quad & \partial_t^2 u = \Delta_e u + b \nabla \phi - M \nabla \theta \quad \text{in } \Omega \times R_+, \\
 & \partial_t^2 \phi = a \Delta \phi - b \operatorname{div} u - \gamma \phi - r \partial_t \phi + M_1 \theta \quad \text{in } \Omega \times R_+, \\
 & \partial_t \theta = d \Delta \theta - M \operatorname{div} \partial_t u - M_1 \partial_t \phi \quad \text{in } \Omega \times R_+, \\
 & u = 0, \quad \phi = 0, \quad \theta = 0 \quad \text{on } \partial \Omega \times R_+, \\
 & u(0) = u^0, \partial_t u(0) = u^1, \phi(0) = \phi^0, \partial_t \phi(0) = \phi^1, \theta(0) = \theta^0 \quad \text{in } \Omega.
 \end{aligned}$$

In the above we denoted: $\Omega \subset R^n$, $n = 2, 3$, is open bounded set, the regularity of $\partial \Omega$ will be precised later, $\Delta_e = \mu \Delta I + (\mu + \lambda) \nabla \operatorname{div}$ is Lamé operator, $R_+ := (0, \infty)$ and coefficients $a, b, d, r, M, M_1, r, \mu > 0$ (more precise constraints on coefficients will be introduced later).

Interpretation of u, ϕ, θ and mechanical justification of (1.1) is given in [4] and was recalled in [8]. We recall that $u \in R^n$ is the displacement vector for

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media occupying domain Ω , ϕ denotes the change of volume fraction relative to equilibrium configuration, θ is the temperature.

In the deep papers [2], [11] the speed of energy decay for thermoelastic system is studied (ϕ is excluded from system (1.1)). Some theorems proved in these papers will be recalled in Section 3 of our paper.

Methods developed in our paper allow us only to establish the exponential decaying of energy of solution of (1.1). In the paper [2] the possibility of exponential as well as polynomial speed of decay of energy was shown.

The system (1.1) in one dimensional case was studied in [12] and it was proved that when $r > 0$, the exponential decay takes place, and when $r = 0$ this effect does not occur. It is known, that for thermoelasticity system (without ϕ) for $n = 1$ the energy decays exponentially (see references in [7]). These facts mean that the interaction between ϕ and θ works against the damping – therefore the problem stated in this paper is interesting.

To the authors knowledge there are no papers concerning the speed of decaying of energy of thermoelastic system with boundary conditions other than Dirichlet conditions (see [2], [11] and literature cited therein). Because the methods we use in this paper rely on the results of [11], we consider the Dirichlet boundary conditions as well. The problem with other boundary conditions remains completely open.

By Δ we shall denote the Laplace operator with domain $H^2(\Omega) \cap H_0^1(\Omega)$, and the range space $L^2(\Omega)$. Let $P := \nabla (\Delta^{-1}) \operatorname{div}$, where the operator div is considered as acting: $\operatorname{div} : L^2(\Omega)^n \rightarrow H^{-1}(\Omega)$. From [15] we know that P is the orthogonal projection operator in $(L^2(\Omega))^n$ onto the subspace $\{\nabla \psi : \psi \in H_0^1(\Omega)\}$; the proof that this subspace is closed in $(L^2(\Omega))^n$ is done in [5].

The decoupled system corresponding to system (1.1) will have the following form:

$$\begin{aligned}
 (1.2) \quad & \partial_t^2 \bar{u} = \Delta_\epsilon \bar{u} + b \nabla \bar{\phi} - \frac{M^2}{d} P \partial_t \bar{u} \quad \text{in } \Omega \times R_+, \\
 & \partial_t^2 \bar{\phi} = a \Delta \bar{\phi} - b \operatorname{div} \bar{u} - \gamma \bar{\phi} - r \partial_t \bar{\phi} \quad \text{in } \Omega \times R_+, \\
 & \partial_t \bar{\theta} = d \Delta \bar{\theta} - M \operatorname{div} \partial_t \bar{u} - M_1 \partial_t \bar{\phi} \quad \text{in } \Omega \times R_+.
 \end{aligned}$$

Initial and boundary conditons are of the same form as in the system (1.1).

The motivation for introducing the decoupled system (1.2) as a perturbation of (1.1) is the same as in the context of the classical system of thermoelasticity [10], [11], [15]. We drop the term $M_1 \theta$ in the second equation in (1.1) because θ as a solution of parabolic equation has a good regularity

properties. Then, keeping the leading terms in the third equation, we get:

$$\begin{aligned}\Delta\theta &= \frac{M}{d}\operatorname{div}\partial_t u, \quad \text{in } R_+ \times \Omega, \\ \theta &= 0 \quad \text{on } R_+ \times \partial\Omega.\end{aligned}$$

Because of the structure of P described above, this means that $\nabla\theta = \frac{M}{d}P\partial_t u$. By replacing $\nabla\theta$ in the first equation in (1.1) by $\frac{M}{d}P\partial_t u$ we obtain the first equation in (1.2).

In Section 4 we show that for appropriate domains Ω and the parameter $b > 0$ sufficiently small, the energy of solutions of (1.2) decays with exponential speed when $t \rightarrow \infty$. The proof will be based on a concept of expansion of the solution of (1.2) into a series of functions (independent of b) multiplied by powers of b .

Besides this, we show that solutions of system (1.2) are described by the c_0 -semigroup, and we prove the estimation for $\|\operatorname{div}\bar{u}(\cdot)\|_{L^2(\partial\Omega \times [0,t])}$, $t > 0$. The latter result will be necessary in Section 5 for proving the compactness of $S(t) - \bar{S}(t)$, $t \in [0, T]$, $T > 0$, where $S(\cdot), \bar{S}(\cdot)$ are c_0 -semigroups connected with systems (1.1), (1.2). These results from Sections 4, 5 will be used in Section 6 for deriving the main result of the paper - about exponential stabilization of solutions of (1.1), provided $b > 0$ is sufficiently small.

2. Assumptions on coefficients, basic spaces and recalling main results from [8]

Let us denote

$$\begin{aligned}\epsilon_{ij}(u) &:= \frac{1}{2}(\partial_j u_i + \partial_i u_j), \\ \sigma_{ij}(u) &:= \lambda \sum_{l=1}^n \epsilon_{ll}(u) \delta_{ij} + 2\mu \epsilon_{ij}(u), \quad i, j \in \{1, \dots, n\}, \\ \sigma(u) : \epsilon(v) &:= \sum_{i,j=1}^n \sigma_{ij}(u) \epsilon_{ij}(v).\end{aligned}$$

We remind that $(\Delta_e u)_i = \sum_{j=1}^n \partial_j \sigma_{ij}(u)$, $i \in \{1, \dots, n\}$.

ASSUMPTION 2.1. We require $\lambda + \mu > 0$, $(\lambda + \mu)\gamma > b^2$ when $n = 2$ and $3\lambda + 2\mu > 0$, $(3\lambda + 2\mu)\gamma > 3b^2$ when $n = 3$.

It was proved in [7] that

PROPOSITION 2.2. *If b, γ, ν, λ satisfy Assumption 2.1 then there exist constants $c_1, c_2 > 0$ such that*

$$\sigma(u) : \epsilon(u) \geq c_1 \sum_{i,j=1}^n \epsilon_{ij}^2(u),$$

$$\sigma(u) : \epsilon(u) + 2b\phi \operatorname{div} u + \gamma\phi^2 \geq c_2 \left(\phi^2 + \sum_{i,j=1}^n \epsilon_{ij}^2(u) \right).$$

We define spaces $V = H_0^1(\Omega)^n \times H_0^1(\Omega)$, $H = V \times (L^2(\Omega))^{n+1} \times L^2(\Omega)$. It was proved in [7] that

PROPOSITION 2.3. *The bilinear form*

$$\left\langle \begin{pmatrix} u^1 \\ \phi^1 \end{pmatrix}, \begin{pmatrix} u^2 \\ \phi^2 \end{pmatrix} \right\rangle_V := \int_{\Omega} [\sigma(u^1) : \epsilon(u^2) + a \nabla \phi^1 \cdot \nabla \phi^2 + \gamma \phi^1 \phi^2 + b \phi^1 \operatorname{div} v^2 + b \phi^2 \operatorname{div} u^1]$$

is the scalar product in V and V is the Hilbert space.

