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FEKETE–SZEGÖ PROBLEM FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

Abstract. In this present investigation, authors introduce certain subclasses of starlike and convex functions of complex order b, using a linear multiplier differential operator $D^m_{\lambda,\mu} f(z)$. In this paper, for these classes the Fekete–Szegö problem is completely solved. Various new special cases of our results are also pointed out.

1. Introduction

Let $\mathcal{A}$ denote the family of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Further, let $\mathcal{S}$ denote the class of functions which are univalent in $\mathcal{U}$. It is well-known that for $f \in \mathcal{S}$, $\left|a_3 - a_2^2\right| \leqslant 1$. A classical theorem of Fekete–Szegö (see [7]) states that for $f \in \mathcal{S}$ given by (1.1)

$$\left|a_3 - \eta a_2^2\right| \leqslant \begin{cases} 3 - 4\eta & \text{if } \eta \leqslant 0, \\ 1 + 2\exp\left(\frac{-2\eta}{1-\eta}\right) & \text{if } 0 < \eta < 1, \\ 4\eta - 3 & \text{if } \eta \geqslant 1. \end{cases}$$

This inequality is sharp in the sense that for each η there exists a function in $\mathcal{S}$ such that equality holds. Later, Pfluger (see [17]) has considered the complex values of η and provided

$$\left|a_3 - \eta a_2^2\right| \leqslant 1 + 2\left|\exp\left(\frac{-2\eta}{1-\eta}\right)\right|.$$

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Up to this time, several authors have attempted to extend the above inequality to more general classes of analytic functions.

Given $0 \leqslant \alpha < 1$, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathcal{U}$ if

$$\Re \frac{zf'(z)}{f(z)} > \alpha, \ \ z \in \mathcal{U}, \ 0 \leqslant \alpha < 1.$$

On the other hand, a function $f \in \mathcal{A}$ is said to be in the class of convex functions of order α in $\mathcal{U}$, denoted by $\mathcal{C}(\alpha)$, if

$$\Re \left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ \ z \in \mathcal{U}, \ 0 \leqslant \alpha < 1.$$

A notions of α-starlikeness and α-convexity were generalized onto a complex order α by Nasr and Aouf (see [13]), Wiatrowski (see [21]), Nasr and Aouf (see [14]). In particular, the classes $\mathcal{S}^* = \mathcal{S}^*(0)$ and $\mathcal{C} = \mathcal{C}(0)$ are the familiar classes of starlike and convex functions in $\mathcal{U}$, respectively.

The *linear multiplier differential operator* $D^{m,\alpha}_{\lambda,\mu} f$ was defined by the authors in (see [6]) as follows

$$\begin{aligned} D^0_{\lambda,\mu} f(z) &= f(z), \\ D^1_{\lambda,\mu} f(z) &= D_{\lambda,\mu} f(z) = \lambda\mu z^2 (f(z))'' + (\lambda - \mu) z (f(z))' + (1 - \lambda + \mu) f(z), \\ D^2_{\lambda,\mu} f(z) &= D_{\lambda,\mu}\left(D^1_{\lambda,\mu} f(z)\right), \\ &\ \vdots \\ D^m_{\lambda,\mu} f(z) &= D_{\lambda,\mu}\left(D^{m-1}_{\lambda,\mu} f(z)\right), \end{aligned}$$

where $\lambda \geqslant \mu \geqslant 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

If f is given by (1.1) then from the definition of the operator $D^m_{\lambda,\mu} f(z)$ it is easy to see that

$$D^m_{\lambda,\mu} f(z) = z + \sum_{n=2}^{\infty} [1 + (\lambda\mu n + \lambda - \mu)(n-1)]^m a_n z^n. \tag{1.2}$$

It should be remarked that the $D^{m,\alpha}_{\lambda,\mu}$ is a generalization of many other linear operators considered earlier. In particular, for $f \in \mathcal{A}$ we have the following:

- $D^m_{1,0} f(z) \equiv D^m f(z)$ the operator investigated by Sălăgean (see [20]).
- $D^m_{\lambda,0} f(z) \equiv D^m_\lambda f(z)$ the operator studied by Al-Oboudi (see [2]).
- $D^m_{\lambda,\mu} f(z)$ the operator firstly considered for $0 \leqslant \mu \leqslant \lambda \leqslant 1$, by Răducanu and Orhan (see [19]).

Now, by making use of the differential operator $D^m_{\lambda,\mu}$, we define a new subclass of analytic functions.

