DEMONSTRATIO MATHEMATICA
Vol. XLV No 4 2012

10.1515/dema-2013-0421

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CARDINAL INVARIANTS CONCERNING CLOSED GRAPH FUNCTIONS

Abstract. Cardinal invariants connected with sums, products and quotients of real functions concerning the family of closed graph functions and the complement in $\mathbb{R}^{\mathbb{R}}$ of this family are investigate.

1. Introduction

The letters $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the real line, the set of rationals and the set of positive integers, respectively. The family of all functions from a set X into Y is denoted by Y^X. The word *function* denotes a mapping from $\mathbb{R}$ to $\mathbb{R}$ unless otherwise explicitly stated. For each set $A \subset \mathbb{R}$ the symbol χ_A denote the characteristic function of A. We consider cardinals as ordinals not in one-to-one correspondence with the smaller ordinals. The symbol $\operatorname{card} X$ stands for the cardinality of a set X. We write $\mathfrak{c} = \operatorname{card} \mathbb{R}$. For each set $A \subset \mathbb{R}$ the symbol $\operatorname{cl} A$ denotes the closure of A.

Let $f \colon \mathbb{R} \to \mathbb{R}$. The symbol $C(f)$ denotes the set of points of continuity of f. For each $y \in \mathbb{R}$ let $[f = y] = \big\{x \in \mathbb{R} \colon f(x) = y\big\}$. Similarly we define the symbols $[f > y]$, $[f < y]$. We say that f has the *closed graph* ($f \in \mathcal{U}$), if the set $\{(x, f(x)) : x \in \mathbb{R}\}$ is closed in $\mathbb{R}^2$. We say that f is a *piecewise continuous* function ($f \in \mathcal{P}$), if there are closed sets $F_n \subset \mathbb{R}$ such that $\bigcup_{n \in \mathbb{N}} F_n = \mathbb{R}$ and the restriction $f \restriction F_n$ is continuous for each $n \in \mathbb{N}$. It is known that a function $f \in \mathbb{R}^{\mathbb{R}}$ is piecewise continuous iff f is a *Baire-one-star function* i.e. for each closed set $F \subset \mathbb{R}$ there is an open set G such that $F \cap G \neq \emptyset$ and $f \restriction (F \cap G)$ is continuous (see [9]).

If $\mathcal{A} \subset \mathbb{R}^{\mathbb{R}}$, denote by

$$\mathcal{A} - \mathcal{A} \stackrel{\text{df}}{=} \big\{f - g : f, g \in \mathcal{A}\big\},$$

$$\mathcal{A}/\mathcal{A} \stackrel{\text{df}}{=} \big\{f/g : f, g \in \mathcal{A} \;\&\; g(x) \neq 0 \text{ for each } x \in \mathbb{R}\big\}.$$

2000 *Mathematics Subject Classification*: Primary 26A15, Secondary 54C08.
Key words and phrases: closed graph function, cardinal invariants.

In 1991 T. Natkaniec [11] defined the following two cardinal functions for every $\mathcal{A} \subset \mathbb{R}^{\mathbb{R}}$

$$\mathrm{a}(\mathcal{A}) \stackrel{\text{df}}{=} \min\big(\{\operatorname{card}\mathcal{F} : \mathcal{F} \subset \mathbb{R}^{\mathbb{R}} \,\&\, \neg(\exists_g \forall_{f\in\mathcal{F}} f + g \in \mathcal{A})\} \cup \{(2^{\mathfrak{c}})^{+}\}\big),$$

$$\mathrm{m}(\mathcal{A}) \stackrel{\text{df}}{=} \min\big(\{\operatorname{card}\mathcal{F} : \mathcal{F} \subset \mathbb{R}^{\mathbb{R}} \,\&\, \neg(\exists_{g\in\mathbb{R}^{\mathbb{R}}\setminus\{\chi_\emptyset\}} \forall_{f\in\mathcal{F}} f \cdot g \in \mathcal{A})\} \cup \{(2^{\mathfrak{c}})^{+}\}\big).$$

The extra assumption that $g \neq \chi_\emptyset$ is added in the definition of m since otherwise for every family $\mathcal{A} \subset \mathbb{R}^{\mathbb{R}}$ containing constant zero function $\chi_\emptyset$ we would have $\mathrm{m}(\mathcal{A}) = (2^{\mathfrak{c}})^{+}$.