Let $\xi^i := (u^i, \phi^i, v^i, \psi^i, \theta^i)^T \in H$, $i = 1, 2$. The bilinear form

$$(\xi^1, \xi^2) := \left\langle \begin{pmatrix} u^1 \\ \phi^1 \end{pmatrix}, \begin{pmatrix} u^2 \\ \phi^2 \end{pmatrix} \right\rangle_V + \int_{\Omega} [v^1 \cdot v^2 + \psi^1 \psi^2 + \theta^1 \theta^2]$$

is the inner product in H and H is the Hilbert space.

The norm generated by this scalar product will be denoted by $\|\cdot\|$. We shall also use the dense subspace $X \subset H$, $X := (H^2(\Omega) \cap H_0^1(\Omega))^n \times (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)^n \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))$.

We rewrite (1.1) as the ordinary differential equation in H :

$$(2.1) \quad \begin{aligned} \frac{d\xi}{dt} &= L\xi, \quad t > 0 \\ \xi(0) &= \xi^0, \end{aligned}$$

where $\xi = (u, \phi, v, \psi, \theta) \in H$, $v \equiv \partial_t u$, $\psi \equiv \partial_t \phi$, $\xi^0 = (u^0, \phi^0, u^1, \phi^1, \theta^0) \in H$, and

$$L := \begin{pmatrix} 0 & , & 0 & , & I & , & 0 & , & 0 \\ 0 & , & 0 & , & 0 & , & I & , & 0 \\ \Delta_e & , & b\nabla & , & 0 & , & 0 & , & -M\nabla \\ -b\operatorname{div} & , & (a\Delta - \gamma I) & , & 0 & , & -rI & , & M_1 I \\ 0 & , & 0 & , & -M\operatorname{div} & , & -M_1 I & , & d\Delta \end{pmatrix},$$

with domain X . We see that $L : X \rightarrow H$.

It was proved in [8] (Theorem 2.7):

THEOREM 2.4. *Let the coefficients satisfy Assumption 2.1, $\partial\Omega$ has regularity of class C^2 .*

Then on H the operator L generates the c_0 -semigroup of contractions $S(t)$, $t \geq 0$. Moreover, when $(u(t), \phi(t), v(t), \psi(t), \theta(t))^T := S(t)\xi^0$ then $v(t) = \partial_t u(t)$, $\psi(t) = \partial_t \phi(t)$, $t > 0$, and $(u(t), \phi(t), \theta(t))$, $t \geq 0$ for $\xi^0 \in X$ is the unique strong solution of (1.1), and for $\xi^0 \in H$ is the unique weak solution of (1.1).

We recall from [8] that $S(t)\xi^0 \in H$ when $\xi^0 \in H$, $S(t)\xi^0 \in X$ when $\xi^0 \in X$, $S(t)\xi^0 \in D(L^k)$ when $\xi^0 \in D(L^k)$, $k \in \mathbb{N}$, $t > 0$.

For strong and weak solutions their energy is given by the formula (see [8]):

$$E(t) := \frac{1}{2} \|S(t)\xi^0\|^2 \equiv \frac{1}{2} \|(u(t), \phi(t), \partial_t u(t), \partial_t \phi(t), \theta(t))\|^2.$$

Because of contractivity property of the semigroup S we have $E(t_2) \leq E(t_1)$ when $0 \leq t_1 \leq t_2$.

3. Some necessary facts proved in papers of Lebeau–Zuazua [11] and Burq–Lebeau [2]

The problem of speed of vanishing of energy when $t \rightarrow \infty$ for the system

$$\begin{aligned} (3.1) \quad & \partial_t^2 u = \Delta_e u - \alpha \nabla \theta \quad \text{in } \Omega \times R_+, \\ & \partial_t \theta = \Delta \theta - \beta \operatorname{div} \partial_t u \quad \text{in } \Omega \times R_+, \\ & u = 0, \theta = 0 \quad \text{on } \partial\Omega \times R_+, \\ & u(0) = u^0, \partial_t u(0) = u^1, \theta(0) = \theta^0 \quad \text{in } \Omega, \end{aligned}$$

was analyzed in papers [2], [11]. The coefficients μ, λ of operator Δ_e in (3.1) must satisfy conditions $\mu > 0, \lambda + 2\mu > 0, \lambda \neq -\mu$, and the coefficients $\alpha, \beta > 0$.

It is easily seen that μ, λ satisfying the Assumption 2.1 will also satisfy the above assumptions.

The decoupled system corresponding to (3.1) was considered in [11]:

$$\begin{aligned} (3.2) \quad & \partial_t^2 \bar{u} = \Delta_e \bar{u} - \alpha \beta P(\partial_t \bar{u}) \quad \text{in } \Omega \times R_+, \\ & \partial_t \bar{\theta} = \Delta \bar{\theta} - \beta \operatorname{div} \partial_t \bar{\theta} \quad \text{in } \Omega \times R_+, \\ & \bar{u} = 0, \bar{\theta} = 0 \quad \text{on } \partial\Omega \times R_+, \\ & \bar{u}(0) = u^0, \partial_t \bar{u}(0) = u^1, \bar{\theta}(0) = \theta^0 \quad \text{in } \Omega. \end{aligned}$$

It was proved in [11] that solutions of systems (3.1), (3.2) are described by appropriate c_0 -semigroups. Energy of solutions of (3.1) is defined by the formula:

$$(3.3) \quad E_0(t) = \frac{1}{2} \int_{\Omega} \left[|\partial_t u(x, t)|^2 + \sigma(u(x, t)) : \epsilon(u(x, t)) + \frac{\alpha}{\beta} \theta(x, t)^2 \right] dx.$$

The energy of solutions $(\bar{u}, \bar{\theta})$ of (3.2) is defined by the same formula, with u, θ replaced by $\bar{u}, \bar{\theta}$, and the energy of $(\bar{u}, \bar{\theta})$ we shall denote by $\bar{E}_0(\cdot)$.

DEFINITION 3.1. We say that system (3.1) or system (3.2) has the property of uniform decaying of energy, when there exist constants $c > 0, \omega > 0$ such that

$$\begin{aligned} E_0(t) &\leq ce^{-\omega t} E_0(0) \quad \text{for every } t > 0, \\ (\bar{E}_0(t) &\leq ce^{-\omega t} \bar{E}_0(0) \quad \text{for every } t > 0). \end{aligned}$$

In the above

$$\begin{aligned} E(0)_0 &= \frac{1}{2} \int_{\Omega} \left[|u^1(x)|^2 + \sigma(u^0(x)) : \epsilon(u^0(x)) + \frac{\alpha}{\beta} \theta^0(x)^2 \right] dx, \\ \bar{E}_0(0) &= E_0(0). \end{aligned}$$

CONDITION (C). We say that open, bounded set $\Omega \subset R^n$ satisfies Condition (C) if for every $s > 0$ the system:

$$\begin{aligned} -\Delta v &= sv \quad \text{in } \Omega, \\ \operatorname{div} v &= 0 \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

has in $(H_0^1(\Omega))^n$ the unique solution $v = 0 \in R^n$.

The information concerning domains satisfying Condition (C) and the references on this subject can be found in [11].

Let φ be the solution of the system

$$\begin{aligned} (3.4) \quad \partial_t^2 \varphi &= \Delta_e \varphi \quad \text{in } \Omega \times R_+, \\ \varphi &= 0 \quad \text{on } \partial\Omega \times R_+, \\ \varphi(0) &= \varphi^0, \partial_t \varphi(0) = \varphi^1 \quad \text{in } \Omega. \end{aligned}$$

In the paper [11] (see Theorem 1.1 and its proof) it was proved:

THEOREM 3.2. Assume $n = 2$ or $n = 3$. In the class of domains Ω satisfying Condition (C), the uniform decaying of energies $E(\cdot), \bar{E}(\cdot)$ hold if and only if there exist $T > 0$ and $C > 0$ such that

$$(3.5) \quad \|\varphi^0\|_{(L^2(\Omega))^n}^2 + \|\varphi^1\|_{(H^{-1}(\Omega))^n}^2 \leq C \int_0^T \|\operatorname{div} \varphi(t)\|_{H^{-1}(\Omega)}^2 dt,$$

holds for every solution of the system (3.4).