Definition 1. Let b be a nonzero complex number, and let $f \in \mathcal{A}$, such that $D^m_{\lambda,\mu} f(z) \neq 0$ for $z \in \mathcal{U} - \{0\}$. We say that f belongs to $\mathcal{S}_m(b, \lambda, \mu)$ if

$$\Re\left(1 + \frac{1}{b}\left(\frac{z(D^m_{\lambda,\mu} f(z))'}{D^m_{\lambda,\mu} f(z)} - 1\right)\right) > 0, \;\; 0 \leqslant \mu \leqslant \lambda, \; m \in \mathbb{N}, \; z \in \mathcal{U}.$$

By giving specific values to the parameters m, b, λ and μ, we obtain the following important subclasses studied by various authors in earlier works, for instance, $\mathcal{S}_m(1 - \alpha, 1, 0) = \mathcal{S}_m(\alpha)$ (Sălăgean (see [20])), $\mathcal{S}_0(b, 1, 0) = \mathcal{S}^*(1 - b)$ (Nasr and Aouf (see [13])), $\mathcal{S}_1(b, 1, 0) = \mathcal{C}(1 - b)$ (Wiatrowski (see [21]), Nasr and Aouf (see [14])). For special values of $\lambda = 1$ and $\mu = 0$ from the general class $\mathcal{S}_m(b, \lambda, \mu)$ the new class $\mathcal{S}_m(b)$ can be obtained.

Actually, many authors have considered the Fekete–Szegö problem for various sublasses of $\mathcal{A}$, the upper bound for $\left|a_3 - \eta a_2^2\right|$ is investigated by many different authors (see [1, 3–5, 7, 9–12, 17]) and (see also recent investigations on this subject by [6, 8, 15, 16]). In the present paper we concentrate on the Fekete–Szegö problem for the subclasses $\mathcal{S}_m(b, \lambda, \mu)$ and $\mathcal{C}_m(b, \lambda, \mu)$.

2. Main results

We denote by $\mathcal{P}$ a class of analytic function in $\mathcal{U}$ with $p(0) = 1$ and $\Re p(z) > 0$. In order to derive our main results, we have to recall here the following Lemma (see [18]).

Lemma 1. *Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \dots$, then*

$$|c_n| \leqslant 2, \quad for\ n \geqslant 1.$$

If $|c_1| = 2$ then $p(z) \equiv p_1(z) = (1 + \gamma_1 z)/(1 - \gamma_1 z)$ with $\gamma_1 = c_1/2$. Conversely, if $p(z) \equiv p_1(z)$ for some $|\gamma_1| = 1$, then $c_1 = 2\gamma_1$ and $|c_1| = 2$. Furthermore, we have

$$\left|c_2 - \frac{c_1^2}{2}\right| \leqslant 2 - \frac{|c_1|^2}{2}.$$

If $|c_1| < 2$ and $\left|c_2 - \frac{c_1^2}{2}\right| = 2 - \frac{|c_1|^2}{2}$, then $p(z) \equiv p_2(z)$, where

$$p_2(z) = \frac{1 + z\frac{\gamma_2 z + \gamma_1}{1 + \bar{\gamma}_1 \gamma_2 z}}{1 - z\frac{\gamma_2 z + \gamma_1}{1 + \bar{\gamma}_1 \gamma_2 z}},$$

and $\gamma_1 = c_1/2$, $\gamma_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$. Conversely, if $p(z) \equiv p_2(z)$ for some $|\gamma_1| < 1$ and $|\gamma_2| = 1$ then $\gamma_1 = c_1/2$, $\gamma_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$ and $\left|c_2 - \frac{c_1^2}{2}\right| \leqslant 2 - \frac{|c_1|^2}{2}$.

Now, we consider functional $\left|a_3 - \eta a_2^2\right|$ for b nonzero complex number and $\eta \in \mathbb{C}$.