The values of functions a and m for different classes of real functions has been studied in several papers (see e.g. [3] and [4]).

The following cardinal function connected with quotients of functions has been defined in [5] (compare also [6]) for every $\mathcal{A} \subset \mathbb{R}^{\mathbb{R}}$.

$$\mathrm{q}(\mathcal{A}) \stackrel{\text{df}}{=} \min\big(\{\operatorname{card}\mathcal{F} : \mathcal{F} \subset {}^{\mathcal{A}}\!/_{\mathcal{A}} \,\&\, \neg(\exists_g \forall_{f\in\mathcal{F}}\, f/g \in \mathcal{A})\} \cup \{(\operatorname{card}{}^{\mathcal{A}}\!/_{\mathcal{A}})^{+}\}\big).$$

In the above definition it is quite natural to restrict ourselves to subfamilies of ${}^{\mathcal{A}}\!/_{\mathcal{A}}$ only. Indeed, if there is a function g such that both f/g and $1/g$ are in $\mathcal{A}$, then $f \in {}^{\mathcal{A}}\!/_{\mathcal{A}}$.

In 1996 Jordan (see [7] or [8]) examined the values of $\mathrm{a}(\neg\mathcal{A})$, where $\neg\mathcal{A} \stackrel{\text{df}}{=} \mathbb{R}^{\mathbb{R}} \setminus \mathcal{A}$ and classes $\mathcal{A}$ are chosen from the classes of Darboux-like functions. Notice that $\mathrm{a}(\neg\mathcal{A})$ has the following interpretation:

$\mathrm{a}(\neg\mathcal{A})$ is the smallest cardinality of a family $\mathcal{B} \subset \mathbb{R}^{\mathbb{R}}$ such that $\mathcal{A} - \mathcal{B} = \mathbb{R}^{\mathbb{R}}$, where $\mathcal{A} - \mathcal{B} = \{f - g : f \in \mathcal{A} \,\&\, g \in \mathcal{B}\}$.

The purpose of this paper is to find the values of cardinal functions a, m and q for the families $\mathcal{U}$ and $\neg\mathcal{U}$, where $\neg\mathcal{U} \stackrel{\text{df}}{=} \mathbb{R}^{\mathbb{R}} \setminus \mathcal{U}$. We obtained the following results:

- $\mathrm{a}(\mathcal{U}) = 2 = \mathrm{m}(\mathcal{U})$ (see Theorems 2.2 and 2.3);
- $\mathrm{q}(\mathcal{U}) = \omega$ (see Theorem 2.7);
- $\mathrm{a}(\neg\mathcal{U}) = 2^{\mathfrak{c}} = \mathrm{q}(\neg\mathcal{U})$ (see Corollaries 3.2 and 3.7);
- $\mathrm{m}(\neg\mathcal{U}) = 1$ (this equality is obvious).

2. The family $\mathcal{U}$

Before we start our examination, we recall some basic facts about cardinal functions a and m (see [4] or [7, Proposition 1]), which will be applied in this paper.

Proposition 2.1. *Let $\mathcal{A} \subset \mathbb{R}^{\mathbb{R}}$. Then*

(a) *if $\mathcal{A} \neq \emptyset$ then $\mathrm{a}(\mathcal{A}) = 2$ if and only if $\mathcal{A} - \mathcal{A} \neq \mathbb{R}^{\mathbb{R}}$;*
(b) *$\mathrm{a}(\mathcal{A}) \le 2^{\mathfrak{c}}$ if and only if $\mathcal{A} \neq \mathbb{R}^{\mathbb{R}}$;*
(c) *$\mathrm{m}(\mathcal{A}) \ge 2$ if $\chi_\emptyset, \chi_{\mathbb{R}} \in \mathcal{A}$.*

Since $\mathcal{U}-\mathcal{U}=\mathcal{P}$ (see [2] or [1, Theorem 2]), then from Proposition 2.1(a) we conclude that

Theorem 2.2. $\mathrm{a}(\mathcal{U})=2$.