The characterization of domains Ω for which the inequality (3.5) holds was done in papers [11] when $n = 2$, and [2] when $n = 3$. In the case $n = 2$ it is required that the transversal bicharacteristic rays behave appropriately

when their space component meet $\partial\Omega$. These rays are the trajectories of Hamiltonian flow corresponding to principal symbol of wave operator describing transversal elasticity waves, broken appropriately when their space components meet $\partial\Omega$.

When $n = 3$ the condition on Ω also concerns every transversal ray when its space component meet $\partial\Omega$. But in this case the condition implies the restriction on propagation of wave front of polarization for every solution of the system (3.4).

In both works, mentioned above, the methods of microlocal analysis are applied. In the second work, also the microlocal defect measure of compactness is used and it involves the C^∞ regularity of $\partial\Omega$. Moreover, in both works, the considerations are proved under the assumption $\lambda + 2\mu > \mu$. Let us remark, that when μ, λ satisfy conditions of 2.1 and $\lambda \in (-\frac{2}{3}\mu, -\frac{\mu}{3}) \cup R_+$, then $\lambda + 2\mu > \mu$ holds.

4. Analysis of system (1.2)

In this section we assume C^2 regularity of $\partial\Omega$. We are going to make preparations to use the linear semigroups theory to analyze system (1.2).

We rewrite system (1.2) in the form of linear equation in H :

$$(4.0) \quad \frac{d\bar{\xi}}{dt} = \bar{L}\bar{\xi}$$

where $\bar{\xi} := (\bar{u}, \bar{\phi}, \bar{v}, \bar{\psi}, \bar{\theta})^T$,

$$\bar{L} := \begin{bmatrix} 0 & , & 0 & , & I & , & 0 & , & 0 \\ 0 & , & 0 & , & 0 & , & I & , & 0 \\ \Delta_e & , & b\nabla & , & -\frac{M^2}{d}P & , & 0 & , & 0 \\ -b\operatorname{div} & , & (a\Delta - \gamma I) & , & 0 & , & -rI & , & 0 \\ 0 & , & 0 & , & -M\operatorname{div} & , & -M_1I & , & d\Delta \end{bmatrix}.$$

We have $\bar{L} : X \rightarrow H$ and it is easy to see that \bar{L} is a closed operator.

In the further considerations we shall need the following proposition:

PROPOSITION 4.1. *For each $k > 0$ the operator $kI - \bar{L} : X \rightarrow H$ is surjective.*

Proof. For the purpose of the proof we introduce the following objects. The Hilbert space $H_1 := V \times (L^2(\Omega))^{n+1}$, subjected with the inner product

$$(\zeta^1, \zeta^2)_0 := \left\langle \begin{pmatrix} u^1 \\ \phi^1 \end{pmatrix}, \begin{pmatrix} u^2 \\ \phi^2 \end{pmatrix} \right\rangle_V + \int_{\Omega} (v^1 v^2 + \psi^1 \psi^2),$$

where $\zeta^i \in H_1$ have components $\zeta^i \equiv (u^i, \phi^i, v^i, \psi^i)$, $i = 1, 2$, and $\|\cdot\|_0$ will denote the norm in H_1 generated by this inner product.

Then we define the operator

$$L_1 := \begin{bmatrix} 0 & , & 0 & , & I & , & 0 \\ 0 & , & 0 & , & 0 & , & I \\ \Delta_e & , & b\nabla & , & -\frac{M^2}{d}P & , & 0 \\ -b\operatorname{div} & , & (a\Delta - \gamma I) & , & 0 & , & -rI \end{bmatrix},$$

whose domain is equal $X_1 := (H^2(\Omega) \cap H_0^1(\Omega))^n \times (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)^n \times H_0^1(\Omega)$.

It is easy to observe that X_1 is a dense linear subspace of H_1 , $L_1 : X_1 \rightarrow H_1$. For $\xi \equiv (\zeta, \theta) \in X$ we have $\overline{L}\xi = (L_1\zeta, d\Delta\theta - M\operatorname{div}v - M_1\psi)^T$, where v, ψ are suitable components of ζ , and moreover $\|\xi\|^2 = \|\zeta\|_0^2 + \int_{\Omega} \theta^2$.

The idea of further considerations is the following: first we prove maximal dissipativity of L_1 , which allows us to deduce immediately (see [6]) the surjectivity of $kI - L_1$. Then in a simple way we obtain the assertion. Therefore we are going to prove maximal dissipativity of L_1 .

After calculations we achieve

$$(L_1\zeta, \zeta)_0 = -r \int_{\Omega} \psi^2 - \frac{M^2}{d} \int_{\Omega} |Pv|^2 \leq 0, \quad \zeta \in X_1,$$

which gives the dissipativity of L_1 . To proceed on we prove first that $\ker(L_1) = \{0\}$, and second that $L_1 : X_1 \rightarrow H_1$ is surjective.

So, let $L_1\zeta = 0$, $\zeta \in X_1$. Then $v = 0$, $\psi = 0$ and $(u, \phi)^T$ should satisfy $E \begin{pmatrix} u \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, where

$$E := \begin{pmatrix} -\Delta_e & , & -b\nabla \\ b\operatorname{div} & , & -(a\Delta - \gamma I) \end{pmatrix}, \quad E : D(E) \rightarrow (L^2(\Omega))^n \times L^2(\Omega),$$

$$D(E) := (H^2(\Omega) \cap H_0^1(\Omega))^n \times (H^2(\Omega) \cap H_0^1(\Omega)).$$

Hence for the solution $(u, \phi)^T \in D(E)$ of this system we will have

$$0 = \left(E \begin{pmatrix} u \\ \phi \end{pmatrix}, \begin{pmatrix} u \\ \phi \end{pmatrix} \right)_{L^2} = \left\| \begin{pmatrix} u \\ \phi \end{pmatrix} \right\|_V^2,$$

which gives $u = 0$, $\phi = 0$, and the injectivity of L_1 .

Then let $g \equiv (g^1, g^2, g^3, g^4)^T \in H_1$ and consider the system $L_1\zeta = g$.

From this equation we immediately get that $v = g^1$, $\psi = g^2$ and $(u, \phi)^T$ should solve the system

$$(4.1) \quad E \begin{pmatrix} u \\ \phi \end{pmatrix} = - \begin{pmatrix} g^3 + \frac{M^2}{d} P g^1 \\ g^4 + r g^2 \end{pmatrix} =: \tilde{g} \in (L^2(\Omega))^n \times L^2(\Omega).$$

From Proposition 2.2, Korn inequality and then Poincaré inequality we can achieve

$$\left\| \begin{pmatrix} u \\ \phi \end{pmatrix} \right\|_V^2 \geq c \left\| \begin{pmatrix} u \\ \phi \end{pmatrix} \right\|_{L^2}^2, \quad \text{for each } (u, \phi) \in V,$$

$c > 0$ is constant.

This immediately yields

$$\left(E \begin{pmatrix} u \\ \phi \end{pmatrix}, \begin{pmatrix} u \\ \phi \end{pmatrix} \right)_{L^2} \geq c \left\| \begin{pmatrix} u \\ \phi \end{pmatrix} \right\|_{L^2}^2, \quad (u, \phi)^T \in V.$$

Hence, we can apply the Lax–Milgram theory and obtain the existence of the weak solution $(u, \phi) \in V$ for the equation (4.1). Then, we apply the theory of regularity of weak solutions for elliptic systems [13] (Theorem 4.18), and claim that the weak solution $(u, \phi) \in D(E)$. Summarizing, the system $L_1 \zeta = g$ has solution $\zeta \in X_1$, which gives surjectivity of L_1 .

Therefore, we can consider $L_1^{-1} : H_1 \rightarrow X_1$. We prove that L_1^{-1} treated as the operator from H_1 into H_1 is continuous. It is equivalent with establishing inequality $\|\zeta\|_0 \leq c_1 \|g\|_0$, when $g \in H_1$ and $L_1 \zeta = g$, $c_1 > 0$ is a constant independent of g .