Theorem 1. *Let b be a nonzero complex number and $\eta \in \mathbb{C}$, $0 \leqslant \mu \leqslant \lambda$. If f of the form (1.1) is in $\mathcal{S}_m(b,\lambda,\mu)$, then*

$$|a_2| \leqslant \frac{2|b|}{A^m}, \tag{2.1}$$

$$|a_3| \leqslant \frac{|b|}{B^m} \max\{1, |1+2b|\} \tag{2.2}$$

and

$$\left|a_3 - \eta a_2^2\right| \leqslant \frac{|b|}{B^m} \max\left\{1, \left|1+2b-4\eta b\frac{B^m}{A^{2m}}\right|\right\}, \tag{2.3}$$

where $A = [1+(2\lambda\mu+\lambda-\mu)]$ and $B = [1+2(3\lambda\mu+\lambda-\mu)]$. Consider the functions

$$\frac{z(D^m_{\lambda,\mu}f(z))'}{D^m_{\lambda,\mu}f(z)} = 1+b[p_1(z)-1] \tag{2.4}$$

and

$$\frac{z(D^m_{\lambda,\mu}f(z))'}{D^m_{\lambda,\mu}f(z)} = 1+b[p_2(z)-1] \tag{2.5}$$

where p_1, p_2 are given in Lemma 1. Equality in (2.1) holds if (2.4); in (2.2) if (2.4) and (2.5); for each η in (2.3) if (2.4) and (2.5).

Proof. Denote $D^m_{\lambda,\mu}f(z) = z+\beta_2 z^2+\beta_3 z^3+\ldots$, then

$$\beta_2 = A^m a_2, \quad \beta_3 = B^m a_3. \tag{2.6}$$

By the definition of the class $\mathcal{S}_m(b,\lambda,\mu)$ there exists $p \in \mathcal{P}$ such that

$$\frac{z(D^m_{\lambda,\mu}f(z))'}{D^m_{\lambda,\mu}f(z)} = 1+b(p(z)-1),$$

so that

$$\left(\frac{z\left(1+2\beta_2 z+3\beta_3 z^2+\ldots\right)}{z+\beta_2 z^2+\beta_3 z^3+\ldots}\right) = 1-b+b(1+c_1 z+c_2 z^2+\ldots),$$

which implies the equality

$$z+2\beta_2 z^2+3\beta_3 z^3+\cdots = z+(bc_1+\beta_2)z^2+(bc_2+\beta_2 bc_1+\beta_3)z^3+\ldots$$

Equating the coefficients of both sides we have

$$\beta_2 = bc_1, \qquad \beta_3 = \frac{b^2c_1^2}{2}+\frac{bc_2}{2}, \tag{2.7}$$

so that, on account of (2.6) and (2.7)

$$a_2 = \frac{bc_1}{A^m}, \qquad a_3 = \frac{b}{2B^m}(bc_1^2+c_2). \tag{2.8}$$

Taking into account (2.8) and Lemma 1, we obtain

$$|a_2| = \left|\frac{b}{A^m} c_1\right| \leqslant \frac{2|b|}{A^m}, \tag{2.9}$$

and

$$\begin{aligned} |a_3| &= \left|\frac{b}{2B^m}\left[c_2 - \frac{c_1^2}{2} + \frac{1+2b}{2} c_1^2\right]\right| \\ &\leqslant \frac{|b|}{2B^m}\left[2 - \frac{|c_1|^2}{2} + |1+2b| \frac{|c_1|^2}{2}\right] \\ &= \frac{|b|}{B^m}\left[1 + |c_1|^2 \frac{|1+2b|-1}{4}\right] \\ &\leqslant \frac{|b|}{B^m} \max\{1, [1 + |1+2b| - 1]\}. \end{aligned}$$

Thus, we have

$$|a_3| \leqslant \frac{|b|}{B^m} \max\{1, |1+2b|\}.$$

Then, with the aid of Lemma 1, we obtain

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &= \left|\frac{b}{2B^m}(bc_1^2 + c_2) - \eta \frac{b^2 c_1^2}{A^{2m}}\right| \\ &\leq \frac{|b|}{2B^m}\left(\left|c_2 - \frac{c_1^2}{2}\right| + \frac{|c_1|^2}{2}\left|1 + 2b - \frac{4\eta b B^m}{A^{2m}}\right|\right) \\ &\leq \frac{|b|}{2B^m}\left(2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left|1 + 2b - \frac{4\eta b B^m}{A^{2m}}\right|\right) \\ &= \frac{|b|}{B^m}\left[1 + \frac{|c_1|^2}{4}\left(\left|1 + 2b - \frac{4\eta b B^m}{A^{2m}}\right| - 1\right)\right] \\ &\leqslant \frac{|b|}{B^m} \max\left\{1, \left|1 + 2b - \frac{4\eta b B^m}{A^{2m}}\right|\right\}. \end{aligned} \tag{2.10}$$

We now obtain sharpness of the estimates in (2.1), (2.2) and (2.3).