Theorem 2.3. $\mathrm{m}(\mathcal{U})=2$.

Proof. The inequality $\mathrm{m}(\mathcal{U})\geq 2$ follows from Proposition 2.1(c). To see that $\mathrm{m}(\mathcal{U})\leq 2$ take $\mathcal{F}\overset{\mathrm{df}}{=}\{\chi_{\mathbb{Q}},\chi_{\mathbb{R}}\}$. Let $g\in\mathbb{R}^{\mathbb{R}}\setminus\{\chi_{\emptyset}\}$. It is enough to show that $f\cdot g\notin\mathcal{U}$ for some $f\in\mathcal{F}$. If $g\in\mathcal{U}$, then $\chi_{\mathbb{Q}}\cdot g\notin\mathcal{U}$, since $\mathrm{cl}[\chi_{\mathbb{Q}}\cdot g=0]=\mathbb{R}$ and $\chi_{\mathbb{Q}}\cdot g\neq\chi_{\emptyset}$. Otherwise, evidently $\chi_{\mathbb{R}}\cdot g\notin\mathcal{U}$. ∎

Now, recall the characterization of ${}^{\mathcal{U}}\!/_{\mathcal{U}}$ given by Borsík [1, Theorem 4].

Proposition 2.4. *A function f belongs to ${}^{\mathcal{U}}\!/_{\mathcal{U}}$ iff f belongs to $\mathcal{P}$ and the set $[f=0]$ is closed.*

Lemma 2.5. *Let $f,h\in{}^{\mathcal{U}}\!/_{\mathcal{U}}$. There is a function $g:\mathbb{R}\to(0,\infty)$ such that $f/g\in\mathcal{U}$ and $h/g\in\mathcal{U}$.*

Proof. Let $f,h\in{}^{\mathcal{U}}\!/_{\mathcal{U}}$. By Proposition 2.4 the sets $A\overset{\mathrm{df}}{=}[f=0]$, $B\overset{\mathrm{df}}{=}[h=0]$ are closed and $f,h\in\mathcal{P}$. Let $\{F_n : n\in\mathbb{N}\}$ be an increasing sequence of closed subsets of $\mathbb{R}$ such that $F_1\overset{\mathrm{df}}{=}\emptyset$, the restrictions $f\restriction F_n$ and $h\restriction F_n$ are continuous for each $n\geq 2$ and $\bigcup_{n\in\mathbb{N}}F_n=\mathbb{R}$. Put $E_n\overset{\mathrm{df}}{=}F_{n+1}\setminus F_n$ for each $n\in\mathbb{N}$. Define $g:\mathbb{R}\to(0,\infty)$ by the formula

$$g(x)=\begin{cases}1, & \text{if } x\in A\cap B,\\ \varrho(x,F_n\cup B)\min\{1,|h(x)|\}, & \text{if } x\in E_n\cap A\setminus B,\ n\in\mathbb{N},\\ \varrho(x,F_n\cup A)\min\{1,|f(x)|\}, & \text{if } x\in E_n\cap B\setminus A,\ n\in\mathbb{N},\\ \varrho(x,F_n\cup A\cup B)\min\{1,|f(x)|,|h(x)|\}, & \text{if } x\in E_n\setminus(A\cup B),\ n\in\mathbb{N}.\end{cases}$$

We will show that $f/g\in\mathcal{U}$ (analogously we can prove that $h/g\in\mathcal{U}$). Let $x_0\in\mathbb{R}$, $x_n\to x_0$ and $(f/g)(x_n)\to y_0$ ($y_0\in\mathbb{R}$). We consider three cases.