Taking into account the observations made above, for components $\begin{pmatrix} u \\ \phi \end{pmatrix}$ of ζ we carry the following estimations

$$\begin{aligned} \left\| \begin{pmatrix} u \\ \phi \end{pmatrix} \right\|_V^2 &= \left(E \begin{pmatrix} u \\ \phi \end{pmatrix}, \begin{pmatrix} u \\ \phi \end{pmatrix} \right)_{L^2} \leq \left| \left(\tilde{g}, \begin{pmatrix} u \\ \phi \end{pmatrix} \right)_{L^2} \right| \\ &\leq \|\tilde{g}\|_{L^2} \left\| \begin{pmatrix} u \\ \phi \end{pmatrix} \right\|_{L^2} \leq c^{-\frac{1}{2}} \|\tilde{g}\|_{L^2} \left\| \begin{pmatrix} u \\ \phi \end{pmatrix} \right\|_V. \end{aligned}$$

Because

$$\|\tilde{g}\|_{L^2} \leq c^{-\frac{1}{2}} \left\| \begin{pmatrix} g^1 \\ g^2 \end{pmatrix} \right\|_V + \max \left\{ \frac{M^2}{d}, r \right\} \left\| \begin{pmatrix} g^3 \\ g^4 \end{pmatrix} \right\|_{L^2},$$

we have showed

$$\left\| \begin{pmatrix} u \\ \phi \end{pmatrix} \right\|_V \leq c_2 \left(\left\| \begin{pmatrix} g^1 \\ g^2 \end{pmatrix} \right\|_V + \left\| \begin{pmatrix} g^3 \\ g^4 \end{pmatrix} \right\|_{L^2} \right),$$

with constant $c_2 > 0$ which can be easily calculated.

For components $\begin{pmatrix} v \\ \psi \end{pmatrix}$ of ζ we derive very simply that

$$\left\| \begin{pmatrix} v \\ \psi \end{pmatrix} \right\|_{L^2} = \left\| \begin{pmatrix} g^1 \\ g^2 \end{pmatrix} \right\|_{L^2} \leq c^{-\frac{1}{2}} \left\| \begin{pmatrix} g^1 \\ g^2 \end{pmatrix} \right\|_V.$$

That proves the desired inequality. The norm of $L^{-1} : H_1 \rightarrow H_1$ we denote by $\|L^{-1}\|$.

Let $\lambda \in (0, \frac{1}{\|L_1^{-1}\|})$, $f \in H_1$ and consider the system $(\lambda I - L_1)\zeta = f$.

We write this equation in the equivalent form $(I - \lambda L_1^{-1})\zeta = -L_1^{-1}f$. Because $\lambda\|L_1^{-1}\| < 1$, the solution of the latter equation obeys the form $\zeta = -L_1^{-1}(\sum_{k=0}^{\infty} (\lambda L_1^{-1})^k f)$, and hence ζ belongs to X_1 .

The proof of maximal dissipativity is finished and we can claim [6] that for each $k > 0$, the operator $kI - L_1 : X_1 \rightarrow H_1$ is surjective. Now let $k > 0$, $f \in H$ and consider the system $(kI - \bar{L})\xi = f$. We can write this system in the form

$$\begin{aligned} (kI - L_1)\zeta &= f', \\ k\theta - d\nabla\theta &= f^5 - M\operatorname{div}v - M_1\psi, \end{aligned}$$

where $\xi \equiv (\zeta, \theta)$, $\zeta \equiv (u, \phi, v, \psi)$, $f \equiv (f', f^5)$, $f' \in H_1$, $f^5 \in L^2(\Omega)$. Because for each $k > 0$, $g \in L^2(\Omega)$, the equation $k\theta - d\nabla\theta = g$ has solution $\theta \in H^2(\Omega) \cap H_0^1(\Omega)$, we can finish the proof. ■

THEOREM 4.2. *Operator \bar{L} is the generator of c_0 -semigroup in H , which we denote by $\bar{S}(t), t \geq 0$.*

Proof. After elementary calculations we obtain that for $\bar{\xi} \in X$

$$\begin{aligned} (\bar{L}\bar{\xi}, \bar{\xi}) &= (L_1\zeta, \zeta)_0 + (d\Delta\bar{\theta} - M\operatorname{div}\bar{v} - M_1\bar{\psi}, \bar{\theta})_{L^2} = \\ &= -r \int_{\Omega} \bar{\psi}^2 - \frac{M^2}{d} \int_{\Omega} |P\bar{v}|^2 - d \int_{\Omega} |\nabla\bar{\theta}|^2 - \int_{\Omega} (M\operatorname{div}\bar{v} + M_1\bar{\psi}) \bar{\theta} \leq \\ &\leq -d \int_{\Omega} |\nabla\bar{\theta}|^2 + \frac{1}{2}M_1 \left(\|\bar{\psi}\|_{L^2(\Omega)}^2 + \|\bar{\theta}\|_{L^2(\Omega)}^2 \right) + M \int_{\Omega} \bar{v} \cdot \nabla\bar{\theta} \leq \\ &\leq \frac{M}{\epsilon} \|\bar{v}\|_{L^2(\Omega)}^2 + \frac{1}{2}M_1 \left(\|\bar{\psi}\|_{L^2(\Omega)}^2 + \|\bar{\theta}\|_{L^2(\Omega)}^2 \right), \end{aligned}$$

when $\epsilon > 0$ is sufficiently small.

Therefore, we showed that there exists constant $c > 0$ such that the inequality

$$(\bar{L}\bar{\xi}, \bar{\xi}) \leq c\|\bar{\xi}\|^2, \quad \bar{\xi} \in X$$

holds.

From that estimation we derive that for each $k > c$, $((kI - \bar{L})\bar{\xi}, \bar{\xi}) \geq (k - c)\|\bar{\xi}\|^2$, $\bar{\xi} \in X$. From that we can deduce injectivity of $(kI - \bar{L})$, and

from the Proposition 4.1 the existence of the resolvent operator $R(k; \bar{L}) : H \rightarrow X$ when $k > c$, and the estimation

$$\|R(k; \bar{L})\|_{H \rightarrow H} \leq (k - c)^{-1}.$$

Using Theorem 5.3, Chapter 1 of [14] we obtain that \bar{L} generates c_0 -semigroup in H . ■

From classical theorems of semigroup theory we also obtain

COROLLARY 4.3. *If $\xi^0 \in X$ then $\bar{\xi}(t) := \bar{S}(t)\xi^0 \in X$, $t \geq 0$, is the solution of equation (4.0) satisfying initial condition $\bar{\xi}(0) = \xi^0$. Moreover (similarly as in [8]) if we define $(\bar{u}(t), \bar{\phi}(t), \bar{v}(t), \bar{\psi}(t), \bar{\theta}(t))^T \equiv \bar{\xi}(t)$ then $\partial_t \bar{u}(t) = \bar{v}(t)$, $\partial_t \bar{\phi}(t) = \bar{\psi}(t)$ and $(\bar{u}(\cdot), \bar{\phi}(\cdot), \bar{\theta}(\cdot))$ is the strong solution of initial-boundary problem (1.2) when $\xi^0 \in X$ and the weak solution when $\xi^0 \in H$.*

The energy $\bar{E}(t)$ of the solution $\bar{\xi}(t) \equiv \bar{S}(t)\xi^0$, $t > 0$ to the system (4.0) will be defined by $\bar{E}(t) := \frac{1}{2} \|\bar{\xi}(t)\|^2$.

For the fixed value of coefficient $b > 0$ in the system (1.2) we denote the corresponding c_0 -semigroup by $\bar{S}_b(\cdot)$.

Now we formulate the main result of this section.

THEOREM 4.4. *Let domain Ω satisfy the Condition (C) and guarantee that inequality (3.5) holds. Then exists $b_0 > 0$ such, that for every $b \in (0, b_0)$ the semigroup $\bar{S}_b(t)$ has the property of uniform decaying.*

Proof. An approximation argument shows that we can assume that $\xi^0 \in X$.

Let $\bar{\xi}(t) := \bar{S}_b(t)\xi^0$; $t > 0$ be a solution of (4.0). We look for $\bar{\xi}(t)$ having the form

$$\xi(t) = \sum_{l=0}^{\infty} b^l \xi_l(t); \quad \xi_l(t) \equiv (u_l(t), \phi_l(t), \partial_t u_l(t), \partial_t \phi_l(t), \theta_l(t)), \quad l \in \{0\} \cup N.$$

After the formal substitution into the equation (4.0) we derive equations for (u_l, ϕ_l, θ_l) , $l \in \{0\} \cup N$.