Firstly, in (2.1) the equality holds if $c_1 = 2$. Equivalently, we have $p(z) \equiv p_1(z) = (1+z)/(1-z)$. Therefore, the extremal function in $\mathcal{S}_m(b, \lambda, \mu)$ is given by

$$\frac{z(D_{\lambda,\mu}^m f(z))'}{D_{\lambda,\mu}^m f(z)} = \frac{1 + (2b-1)z}{1-z}. \tag{2.11}$$

Next, in (2.2), for first case, the equality holds if $c_1 = c_2 = 2$. Therefore, the extremal functions in $\mathcal{S}_m(b, \lambda, \mu)$ is given by (2.11) and for second case, the equality holds if $c_1 = 0,\ c_2 = 2$. Equivalently, we have $p(z) \equiv p_2(z) =$

$(1+z^2)/(1-z^2)$. Therefore, the extremal function in $\mathcal{S}_m(b,\lambda,\mu)$ is given by

$$\frac{z(D^m_{\lambda,\mu}f(z))'}{D^m_{\lambda,\mu}f(z)}=\frac{1+(2b-1)z^2}{1-z^2}. \tag{2.12}$$

Finally, in (2.3), the equality holds. Obtained extremal function for (2.2) is also valid for (2.3).

Thus, the proof of Theorem 1 is completed. ∎

We next consider the case, when η and b are real. Then we have:

Theorem 2. *Let $b>0$ and let $f\in\mathcal{S}_m(b,\lambda,\mu)$. Then for $\eta\in\mathbb{R}$ we have*

$$\left|a_3-\eta a_2^2\right|\leqslant\begin{cases}\frac{b}{B^m}\left\{1+2b\left[1-\frac{2\eta B^m}{A^{2m}}\right]\right\} & \text{if } \eta\leqslant\frac{A^{2m}}{2B^m},\\ \frac{b}{B^m} & \text{if } \frac{A^{2m}}{2B^m}\leqslant\eta\leqslant\frac{(1+2b)A^{2m}}{4bB^m},\\ \frac{b}{B^m}\left[\frac{4\eta bB^m}{A^{2m}}-2b-1\right] & \text{if } \eta\geqslant\frac{(1+2b)A^{2m}}{4bB^m},\end{cases}$$

where $A=[1+(2\lambda\mu+\lambda-\mu)]$ and $B=[1+2(3\lambda\mu+\lambda-\mu)]$. For each η, the equality holds for functions in (2.4) and (2.5).

Proof. First, let $\eta\leqslant\frac{A^{2m}}{2B^m}\leqslant\frac{(1+2b)A^{2m}}{4bB^m}$. In this case (2.8) and Lemma 1 give

$$\begin{aligned}\left|a_3-\eta a_2^2\right|&\leqslant\frac{b}{2B^m}\left[2-\frac{|c_1|^2}{2}+\frac{|c_1|^2}{2}\left(1+2b-\frac{4\eta bB^m}{A^{2m}}\right)\right]\\&\leqslant\frac{b}{B^m}\left[1+2b\left(1-\frac{2\eta B^m}{A^{2m}}\right)\right].\end{aligned}$$

Let, now $\frac{A^{2m}}{2B^m}\leqslant\eta\leqslant\frac{(1+2b)A^{2m}}{4bB^m}$. Then, using the above calculations, we obtain

$$\left|a_3-\eta a_2^2\right|\leqslant\frac{b}{B^m}.$$

Finally, if $\eta\geqslant\frac{(1+2b)A^{2m}}{4bB^m}$, then

$$\begin{aligned}\left|a_3-\eta a_2^2\right|&\leqslant\frac{b}{2B^m}\left[2-\frac{|c_1|^2}{2}+\frac{|c_1|^2}{2}\left(\frac{4\eta bB^m}{A^{2m}}-1-2b\right)\right]\\&=\frac{b}{2B^m}\left[2+\frac{|c_1|^2}{2}\left(\frac{4\eta bB^m}{A^{2m}}-2-2b\right)\right]\\&\leqslant\frac{b}{B^m}\left[\frac{4\eta bB^m}{A^{2m}}-2b-1\right].\end{aligned}$$

Thus, the proof of Theorem 2 is completed. ∎

Finally, we consider the case, when b is a nonzero complex number and $\eta\in\mathbb{R}$. Then we get:

Theorem 3. *Let b be a nonzero complex number and let $f \in \mathcal{S}_m(b, \lambda, \mu)$. Then for $\eta \in \mathbb{R}$ we have*

$$\left|a_3 - \eta a_2^2\right| \leqslant \begin{cases} \frac{4|b|^2}{A^{2m}}\left[\Re(k_1) - \eta\right] + \frac{|b||\sin\theta|}{B^m} & \text{if } \eta \leqslant N_1, \\ \frac{|b|}{B^m} & \text{if } N_1 \leqslant \eta \leqslant R_1, \\ \frac{4|b|^2}{A^{2m}}\left[\eta - \Re(k_1)\right] + \frac{|b||\sin\theta|}{B^m} & \text{if } \eta \geqslant R_1, \end{cases}$$

where $A = [1 + (2\lambda\mu + \lambda - \mu)]$, $B = [1 + 2(3\lambda\mu + \lambda - \mu)]$, $|b| = be^{i\theta}$, $k_1 = \frac{A^{2m}}{2B^m} + \frac{A^{2m}e^{i\theta}}{4|b|B^m}$, $\ell_1 = \frac{A^{2m}}{4|b|B^m}$, $N_1 = \Re(k_1) - \ell_1(1 - |\sin\theta|)$ and $R_1 = \Re(k_1) + \ell_1(1 - |\sin\theta|)$. For each η there is a function in $\mathcal{S}_m(b, \lambda, \mu)$ such that the equality holds.

Proof. From the inequality (2.10), we have

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &= \frac{|b|}{2B^m}\left(\left|c_2 - \frac{c_1^2}{2}\right| + \frac{|c_1|^2}{2}\left|1 + 2b - \frac{4\eta b B^m}{A^{2m}}\right|\right) \\ &\leqslant \frac{|b|}{2B^m}\left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left|1 + 2b - \frac{4\eta b B^m}{A^{2m}}\right|\right] \\ &= \frac{|b|}{2B^m}\left[\frac{|c_1|^2}{2}\left(\left|1 + 2b - \frac{4\eta b B^m}{A^{2m}}\right| - 1\right) + 2\right] \\ &= \frac{|b|}{B^m} + \frac{|b|}{4B^m}\left[\left|\frac{4\eta b B^m}{A^{2m}} - 2b - 1\right| - 1\right]|c_1|^2 \\ &= \frac{|b|}{B^m} + \frac{|b|^2}{A^{2m}}\left[\left|\eta - \frac{A^{2m}}{2B^m} - \frac{A^{2m}}{4bB^m}\right| - \frac{A^{2m}}{4\,|b|\,B^m}\right]|c_1|^2 . \end{aligned}$$

If we write $|b| = be^{i\theta}$ (or $b = |b|\,e^{-i\theta}$), $\frac{A^{2m}}{2B^m} + \frac{A^{2m}e^{i\theta}}{4|b|B^m} = k_1$ and $\frac{A^{2m}}{4|b|B^m} = \ell_1$ in last equation, we get

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &\leqslant \frac{|b|}{B^m} + \frac{|b|^2}{A^{2m}}\left[|\eta - k_1| - \ell_1\right]|c_1|^2 \\ &\leqslant \frac{|b|}{B^m} + \frac{|b|^2}{A^{2m}}\left[|\eta - \Re(k_1)| + \ell_1|\sin\theta| - \ell_1\right]|c_1|^2 \\ &= \frac{|b|}{B^m} + \frac{|b|^2}{A^{2m}}\left[|\eta - \Re(k_1)| - \ell_1(1 - |\sin\theta|)\right]|c_1|^2 . \end{aligned} \tag{2.13}$$

We consider the following cases for (2.13). Suppose $\eta \leqslant \Re(k_1)$. Then

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &\leqslant \frac{|b|}{B^m} + \frac{|b|^2}{A^{2m}}\left[\Re(k_1) - \ell_1(1 - |\sin\theta|) - \eta\right]|c_1|^2 \\ &= \frac{|b|}{B^m} + \frac{|b|^2}{A^{2m}}\left[N_1 - \eta\right]|c_1|^2 . \end{aligned} \tag{2.14}$$

Let $\eta \leqslant N_1 = \Re(k_1) - \ell_1\,(1 - |\sin\theta|)$. By using Lemma 1 and $\ell_1 = \frac{A^{2m}}{4|b|B^m}$ in inequality (2.14), we get

$$\begin{aligned}\left|a_3 - \eta a_2^2\right| &\leqslant \frac{|b|}{B^m} + \frac{4\,|b|^2}{A^{2m}}\left(\Re(k_1) - \eta\right) - \frac{4\,|b|^2}{A^{2m}}\frac{A^{2m}}{4\,|b|\,B^m}\,(1 - |\sin\theta|)\\ &= \frac{|b|}{B^m} + \frac{4\,|b|^2}{A^{2m}}\left(\Re(k_1) - \eta\right) - \frac{|b|}{B^m}\,(1 - |\sin\theta|)\\ &= \frac{4\,|b|^2}{A^{2m}}\left(\Re(k_1) - \eta\right) + \frac{|b|\,|\sin\theta|}{B^m}.\end{aligned}$$