If $x_0\in A$, then there is $n_0\in\mathbb{N}$ such that $x_n\in A$ for each $n>n_0$. Indeed, if $x\in\{x_n : n\in\mathbb{N}\}\setminus A$, then $x\in E_k\setminus A$ for some $k\in\mathbb{N}$ and

$$|(f/g)(x)|\geq|f(x)|/(\varrho(x,A)\cdot\min\{1,|f(x)|\})\geq 1/\varrho(x,A)\geq 1/|x-x_0|.$$

Since $x_n\to x_0$ and $(f/g)(x_n)\to y_0$, where $y_0\in\mathbb{R}$, the set of all such x must be finite. Hence $(f/g)(x_n)=0\to(f/g)(x_0)=0$ for $n>n_0$.

Now, assume that $x_0\in\mathbb{R}\setminus(A\cup B)$. Let $k\in\mathbb{N}$ be such that $x_0\in E_k\setminus(A\cup B)$. We will show that there is $n_0\in\mathbb{N}$ such that $x_n\in E_k\setminus(A\cup B)$ for each $n>n_0$. Since $x_n\to x_0$, $x_0\notin(F_k\cup A\cup B)$ and the set $F_k\cup A\cup B$ is closed, there is $n_1\in\mathbb{N}$ such that $x_n\notin(F_k\cup A\cup B)$ for each $n>n_1$.

Moreover, if $x \in \{x_n : n > n_1\}$ be such that $x \notin F_{k+1}$, then $x \in E_i \setminus (A \cup B)$ for some $i \geq k+1$ and

$$|(f/g)(x)| \geq 1/\varrho(x, F_{k+1}) \geq 1/|x - x_0|.$$

Consequently, the set of all such x must be finite. Since f and g are continuous on $E_k \setminus (A \cup B)$, we have $(f/g)(x_n) \to (f/g)(x_0)$.

Similarly we can show that, if $x_0 \in B \setminus A$, then $(f/g)(x_n) \to (f/g)(x_0)$. ∎

The proof of the next proposition is a repetition of the argumentation used in the proof of [10, Proposition II 3.3].

Proposition 2.6. *Let $g\colon \mathbb{R} \to \mathbb{R}$. If $C(g) \neq \emptyset$, then there is a $q \in \mathbb{Q}$ such that $\chi_{\{q\}} + g \notin \mathcal{U}$.*

Proof. Let g be a function and $x \in C(g)$. Let $\delta > 0$ such that $|g(t) - g(x)| < 2^{-1}$ for $t \in (x - \delta, x + \delta)$. Choose a $q \in \mathbb{Q} \cap (x - \delta, x + \delta)$. Then $(\chi_{\{q\}} + g)(q) > g(x) + 2^{-1}$ and $(\chi_{\{q\}} + g)(t) < g(x) + 2^{-1}$ for every $t \in (x - \delta, x + \delta) \setminus \{q\}$. Consequently $\chi_{\{q\}} + g \notin \mathcal{U}$. ∎

Theorem 2.7. $\mathrm{q}(\mathcal{U}) = \omega$.

Proof. Proceeding similarly as in the proof of Lemma 2.5 we can show that if $f_1, \dots f_k \in {}^{\mathcal{U}}\!/\!_{\mathcal{U}}$, then there is a function $g\colon \mathbb{R} \to (0, \infty)$ such that $f_i/g \in \mathcal{U}$ for each $i \in \{1, \dots, k\}$. Consequently, $\mathrm{q}(\mathcal{U}) \geq \omega$.

Now, we will prove the opposite inequality. Define

$$\mathcal{F} \overset{\text{df}}{=} \big\{\exp \circ \chi_{\{q\}} : q \in \mathbb{Q}\big\}.$$

Evidently $\operatorname{card} \mathcal{F} = \omega$ and $\mathcal{F} \subset {}^{\mathcal{U}}\!/\!_{\mathcal{U}}$. Let $g\colon \mathbb{R} \to \mathbb{R} \setminus \{0\}$. We will show that there is a function $f \in \mathcal{F}$ such that $f/g \notin \mathcal{U}$.