For (u_0, ϕ_0, θ_0) we obtain

$$(4.2) \quad \begin{aligned} \partial_t^2 u_0 &= \Delta_e u_0 - \frac{M^2}{d} P(\partial_t u_0) \quad \text{in } \Omega \times R_+, \\ \partial_t \theta_0 &= d \Delta \theta_0 - M \operatorname{div} \partial_t v_0 - M_1 \partial_t \phi_0 \quad \text{in } \Omega \times R_+, \end{aligned}$$

$$(4.3) \quad \partial_t^2 \phi_0 = a \Delta \phi_0 - \gamma \phi_0 - r \partial_t \phi_0 \quad \text{in } \Omega \times R_+,$$

$u_0(0) = u^0$, $\partial_t u_0(0) = u^1$, $\phi_0(0) = \phi^0$, $\partial_t \phi_0(0) = \phi^1$, $\theta_0(0) = \theta^0$ in Ω , where $(u^0, \phi^0, u^1, \phi^1, \theta^0) \equiv \xi^0$, $u_0 = 0$, $\phi_0 = 0$, $\theta_0 = 0$ on $\partial\Omega \times R_+$.

For $k \in \{0\} \cup N$, $(u_{k+1}, \phi_{k+1}, \theta_{k+1})$ will be the solution of problem:

$$(4.4) \quad \begin{aligned} \partial_t^2 u_{k+1} &= \Delta_e u_{k+1} - \frac{M^2}{d} P(\partial_t u_{k+1}) + \nabla \phi_k \quad \text{in } \Omega \times R_+, \\ \partial_t \theta_{k+1} &= d \Delta \theta_{k+1} - M \operatorname{div} \partial_t v_{k+1} - M_1 \partial_t \phi_{k+1} \quad \text{in } \Omega \times R_+, \end{aligned}$$

$$(4.5) \quad \begin{aligned} \partial_t^2 \phi_{k+1} &= a \Delta \phi_{k+1} - \gamma \phi_{k+1} - r \partial_t \phi_{k+1} - \operatorname{div} u_k \quad \text{in } \Omega \times R_+, \\ u_{k+1}(0) &= 0, \partial_t u_{k+1}(0) = 0, \phi_{k+1}(0) = 0, \partial_t \phi_{k+1}(0) = 0, \theta_{k+1}(0) = 0 \text{ in } \Omega, \\ u_{k+1} &= 0, \phi_{k+1} = 0, \theta_{k+1} = 0 \text{ on } \partial\Omega \times R_+. \end{aligned}$$

For the clarity of the further considerations we introduce simplifying notation and make useful observations.

We define the norms

$$\begin{aligned} \|(\phi, \psi)\|_1 &:= \left(\int_{\Omega} (|\nabla \phi|^2 + r\phi^2 + \psi^2) \right)^{\frac{1}{2}}, \quad (\phi, \psi) \in H_0^1(\Omega) \times L^2(\Omega), \\ \|(u, v, \theta)\|_2 &:= \left(\int_{\Omega} (\sigma(u) : \epsilon(u) + |v|^2 + \theta^2) \right)^{\frac{1}{2}}, \\ &\quad (u, v, \theta) \in H_0^1(\Omega)^n \times L^2(\Omega)^n \times L^2(\Omega). \end{aligned}$$

From the second inequality in Proposition 2.2 one can derive

$$(4.6) \quad \| (u, v, \theta) \|_2^2 + \| (\phi, \psi) \|_1^2 \leq c \| \xi \|^2,$$

where $\xi \equiv (u, \phi, v, \psi, \theta) \in H$, $c > 0$ is a constant.

It can be also noticed that

$$(4.7) \quad \| \xi \| \leq c_0 (\| (u, v, \theta) \|_2 + \| (\phi, \psi) \|_1),$$

c_0 is a constant.

We shall denote

$$h_k(t) := \| (\phi_k(t), \partial_t \phi_k(t)) \|_1, \quad g_k(t) := \| (u_k(t), \partial_t u_k(t), \theta_k(t)) \|_2, \quad k \in \{0\} \cup N.$$

We are ready to begin the essential considerations of this proof.

It is well known that solutions of the considered initial-boundary value problem for equation (4.3) is described by c_0 -semigroup of contractions, which we denote $\Gamma(t)$, $t > 0$. Hence this solution can be written as

$$(\phi_0(t), \partial_t \phi_0(t)) = \Gamma(t) (\phi^0, \phi^1).$$

This can be accomplished classically by way of the Fourier method.

One can show that there exist constants $c_1, q > 0$ such that

$$(4.8) \quad h_0^2(t) \leq c_1 e^{-qt} \| (\phi^0, \phi^1) \|_1^2, \quad t \geq 0.$$

This estimation can be obtained by finding the solution of equation (4.3) by way of the Fourier method.

We rewrite the system (4.2) in the following form:

$$(4.9) \quad \frac{d}{dt} \begin{pmatrix} u_0 \\ \partial_t u_0 \\ \theta_0 \end{pmatrix} = W \begin{pmatrix} u_0 \\ \partial_t u_0 \\ \theta_0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -M_1 \partial_t \phi_0 \end{pmatrix},$$

where we treat $-M \partial_t \phi_0$ as the known function and

$$W := \begin{bmatrix} 0 & , & I & , & 0 \\ \Delta_e & , & -\frac{M}{d^2} P & , & 0 \\ 0 & , & -M \operatorname{div} & , & d \Delta \end{bmatrix}.$$

In [11] it is proved that W is the generator of c_0 -semigroup $S_W(t)$, $t \geq 0$ in space $(H_0^1(\Omega))^n \times L^2(\Omega)^n \times L^2(\Omega)$. The domain of W is equal to

$$(H^2(\Omega) \cap H_0^1(\Omega))^n \times (H_0^1(\Omega))^n \times (H^2(\Omega) \cap H_0^1(\Omega)).$$

The solution of system (4.9) can be written in the form

$$(4.10) \quad \begin{pmatrix} u_0(t) \\ \partial_t u_0(t) \\ \theta_0(t) \end{pmatrix} = S_W(t) \begin{pmatrix} u^0 \\ u^1 \\ \theta^0 \end{pmatrix} + \int_0^t S_W(t-s) \begin{pmatrix} 0 \\ 0 \\ -M_1 \partial_t \phi_0(s) \end{pmatrix} ds.$$

For the validity of this formula one can consult Corollary 2.20 and Corollary 2.11 from Chapter 4 of [14].

From [11] (see Theorem 3.2 in this paper) we obtain that

$$(4.11) \quad \left\| \left[S_W(t) \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \right]^T \right\|_2^2 \leq c_2 e^{-q_1 t} \|(u, v, \theta)\|_2^2, \quad t \geq 0,$$

where $c_1, q_1 > 0$ are constant.

Now using (4.10) and (4.11) we obtain

$$(4.12) \quad g_0(t) \leq c_2 e^{-\frac{q_1}{2}t} \|(u^0, u^1, \theta^0)\|_2 + c_2 \int_0^t e^{-\frac{q_1}{2}(t-s)} \|(0, 0, -M_1 \partial_t \phi_0(s))\|_2 ds.$$

Because $\|(0, 0, -M_1 \partial_t \phi_0(s))\|_2 \leq M_1 \|(\phi_0(s), \partial_t \phi_0(s))\|_1$, from (4.12) and (4.8) we get

$$(4.13) \quad g_0(t) \leq c_2 e^{-\frac{q_1}{2}t} \|(u^0, u^1, \theta^0)\|_2 + c_2 M_1 \|(\phi^0, \phi^1)\|_1 \int_0^t e^{-\frac{q_1}{2}(t-s)} e^{-\frac{q}{2}s} ds.$$

To simplify further estimations we can assume that from here

$$(4.14) \quad 0 < q < q_1,$$

because if the estimation (4.8) is satisfied for larger q it is also satisfied for smaller values of q . Taking into account this assumption we get from (4.13):

$$(4.15) \quad g_0(t) \leq c_3 e^{-\frac{q}{2}t} [\| (u^0, u^1, \theta^0) \| + \| (\phi^0, \phi^1) \|_1],$$

which according to (4.6) gives

$$(4.16) \quad g_0(t)^2 \leq 2c_3^2 c e^{-qt} \|\xi^0\|^2.$$

From estimations (4.8), (4.16) and (4.7) we deduce

$$(4.17) \quad \|\xi_0(t)\| \leq c_4 e^{-\frac{q}{2}t} \|\xi^0\|.$$

For the solution of the initial-boundary problem (4.5) we can write:

$$(\phi_{k+1}(t), \partial_t \phi_{k+1}(t)) = \int_0^t \Gamma(t-s) (0, -\operatorname{div} u_k(s)) ds.$$