If we take $N_1 = \Re(k_1) - \ell_1\,(1 - |\sin\theta|) \leqslant \eta \leqslant \Re(k_1)$, then (2.14) gives

$$\left|a_3 - \eta a_2^2\right| \leqslant \frac{|b|}{B^m}.$$

Let $\eta \geqslant \Re(k_1)$. From (2.13) we get

$$\begin{aligned}\left|a_3 - \eta a_2^2\right| &\leqslant \frac{|b|}{B^m} + \frac{|b|^2}{A^{2m}}\left[\eta - \left(\Re(k_1) + \ell_1(1 - |\sin\theta|)\right)\right]|c_1|^2 \\ &= \frac{|b|}{B^m} + \frac{|b|^2}{A^{2m}}\left[\eta - R_1\right]|c_1|^2 .\end{aligned} \tag{2.15}$$

Let $\eta \leqslant R_1 = \Re(k_1) + \ell_1\,(1 - |\sin\theta|)$. Applying (2.15) we obtain

$$\left|a_3 - \eta a_2^2\right| \leqslant \frac{|b|}{B^m}.$$

Let $\eta \geqslant R_1 = \Re(k_1) + \ell_1\,(1 - |\sin\theta|)$. By using Lemma 1 and $\ell_1 = \frac{A^{2m}}{4|b|B^m}$ in equality (2.15), we get

$$\begin{aligned}\left|a_3 - \eta a_2^2\right| &\leqslant \frac{|b|}{B^m} + \frac{4\,|b|^2}{A^{2m}}\left(\eta - \Re(k_1)\right) - \frac{|b|}{B^m}\,(1 - |\sin\theta|)\\ &\leqslant \frac{4\,|b|^2}{A^{2m}}\left(\eta - \Re(k_1)\right) + \frac{|b|\,|\sin\theta|}{B^m}.\end{aligned}$$

Therefore, the proof is completed. ∎

Corollary 1. *If we take $\lambda = 1$ and $\mu = 0$ in Theorems 1–3, we have following new results, respectively.*

(1) *Let $b \in \mathbb{C}$, $b \neq 0$ and $f \in \mathcal{S}_m(b)$. Then for $\eta \in \mathbb{C}$ we have*

$$|a_2| \leqslant \frac{|b|}{2^{m-1}}, \qquad |a_3| \leqslant \frac{|b|}{3^m}\max\left\{1, |1 + 2b|\right\}$$

and

$$\left|a_3 - \eta a_2^2\right| \leqslant \frac{|b|}{3^m}\max\left\{1, \left|1 + 2b - 4\eta b\left(\frac{3}{4}\right)^m\right|\right\}.$$

Equality holds for the cases $\lambda=1$, $\mu=0$ of (2.4) and (2.5) in Theorem 1.

(2) *Let $b > 0$ and $f \in \mathcal{S}_m(b)$. Then for $\eta \in \mathbb{R}$ we have*

$$\left|a_3 - \eta a_2^2\right| \leqslant \begin{cases} \frac{b}{3^m}\left\{1 + 2b\left[1 - \frac{2\eta B^m}{A^{2m}}\right]\right\} & \text{if } \eta \leqslant \frac{1}{2}\left(\frac{4}{3}\right)^m, \\ \frac{b}{B^m} & \text{if } \frac{1}{2}\left(\frac{4}{3}\right)^m \leqslant \eta \leqslant \frac{(1+2b)}{4b}\left(\frac{4}{3}\right)^m, \\ \frac{b}{B^m}\left[\frac{4\eta b B^m}{A^{2m}} - 2b - 1\right] & \text{if } \eta \geqslant \frac{(1+2b)}{4b}\left(\frac{4}{3}\right)^m. \end{cases}$$

For each η, the equality holds for the cases $\lambda=1$, $\mu=0$ of (2.4) and (2.5).