If $C(g) = \emptyset$, then $C((\exp \circ \chi_{\{0\}})/g) \subset \{0\}$, so $(\exp \circ \chi_{\{0\}})/g \notin \mathcal{U}$.

Otherwise, by Proposition 2.6, there is a $q \in \mathbb{Q}$ such that $\chi_{\{q\}} - \ln|g| \notin \mathcal{U}$. Hence

$$\exp \circ (\chi_{\{q\}} - \ln|g|) = (\exp \circ \chi_{\{q\}})/|g| \notin \mathcal{U},$$

and consequently $(\exp \circ \chi_{\{q\}})/g \notin \mathcal{U}$. ∎

Remark 1. *Let $\mathcal{F}$ be a finite family of piecewise continuous functions. Then there is a closed graph function $g\colon \mathbb{R} \to (0, \infty)$ such that $f + g \in \mathcal{U}$ for each $f \in \mathcal{F}$.*

Proof. Let $f_1, \dots, f_l \in \mathcal{P}$. Let $\{F_n : n \in \mathbb{N}\}$ be an increasing sequence of closed subsets of $\mathbb{R}$ such that $\bigcup_{n \in \mathbb{N}} F_n = \mathbb{R}$ and the restrictions $f_i \restriction F_n$ are continuous for each $n \in \mathbb{N}$ and $i \in \{1, \dots, l\}$. Put $F_0 \overset{\text{df}}{=} \emptyset$ and $E_n \overset{\text{df}}{=} F_n \setminus F_{n-1}$ for each $n \in \mathbb{N}$. Define a function $h : \mathbb{R} \to (0, \infty)$ as $h(x) \overset{\text{df}}{=} 1/\varrho(x, F_{n-1})$ for $x \in E_n$ and $n \in \mathbb{N}$. Finally, let $g \overset{\text{df}}{=} h - \min\{f_1, \dots, f_l, 0\})$. Evidently g is positive.

Fix $i \in \{1, \ldots, l\}$. We will show that $f_i + g \in \mathcal{U}$. Let $x_0 \in \mathbb{R}$, $x_n \to x_0$ and $(f_i + g)(x_n) \to y_0$ ($y_0 \in \mathbb{R}$). Let $k \in \mathbb{N}$ be such that $x_0 \in E_k$. Then there is $n_0 \in \mathbb{N}$ such that $x_n \in E_k$ for each $n > n_0$. Indeed, since $x_n \to x_0$ and $x_0 \notin \operatorname{cl} F_{n-1}$, there is $n_1 \in \mathbb{N}$ such that $x_n \notin F_{n-1}$ for each $n > n_1$. Moreover, if $x \in \{x_n : n > n_1\} \setminus F_k$, then $x \in E_i$ for some $i \geq k+1$ and

$$(f_i + g)(x) \geq (f_i + h - \min\{f_i, 0\})(x) \geq h(x) = 1/\varrho(x, F_{i-1}) \geq 1/|x - x_0|.$$

Consequently, the set of all x must be finite. Since $(f_i + g)\restriction E_k$ is continuous, we have $(f_i + g)(x_n) \to (f_i + g)(x_0) = y_0$. ∎

It is easy to see that for the family $\mathcal{F} \overset{\text{df}}{=} \{\chi_{\{q\}} : q \in \mathbb{Q}\}$ of piecewise continuous functions does not exist a common summand with respect to the family of closed graph functions (see Proposition 2.6 and the proof of Theorem 2.7).

Observe also that using Remark 1 with $\mathcal{F} \overset{\text{df}}{=} \{\chi_\emptyset, f\}$, where $f \in \mathcal{P}$ and the inclusion $\mathcal{U} \subset \mathcal{P}$, we obtain the following characterization of the sum and the difference of closed graph functions $\mathcal{U} + \mathcal{U} = \mathcal{P} = \mathcal{U} - \mathcal{U}$. Of course this characterization is known (was given by Doboš in 1998 [2]).