Using the same type of argumentation as for deriving (4.8) we get:

$$h_{k+1}(t) \leq c_1 \int_0^t e^{-\frac{q}{2}(t-s)} \|(0, \operatorname{div} u_k(s))\|_1 ds.$$

Since $\|(0, \operatorname{div} u_k(s))\|_1 \leq c_5 \|(u_k(s), \partial_t u_k(s), \theta_k(s))\|_2$, this gives

$$(4.18) \quad h_{k+1}(t) \leq c_6 \int_0^t e^{-\frac{q}{2}(t-s)} g_k(s) ds.$$

We can write the solution of the system (4.4) in the form

$$(u_{k+1}, \partial_t u_{k+1}(t), \theta_{k+1}(t))^T = \int_0^t S_W(t-s) \begin{pmatrix} 0 \\ \nabla \phi_k(s) \\ -M_1 \partial_t \phi_{k+1}(s) \end{pmatrix} ds.$$

According to (4.11) this gives

$$(4.19) \quad g_{k+1}(t) \leq c \int_0^t e^{-\frac{q_1}{2}(t-s)} \|(0, \nabla \phi_k(s), -M_1 \partial_t \phi_{k+1}(s))\|_2 ds.$$

Notice that

$$\begin{aligned} \|(0, \nabla \phi_k(s), -M_1 \partial_t \phi_{k+1}(s))\|_2 &\leq M_1 \|(\phi_{k+1}(s), \partial_t \phi_{k+1}(s))\|_1 \\ &\quad + \|(\phi_k(s), \partial_t \phi_k(s))\|_1. \end{aligned}$$

We use this inequality to the right hand side in (4.19) and then apply (4.18). After calculations and making use of (4.14) we get

$$(4.20) \quad g_{k+1}(t) \leq c_7 \int_0^t e^{-\frac{q}{2}(t-s)} [g_k(t) + h_k(t)] ds.$$

After collecting together (4.18), (4.20) and making use of (4.7) we get

$$\|\xi_{k+1}\| \leq c_0(c_6 + c_7) \int_0^t e^{\frac{q}{2}(t-s)} (h_k(s) + g_k(s)) ds.$$

From (4.6) this gives

$$(4.21) \quad \|\xi_{k+1}(t)\| \leq c_8 \int_0^t e^{\frac{q}{2}(t-s)} \|\xi_k(s)\| ds, \quad k \in \{0\} \cup N.$$

From (4.17) and (4.21), by making use of the successive iterations we achieve

$$\|\xi_l(t)\| \leq c_4 c_8^l \frac{t^l}{l!} e^{-\frac{q}{4}t} \|\xi^0\|, \quad l \in \{0\} \cup N.$$

We notice that when $0 < b < \frac{q}{4c_8}$ the sequence $\sum_{l=0}^{\infty} b^l \xi_l(t)$ is convergent in $C([0, \tau]; H)$ for every $\tau > 0$. Let us define $\bar{\xi}^n(t) := \sum_{l=0}^n b^l \xi_l(t)$, $n \in N$, and $b \in (0, \frac{q}{4c_8})$. We notice that $\bar{\xi}^n$ is the solution of the problem

$$\frac{d\bar{\xi}^n}{dt} = \bar{L}\bar{\xi}^n + h_n(t), \quad \bar{\xi}^n(0) = \bar{\xi}^0,$$

where $h_n(t) := (0, b^n \nabla \phi_n(t), 0, -b^n \operatorname{div} u_n(t), 0)^T$.

Therefore, we can write

$$\bar{\xi}^n(t) = \bar{S}(t)\bar{\xi}^0 + \int_0^t \bar{S}(t-s)h_n(s)ds.$$

On the other hand, denoting $\bar{\xi}(t) := \sum_{l=0}^{\infty} b^l \xi_l(t)$, we obtain

$$\|\bar{\xi}(t) - \bar{S}(t)\bar{\xi}^0\| = \left\| \sum_{l=n+1}^{\infty} b^l \xi_l(t) + \int_0^t \bar{S}(t-s)h_n(s)ds \right\|,$$

for every $n \in N$.

We observe that $\|h_n(t)\| \leq b^n \|\bar{\xi}_n(t)\|$ and this implies

$$\lim_{n \rightarrow \infty} \int_0^t \bar{S}(t-s)h_n(s)ds = 0 \quad \text{in } C([0, \tau]; H) \quad \text{for every } \tau > 0.$$

It means that $\overline{S}(t)\xi^0 = \overline{\xi}(t)$. This allows us to claim

$$\|\overline{S}(t)\xi^0\| \leq \sum_{l=0}^{\infty} b^l \|\xi_l(t)\| \leq c_4 e^{-\omega t} \|\xi^0\|,$$

where $\omega := \frac{q}{4} - c_8 b$. The proof is finished. ■

PROPOSITION 4.5. *Let $\xi^0 \in H$, $(\overline{u}(t), \overline{\phi}(t), \partial_t \overline{u}(t), \partial_t \overline{\phi}(t), \overline{\theta}(t))^T \equiv \overline{S}_b(t)\xi^0$, $b \in (0, b_0)$, where b_0 was defined in Theorem 4.4. There exists constant $c > 0$, such that for every $t > 0$*

$$\|\operatorname{div} \overline{u}(\cdot)\|_{L^2(\partial\Omega \times [0, t])}^2 \leq c(1+t) \|\xi^0\|^2.$$

Proof. To simplify notation we denote $G_t := \Omega \times [0, t]$, $\Sigma_t := \partial\Omega \times [0, t]$ and $\sigma(\overline{u})\nu$ denotes vector with coordinates $(\sigma(\overline{u})\nu)_i := \sum_{j=1}^n \sigma_{ij}(\overline{u})\nu_j$, $i = 1, \dots, n$, where ν is an external normal vector to $\partial\Omega$.

We begin with proving the estimation

$$(4.22) \quad \|\sigma(\overline{u})\nu\|_{L^2(\Sigma_t)} \leq c_1(1+t) \|\xi^0\|^2; \quad t > 0,$$

where $c_1 > 0$ is constant. Proof of this estimation will go along the schema given in [1]. An approximation argument shows that we can assume that initial data ξ^0 belongs to $C_0^\infty(\Omega)$ class.

Let $h \in (W^{1,\infty}(\Omega))^n$, the equation for \overline{u} from the system (1.2) we multiply by $(\nabla \overline{u})h$ and integrate on G_t ; the coordinates of vector $(\nabla \overline{u})h$ are $((\nabla \overline{u})h)_i := \sum_{m=1}^n h_m \partial_m \overline{u}_i$, $i = 1, \dots, n$. We obtain

$$(4.23) \quad 0 = \int_{G_t} (\nabla \overline{u})h \cdot \left(\partial_t^2 \overline{u} - \Delta_e \overline{u} - b \nabla \overline{\phi} + \frac{M^2}{d} P(\partial_t \overline{u}) \right) = S_0 + S_1 + S_2,$$

where

$$\begin{aligned} S_0 &:= \int_{G_t} (\nabla \overline{u})h \cdot (\partial_t^2 \overline{u} - \Delta_e \overline{u}), \\ S_1 &:= -b \int_{G_t} (\nabla \overline{u})h \cdot \nabla \overline{\phi}, \\ S_2 &:= \frac{M^2}{d} \int_{G_t} (\nabla \overline{u})h \cdot P(\partial_t \overline{u}). \end{aligned}$$

After typical estimations, using Korn inequality, we obtain

$$\begin{aligned} |S_1| &\leq c_2 \left[\|\nabla \overline{\phi}\|_{L^2(G_t)}^2 + \int_{G_t} \sigma(\overline{u}) : \epsilon(\overline{u}) \right], \\ |S_2| &\leq c_3 \left[\|\partial_t \overline{u}\|_{L^2(G_t)}^2 + \int_{G_t} \sigma(\overline{u}) : \epsilon(\overline{u}) \right]. \end{aligned}$$

To estimate S_0 we take the same approach as in [1]. We write $S_0 = I_t + S_0^1$, where the formula for S_0^1 is given in [1] and $I_t := \int_{\Sigma_t} (\nabla \bar{u}) h \cdot \sigma(\bar{u}) \nu$. In [1] the following estimation was derived

$$|S_0^1| \leq c_4 \left[\|u^1\|_{L^2(\Omega)}^2 + \int_{\Omega} \sigma(u^0) : \epsilon(u^0) + \|\partial_t \bar{u}(t)\|_{L^2(\Omega)}^2 \right. \\ \left. + \int_{\Omega} \sigma(\bar{u}(t)) : \epsilon(\bar{u}(t)) + \|\partial_t \bar{u}\|_{L^2(G_t)}^2 + \int_{G_t} \sigma(\bar{u}) : \epsilon(\bar{u}) \right].$$