(3) *Let $b \in \mathbb{C}$, $b \neq 0$ and $f \in \mathcal{S}_m(b)$. Then for $\eta \in \mathbb{R}$ we have*

$$\left|a_3 - \eta a_2^2\right| \leqslant \begin{cases} \frac{|b|^2}{4^{m-1}}\left[\Re(k_1) - \eta\right] + \frac{|b||\sin\theta|}{3^m} & \text{if } \eta \leqslant N_1, \\ \frac{|b|}{3^m} & \text{if } N_1 \leqslant \eta \leqslant R_1, \\ \frac{|b|^2}{4^{m-1}}\left[\eta - \Re(k_1)\right] + \frac{|b||\sin\theta|}{3^m} & \text{if } \eta \geqslant R_1, \end{cases}$$

where $|b| = be^{i\theta}$, $k_1 = \left(\frac{4}{3}\right)^m - \left(\frac{4}{3}\right)^m \frac{e^{i\theta}}{4|b|}$, $\ell_1 = \left(\frac{4}{3}\right)^m \frac{1}{4|b|}$, $N_1 = \Re(k_1) - \ell_1\left(1 - |\sin\theta|\right)$ and $R_1 = \Re(k_1) + \ell_1\left(1 - |\sin\theta|\right)$. For each η there is a function in $\mathcal{S}_m(b)$ such that the equality holds.

As an analogue to the complex nth starlikeness of a complex order we can introduce the notion of nth convexity of a complex order as follows:

Definition 2. Let b be a nonzero complex number and let $f \in \mathcal{A}$. We say that f belongs to $\mathcal{C}_m(b, \lambda, \mu)$ if

$$Re\left(1 + \frac{1}{b}\left(z\frac{(D^m_{\lambda,\mu} f(z))''}{(D^m_{\lambda,\mu} f(z))'}\right)\right) > 0, \quad 0 \leqslant \mu \leqslant \lambda, \; m \in \mathbb{N}, \; z \in \mathcal{U}.$$

We easily obtain bounds of coefficients and a solution of the Fekete–Szegö problem in $\mathcal{C}_m(b, \lambda, \mu)$. For special values of $\lambda = 1$ and $\mu = 0$ from the general class $\mathcal{C}_m(b, \lambda, \mu)$, the new class $\mathcal{C}_m(b)$ can be obtained.

Theorem 4. *Let b be a nonzero complex number and $\eta \in \mathbb{C}$, $0 \leqslant \mu \leqslant \lambda$. If f of the form (1.1) is in $\mathcal{C}_m(b, \lambda, \mu)$, then*

$$|a_2| \leqslant \frac{|b|}{A^m}, \qquad |a_3| \leqslant \frac{|b|}{3B^m}\max\{1, |1 + 2b|\}$$

and

$$\left|a_3 - \eta a_2^2\right| \leqslant \frac{|b|}{3B^m}\max\left\{1, \left|1 + 2b - \eta\frac{3bB^m}{A^{2m}}\right|\right\},$$

where $A = [1 + (2\lambda\mu + \lambda - \mu)]$ and $B = [1 + 2(3\lambda\mu + \lambda - \mu)]$. For each η there is a function in $\mathcal{C}_m(b, \lambda, \mu)$ such that equalities hold.

Theorem 5. *Let $b>0$ and let $f\in\mathcal{C}_m(b,\lambda,\mu)$. Then for $\eta\in\mathbb{R}$ we have*

$$\left|a_3-\eta a_2^2\right|\leqslant\begin{cases}\frac{b}{3B^m}\left[1+2b-\eta\frac{3bB^m}{A^{2m}}\right] & \text{if } \eta\leqslant\frac{4A^{2m}}{3B^m},\\ \frac{b}{3B^m} & \text{if } \frac{4A^{2m}}{3B^m}\leqslant\eta\leqslant\frac{(1+2b)A^{2m}}{3bB^m},\\ \frac{b}{3B^m}\left[-1-2b+\eta\frac{3bB^m}{A^{2m}}\right] & \text{if } \eta\geqslant\frac{(1+2b)A^{2m}}{3bB^m},\end{cases}$$

where $A=[1+(2\lambda\mu+\lambda-\mu)]$ and $B=[1+2(3\lambda\mu+\lambda-\mu)]$. For each η there is a function in $\mathcal{C}_m(b,\lambda,\mu)$ such that equality holds.