3. The family $\neg\mathcal{U}$

Theorem 3.1. *Let $\mathcal{A} \subset \mathbb{R}^{\mathbb{R}}$ and $\emptyset \neq \mathcal{A} \neq \mathbb{R}^{\mathbb{R}}$. If $\mathcal{A} \subset \{f \in \mathbb{R}^{\mathbb{R}} : C(f) \neq \emptyset\}$, then* $\mathrm{a}(\neg\mathcal{A}) = 2^{\mathfrak{c}}$.

Proof. The inequality $\mathrm{a}(\neg\mathcal{A}) \leq 2^{\mathfrak{c}}$ follows from Proposition 2.1(b). The inequality $\mathrm{a}(\neg\mathcal{A}) \geq 2^{\mathfrak{c}}$ follows from the proof of [8, Theorem 6]. In this proof Jordan showed that if $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ and $\operatorname{card} \mathcal{F} < 2^{\mathfrak{c}}$, then there is a function $g \in \mathbb{R}^{\mathbb{R}}$ such that $f + g$ is not bounded on any perfect set for every $f \in \mathcal{F}$. Consequently, $C(f+g) = \emptyset$ and $f + g \in \neg\mathcal{A}$ for each $f \in \mathcal{F}$. ∎

In particular, we obtain

Corollary 3.2. $\mathrm{a}(\neg\mathcal{U}) = 2^{\mathfrak{c}}$.

Now, we prove analogous result for $\mathrm{q}(\neg\mathcal{U})$. We start with the useful lemma which can be found in [4, Lemma 2.2].

Lemma 3.3. *If $B \subset \mathbb{R}$ has cardinality $\mathfrak{c}$, $\mathcal{H} \subset \mathbb{Q}^B$, and* $\operatorname{card} \mathcal{H} < 2^{\mathfrak{c}}$, *then there is a $g \in \mathbb{Q}^B$ such that $h \cap g \neq \emptyset$ for every $h \in \mathcal{H}$.*

The next lemma and its proof are similar to [8, Lemma 20] and its proof, respectively.

Lemma 3.4. *Let $A \subset \mathbb{R}$ and* $\operatorname{card} A = \mathfrak{c}$. *If $\mathcal{F} \subset (\mathbb{R} \setminus \{0\})^A$ and* $\operatorname{card} \mathcal{F} < 2^{\mathfrak{c}}$, *there is a function $g\colon A \to \mathbb{R} \setminus \{0\}$ such that $f/g\colon A \to \mathbb{R} \setminus \{0\}$ is unbounded for each $f \in \mathcal{F}$.*

Proof. Let $\{B_n : n \in \mathbb{N}\}$ be a partition of A such that $\operatorname{card} B_n = \mathfrak{c}$ for each $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. For each $f \in \mathcal{F}$ choose $h_n^f \colon B_n \to \mathbb{Q} \setminus \{0\}$ such that

$$(1) \qquad f(x)/h_n^f(x) > n \text{ for every } x \in B_n.$$

Now, by Lemma 3.3 used with the sets B_n and the family $\{h_n^f : f \in \mathcal{F}\}$, there is a $g_n \colon B_n \to \mathbb{Q} \setminus \{0\}$ such that,

$$(2) \qquad (\forall_{f \in \mathcal{F}})\, (\exists_{x \in B_n})\, \big(h_n^f(x) = g_n(x)\big).$$

Let $g \stackrel{\text{df}}{=} \bigcup\{g_n : n \in \mathbb{N}\}$. Then, by (1) and (2), for every $n \in \mathbb{N}$ and $f \in \mathcal{F}$ there is $x \in B_n \subset A$ such that $f(x)/g_n(x) = f(x)/g(x) > n$. Hence, $f/g \colon A \to \mathbb{R} \setminus \{0\}$ is unbounded for every $f \in \mathcal{F}$. ∎