Now, we take h such that $h = \nu$ on $\partial\Omega$. For such h it was computed in [1], that

$$I_t = \int_{\Sigma_t} \sigma(\bar{u}) : \epsilon(\bar{u}).$$

From (4.23) we can estimate

$$|I_t| \leq |S_0^1| + |S_1| + |S_2|.$$

Taking into account estimations given above for S_0^1 , S_1 , S_2 and then using second inequality in Proposition 2.2, we obtain

$$|I_t| \leq c_5 \left[\|\xi^0\|^2 + \|\bar{S}(t)\xi^0\|^2 + \int_0^t \|\bar{S}(r)\xi^0\|^2 dr \right] \\ \leq c_6(1+t)\|\xi^0\|^2.$$

The last inequality is a consequence of decaying property of semigroup $\bar{S}(\cdot)$ proved in Theorem 4.4. Because

$$|\sigma(\bar{u})\nu| \leq c_7 \sum_{i,j=1}^n \epsilon_{ij}^2(\bar{u}) \leq c_8 \sigma(\bar{u}) : \epsilon(\bar{u}),$$

the inequality (4.22) is proved.

In [3], it was proved that on $\partial\Omega$ $\operatorname{div} \bar{u} = ((\nabla \bar{u}) \nu) \cdot \nu$. Also in [3], the existence of continuous, reversible matrix B on $\partial\Omega$ such that $B((\nabla \bar{u}) \nu) = \sigma(\bar{u})\nu$, was proved. From this we obtain

$$\operatorname{div} \bar{u}|_{\partial\Omega} = \nu \cdot B^{-1}(\sigma(u) \cdot \nu).$$

By (4.22) this gives the inequality from the assertion of the Proposition. ■

5. Compactness of the difference $S(t) - \bar{S}(t)$

In this section we prove the following theorem:

THEOREM 5.1. *For every $\tau > 0$, operator $S(\cdot) - \bar{S}(\cdot) : H \rightarrow C([0, \tau], H)$ is compact.*

Proof. We denote

$$\begin{aligned}(u(t), \phi(t), \partial_t u(t), \partial_t \phi(t), \theta(t)) &:= S(t)\xi^0, \\ (\bar{u}(t), \bar{\phi}(t), \partial_t \bar{u}(t), \partial_t \bar{\phi}(t), \bar{\theta}(t)) &:= \bar{S}(t)\xi^0, \\ \tilde{\xi}(t) \equiv (\tilde{u}(t), \tilde{\phi}(t), \partial_t \tilde{u}(t), \partial_t \tilde{\phi}(t), \tilde{\theta}(t)) &:= (S(t) - \bar{S}(t))\xi^0,\end{aligned}$$

where $\xi^0 \equiv (u^0, \phi^0, u^1, \phi^1, \theta^0) \in H$.

For $\xi_0 \in X$ we check that $\tilde{\xi}(\cdot) \in C(R_+; X) \cap C^1(R_+; H)$ and satisfies the equation

$$(5.1) \quad \begin{aligned}\frac{d\tilde{\xi}}{dt} &= L\tilde{\xi} + f(t, \xi^0), \quad t > 0, \\ \tilde{\xi}(0) &= 0,\end{aligned}$$

where

$$f(t, \xi^0) := \left(0, 0, \frac{M^2}{d}P\bar{v}(t) - M\nabla\bar{\theta}(t), M_1\bar{\theta}(t), 0\right), \quad \bar{v}(t) = \partial_t \bar{u}(t), \quad \bar{\xi}(0) = \xi^0.$$

Hence, when $\xi^0 \in X$, we can write (see [14], Corollary 2.2)

$$(5.2) \quad \tilde{\xi}(t) = \int_0^t S(t-s)f(s; \xi^0)ds.$$

We write $f(\cdot) = f_1(\cdot) + f_2(\cdot)$, where $f_1(\cdot) = (0, 0, 0, M_1\bar{\theta}(\cdot), 0)^T$, $f_2(\cdot) = (0, 0, \frac{M^2}{d}P\bar{v}(\cdot) - M\nabla\bar{\theta}(\cdot), 0, 0)$. It is evident that $f_1 \in L^1([0, \tau]; H)$ for each $\tau > 0$, even when $\xi^0 \in H$.

From equation for $\bar{\theta}$ in (1.2), after manipulations we get

$$(5.3) \quad \begin{aligned}(-\Delta)^{\frac{s}{2}}\bar{\theta}(t) &= (-\Delta)^{\frac{s}{2}}Q(t)\theta^0 \\ &\quad - M \int_0^t (-\Delta)^{\frac{1+s}{2}}Q(t-t_1)(-\Delta)^{-\frac{1}{2}}\operatorname{div}\bar{v}(t_1)dt_1 \\ &\quad - M_1 \int_0^t (-\Delta)^{\frac{s}{2}}Q(t-t_1)\bar{\psi}(t_1)dt_1,\end{aligned}$$

where $s \in (0, 1)$, $Q(\cdot)$ denotes analytical semigroup in $L^2(\Omega)$ generated by $d\Delta$.

From (5.3) we get the formula for $\bar{\theta}(\cdot)$ when we put $s = 0$.

If we take into account the estimation

$$(5.4) \quad \|(-\Delta)^\omega Q(t)\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq ct^{-\omega}, \quad t > 0, \quad \omega \in (0, 1),$$

and the boundedness of operator $(-\Delta)^{-\frac{1}{2}}\operatorname{div}$ in $L^2(\Omega)$, we claim that $\nabla\bar{\theta}(\cdot) \in L^1([0, \tau], L^2(\Omega))$; hence $f_2 \in L^1([0, \tau]; H)$, $\tau > 0$.

We have proved that (5.2) makes sense when $\xi^0 \in H$.

Let $O \subset H$ be a bounded set. The assertion of the theorem will be proved if we show that the sets

$$Y_i(\tau, O) := \left\{ \int_0^t S(t-s)f_i(s, \xi^0), 0 \leq t \leq \tau : \xi^0 \right\},$$

are precompact in $C([0, \tau]; H)$, $i = 1, 2$, $\tau > 0$.

Our further considerations will be supported by the following lemma ([10], Lemma 6):

LEMMA 5.2. *Let $E(t)$, $t \geq 0$ be a c_0 -semigroup in a Banach space X and $\{h(s; \alpha), 0 \leq s \leq T : \alpha \in A\} \subset L^1([0, T]; X)$.*

The set $\{\int_0^t E(t-s)h(s; \alpha)ds, 0 \leq t \leq T : \alpha \in A\}$ is precompact in $C([0, T]; X)$ if either

- (1) $\{h(s; \alpha) : \alpha \in A, 0 \leq s \leq T\}$ is a precompact set of X ,
- (2) for any $\epsilon > 0$ there is $\delta(\epsilon) > 0$ and a compact set $K(\epsilon) \subset X$ such that $\int_0^\delta \|h(s; \alpha)\|_X ds \leq \epsilon$, $\alpha \in A$, and $h(s; \alpha) \in K(\epsilon)$ for $\delta \leq s \leq T$, $\alpha \in A$.

First we prove that $Y_1(\tau, O)$ is precompact in $C([0, \tau]; H)$, $\tau > 0$. We show this, if we prove that the set $Q_1 := \{\bar{\theta}(\cdot; \xi^0) : \xi^0 \in O\}$ satisfies condition (2) in Lemma 5.2 – when $X \equiv L^2(\Omega)$, $T \equiv \tau$. So, let us put $s = 0$ in (5.3). We observe that Q_1 is bounded in $C([0, \tau]; L^2(\Omega))$, which yields that the first part of the condition 2 is satisfied. Then, from (5.3), (5.4) we can infer that Q_1 is bounded in $L^1([0, \tau]; H^s(\Omega))$, $s \in (0, 1)$. Because of compactness of the embedding $H^s(\Omega) \subset L^2(\Omega)$, $s > 0$, we conclude that the second part of the condition 2 is satisfied. The desired assertion about $Y_1(\tau, O)$ was proved.