Theorem 6. *Let b be a nonzero complex number and let $f\in\mathcal{C}_m(b,\lambda,\mu)$. Then for $\eta\in\mathbb{R}$ we have*

$$\left|a_3-\eta a_2^2\right|\leqslant\begin{cases}\frac{|b|^2}{A^{2m}}(\Re(k_2)-\eta)+\frac{|b||\sin\theta|}{3B^m} & \text{if } \eta\leqslant N_2,\\ \frac{|b|}{3B^m} & \text{if } N_2\leqslant\eta\leqslant R_2,\\ \frac{|b|^2}{A^{2m}}(\eta-\Re(k_2))+\frac{|b||\sin\theta|}{3B^m} & \text{if } \eta\geqslant R_2,\end{cases}$$

where $A=[1+(2\lambda\mu+\lambda-\mu)]$, $B=[1+2(3\lambda\mu+\lambda-\mu)]$, $|b|=be^{i\theta}$, $k_2=\frac{2A^{2m}}{3B^m}+\frac{A^{2m}e^{i\theta}}{3|b|B^m}$, $\ell_2=\frac{A^{2m}}{3|b|B^m}$, $N_2=\Re(k_2)-\ell_2\left(1-|\sin\theta|\right)$ and $R_2=\Re(k_2)+\ell_2\left(1-|\sin\theta|\right)$. For each η there is a function in $\mathcal{C}_m(b,\lambda,\mu)$ such that equality holds.

Corollary 2. *If we take $\lambda=1$ and $\mu=0$ in Theorems 4–6, we have the following new results.*

(1) *Let $b\in\mathbb{C}$, $b\neq0$ and $f\in\mathcal{C}_m(b)$. Then for $\eta\in\mathbb{C}$ we have*

$$|a_2|\leqslant\frac{|b|}{2^m},\qquad |a_3|\leqslant\frac{|b|}{3^{m+1}}\max\left\{1,|1+2b|\right\}$$

and

$$\left|a_3-\eta a_2^2\right|\leqslant\frac{|b|}{3^{m+1}}\max\left\{1,\left|1+2b-4\eta b\left(\frac{3}{4}\right)^{m+1}\right|\right\}.$$

For each η there is a function in $\mathcal{C}_m(b)$ such that equality holds.

(2) *Let $b>0$ and $f\in\mathcal{C}_m(b)$. Then for $\eta\in\mathbb{R}$ we have*

$$\left|a_3-\eta a_2^2\right|\leqslant\begin{cases}\frac{b}{3^{m+1}}\left[1+2b-4\eta b\left(\frac{3}{4}\right)^{m+1}\right] & \text{if } \eta\leqslant\left(\frac{4}{3}\right)^{m+1},\\ \frac{b}{3^{m+1}} & \text{if } \left(\frac{4}{3}\right)^{m+1}\leqslant\eta\leqslant\frac{(1+2b)}{4b}\left(\frac{4}{3}\right)^{m+1},\\ \frac{b}{3^{m+1}}\left[-1-2b+4\eta b\left(\frac{3}{4}\right)^{m+1}\right] & \text{if } \eta\geqslant\frac{(1+2b)}{4b}\left(\frac{4}{3}\right)^{m+1}.\end{cases}$$

For each η there is a function in $\mathcal{C}_m(b)$ such that equality holds.

(3) *Let* $b \in \mathbb{C}$, $b \neq 0$ *and* $f \in \mathcal{C}_m(b)$. *Then for* $\eta \in \mathbb{R}$ *we have*

$$\left|a_3 - \eta a_2^2\right| \leqslant \begin{cases} \frac{|b|^2}{4^m}(\Re(k_2) - \eta) + \frac{|b||\sin\theta|}{3^{m+1}} & \text{if } \eta \leqslant N_2, \\ \frac{|b|}{3^{m+1}} & \text{if } N_2 \leqslant \eta \leqslant R_2, \\ \frac{|b|^2}{4^m}(\eta - \Re(k_2)) + \frac{|b||\sin\theta|}{3^{m+1}} & \text{if } \eta \geqslant R_2, \end{cases}$$

where $|b| = be^{i\theta}$, $k_2 = \frac{1}{2}\left(\frac{4}{3}\right)^{m+1} - \frac{e^{i\theta}}{4|b|}\left(\frac{4}{3}\right)^{m+1}$, $\ell_2 = \frac{1}{4|b|}\left(\frac{4}{3}\right)^{m+1}$, $N_2 = \Re(k_2) - \ell_2\left(1 - |\sin\theta|\right)$ *and* $R_2 = \Re(k_2) + \ell_2\left(1 - |\sin\theta|\right)$. *For each* η *there is a function in* $\mathcal{C}_m(b)$ *such that equality holds.*

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