Next we go to prove that $Y_2(\tau, 0)$ is precompact in $C([0, \tau]; H)$, $\tau > 0$. We obtain this if we show that the set $Q_2 = \{\frac{M^2}{d}P(\partial_t \bar{u}(\cdot)) - M\nabla \bar{\theta}(\cdot) : \xi^0 \in O\}$ also satisfies condition 2 of Lemma 5.2 – when $X \equiv L^2(\Omega)$, $T = \tau$.

The first topic in our considerations will be to prove that for some $\delta \in (0, 1)$, Q_2 is bounded in $L^1([0, \tau]; (H^\delta(\Omega))^n)$. This will be accomplished in the same way as it was done in [15] (see Appendix).

First, we remark that

$$\frac{M^2}{d}P(\partial_t \bar{v}) - M\nabla \bar{\theta} = \nabla w_1 + \nabla w_2,$$

where w_1 and w_2 are solutions of the following problems:

$$\begin{aligned} \partial_t w_1 &= d\Delta w_1 \quad \text{in } \Omega \times (0, \tau), \\ w_1 &= 0 \quad \text{on } \partial\Omega \times (0, \tau), \\ w_1(0) &= -\left[\frac{M^2}{d}(-\Delta)^{-1}\text{div} u^1 + M\theta^0\right] \quad \text{in } \Omega, \end{aligned}$$

$$\begin{aligned}\partial_t w_2 &= d\Delta w_2 - \frac{M^2}{d}(-\Delta)^{-1} \operatorname{div}(\partial_t^2 \bar{u}) \quad \text{in } \Omega \times (0, \tau), \\ w_2 &= 0 \quad \text{on } \partial\Omega \times (0, \tau), \\ w_2(0) &= 0 \quad \text{in } \Omega.\end{aligned}$$

In the above, we have taken into account the formula for P.

Since the set $\{\frac{M^2}{d}(-\Delta)^{-1}(\operatorname{div} u^1) + M\theta^0 : \xi^0 \in O\}$ is bounded in $L^2(\Omega)$, from the estimation (5.4) we obtain that set $\{w_1 : \xi^0 \in O\}$ is bounded in $L^1([0, \tau]; H^{1+\delta}(\Omega))$, $\tau > 0$.

From the equation for \bar{u} in (1.2) we infer

$$\operatorname{div}(\partial_t^2 \bar{u}) = (\lambda + 2\nu)\Delta \operatorname{div}(\bar{u}) - \frac{M^2}{d} \operatorname{div}(\partial_t \bar{u}) + b\Delta \bar{\phi}.$$

Since the set $\{\bar{\phi} : \xi^0 \in O\}$ is bounded in $C([0, \tau]; H_0^1(\Omega))$, the set

$$\{b(-\Delta)^{-1}(\Delta \bar{\phi}) : \xi^0 \in O\}$$

is bounded in the same space, too. The set

$$\left\{-\frac{M^2}{d} \partial_t \bar{u} : \xi^0 \in O\right\}$$

is bounded in $C([0, \tau]; L^2(\Omega))$, therefore

$$\left\{-\frac{M^2}{d}(-\Delta)^{-1} \operatorname{div}(\partial_t \bar{u}) : \xi^0 \in O\right\}$$

is bounded in $C([0, \tau]; H_0^1(\Omega))$.

Finally, since we have the estimation on $\|\operatorname{div} \bar{u}\|_{L^2(\partial\Omega \times [0, \tau])}$ (Proposition (4.6)), similarly as in [15] we obtain boundedness of

$$\{(\lambda + 2\nu)(-\Delta)^{-1} \Delta \operatorname{div}(\bar{u}) : \xi^0 \in O\} \quad \text{in } L^2(\Omega \times [0, \tau]).$$

Therefore, we claim that the set $\{(-\Delta)^{-1} \operatorname{div}(\partial_t^2 \bar{u}) : \xi^0 \in O\}$ is bounded in $L^2(\Omega \times [0, \tau])$ which allows to claim that $\{w_2 : \xi^0 \in O\}$ is bounded in $L^1([0, \tau]; H^{1+\delta}(\Omega))$, when $\delta \in (0, 1)$. The latter assertion is a consequence of regularity theory for nonhomogeneous heat equation.

Once more – the compactness of embedding $H^\delta(\Omega) \subset L^2(\Omega)$, allows us to make the conclusion that Q_2 satisfies second part of the condition 2.

Taking into account the equation for $w_1(\cdot)$, and then (5.4), after calculations we derive

$$(5.5) \quad \int_0^s \|\nabla w_1(t)\|_{L^2} dt \leq 2cs^{-\frac{1}{2}} m \sup\{\|w_1(0; \xi^0)\|_{L^2} : \xi^0 \in O\}, \quad 0 < s < \tau,$$

where $m := \|\nabla(-\Delta)^{-\frac{1}{2}}\|_{L^2(\Omega) \rightarrow (L^2(\Omega))^n}$.

For brevity, let denote $g(t; \xi^0) := -\frac{m^2}{d}(-\Delta)^{-1} \operatorname{div} \partial_t^2 \bar{u}$. From the above considerations, we have that

$$\|g(\cdot; \xi^0)\|_{L^2([0, \tau] \times \Omega)} \leq M(\tau) < \infty, \quad \xi^0 \in O, \tau > 0,$$

where $M(\tau)$ depends only on $\tau > 0$. Now, taking into account equation for $w_2(\cdot)$, and then (5.3), we derive

$$\begin{aligned} (5.6) \quad \int_0^s \|\nabla w_2(t)\|_{L^2} dt &\leq m \int_0^s \int_0^t \|(-\Delta)^{\frac{1}{2}} Q(t-r) g(r; \xi^0)\|_{L^2} dr dt \\ &\leq cm M(\tau) \int_0^s \int_0^t (t-r)^{-\frac{1}{2}} dr dt = \frac{4}{3} cm M(\tau) s^{\frac{3}{2}}, \end{aligned}$$

where $0 < s < \tau$.

If we collect together (5.5), (5.6) we arrive to the claim that Q_2 satisfies the first part of condition 2 under consideration. The proof is finished. ■

6. Uniform decaying property for system (1.1)

First we recall from [9] the following theorem (Theorem 2).

THEOREM 6.1. *Let $T(t), T_B(t)$ be c_0 -semigroups on a Banach space Y with generators G and $G + B$ respectively. Assume the following hypotheses:*

- (1) $\lim_{t \rightarrow \infty} \|T(t)y\| = 0$ for every $y \in Y$,
- (2) $\|T_B(t)\| \leq M e^{-\omega t}$, $t \geq 0$, where $M, \omega > 0$ are constants,
- (3) $T(t_0) - T_B(t_0)$ is compact for some $t_0 > 0$.

Then there exist constants $M_1, \omega_1 > 0$, such that $\|T(t)\| \leq M_1 e^{-\omega_1 t}$, $t \geq 0$.

We shall also need our main result from the paper [8].

THEOREM 6.2. *If coefficients in the system (1.1) satisfy conditions from Theorem 2.4 and domain Ω satisfies Condition (C), then $\lim_{t \rightarrow \infty} E(t) = 0$.*

We recall that $E(t) = \frac{1}{2} \|S(t)\xi^0\|^2$, where $S(t)\xi^0$ is the solution for (2.1) and in view of Theorem 2.4 is the solution of (1.1).

Let $S_b(\cdot) \equiv S(\cdot)$, where $b > 0$ is the parameter standing in systems (1.1), (1.2). The main result of this paper is the following:

THEOREM 6.3. *Let the assumptions of Theorem 4.4 and Theorem 6.2 hold, and $b \in (0, b_0)$, where b_0 was defined in Theorem 4.4. Then the semigroup $S_b(\cdot)$ has the property of uniform decaying.*

Proof. We will derive the assertion from Theorem 6.1, when applied to semigroups $S_b(t), \bar{S}_b(t)$, $b \in (0, b_0)$. From Theorem 6.2 we deduce that $S_b(t)$, $t \geq 0$ satisfies the condition (i) of Theorem 6.1. From Theorem 4.4, $\bar{S}_b(t)$,

$t \geq 0$, satisfies the condition (ii) of Theorem 6.1. From Theorem 5.1 we deduce that the difference $S_b(t) - \bar{S}_b(t)$ satisfies, for every $t > 0$, condition (iii) from Theorem 6.1. From the conclusion of Theorem 6.1 we obtain the assertion. ■

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