Theorem 3.5. *Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}} \setminus \{\chi_\emptyset\}$ and $\operatorname{card} \mathcal{F} < 2^{\mathfrak{c}}$. There is a function $g \colon \mathbb{R} \to \mathbb{R} \setminus \{0\}$ such that $f/g \in \neg\mathcal{U}$ for each function $f \in \mathcal{F}$.*

Proof. First recall that, if $f \in \mathcal{F}$ and $\operatorname{cl}[f = 0] = \mathbb{R}$, then $f/g \in \neg\mathcal{U}$ for each $g \colon \mathbb{R} \to \mathbb{R} \setminus \{0\}$. So we may assume that $\operatorname{cl}[f = 0] \neq \mathbb{R}$ for each $f \in \mathcal{F}$.

Choose a partition $\{S_\alpha : \alpha < \mathfrak{c}\}$ of $\mathbb{R}$ into pairwise disjoint $\mathfrak{c}$-dense sets and let $\{I_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of the open intervals in $\mathbb{R}$. Let $A_\alpha \stackrel{\text{df}}{=} S_\alpha \cap I_\alpha$. Note that $\operatorname{card} A_\alpha = \mathfrak{c}$ and $A_\alpha \cap A_\beta = \emptyset$ for $\alpha < \beta < \mathfrak{c}$. Fix $\alpha < \mathfrak{c}$. Let $\mathcal{F}_\alpha \stackrel{\text{df}}{=} \{f{\restriction}A_\alpha : f \in \mathcal{F} \ \&\ A_\alpha \subset [f \neq 0]\}$. Evidently $\operatorname{card} \mathcal{F}_\alpha < 2^{\mathfrak{c}}$. By Lemma 3.4 there is some $g_\alpha \colon A_\alpha \to \mathbb{R} \setminus \{0\}$ such that f/g_α is not bounded on A_α for every $f \in \mathcal{F}$. Let $g \colon \mathbb{R} \to \mathbb{R} \setminus \{0\}$ extend $\bigcup\{g_\alpha : \alpha < \mathfrak{c}\}$. Observe that for each $f \in \mathcal{F}$ there is a nondegenerate interval I_f such that $C((f/g){\restriction}I_f) = \emptyset$. So $f/g \in \neg\mathcal{U}$ for every $f \in \mathcal{F}$. ∎

Corollary 3.6. $^{\neg\mathcal{U}}/_{\neg\mathcal{U}} = \mathbb{R}^{\mathbb{R}} \setminus \{\chi_\emptyset\}$.

Proof. Clearly $^{\neg\mathcal{U}}/_{\neg\mathcal{U}} \subset \mathbb{R}^{\mathbb{R}} \setminus \{\chi_\emptyset\}$. Let $f \in \mathbb{R}^{\mathbb{R}} \setminus \{\chi_\emptyset\}$. By Theorem 3.5, there is a function g such that $f/g \in \neg\mathcal{U}$ and $1/g \in \neg\mathcal{U}$. Hence $f = (f/g)/(1/g) \in {}^{\neg\mathcal{U}}/_{\neg\mathcal{U}}$. ∎

Corollary 3.7. $\mathrm{q}(\neg\mathcal{U}) = 2^{\mathfrak{c}}$.

Proof. The inequality $\mathrm{q}(\neg\mathcal{U}) \geq 2^{\mathfrak{c}}$ follows from Theorem 3.5 and Corollary 3.6. Now we will show the other inequality.

Let $\mathcal{F} \stackrel{\text{df}}{=} (\mathbb{R} \setminus \{0\})^{\mathbb{R}}$. Clearly $\operatorname{card} \mathcal{F} = 2^{\mathfrak{c}}$ and $\mathcal{F} \subset {}^{\neg\mathcal{U}}/_{\neg\mathcal{U}}$ (see Corollary 3.6). If $g \colon \mathbb{R} \to \mathbb{R} \setminus \{0\}$, then evidently $g \in \mathcal{F}$ and $g/g = \chi_{\mathbb{R}} \notin \neg\mathcal{U}$. Consequently $\mathrm{q}(\neg\mathcal{U}) \leq 2^{\mathfrak{c}}$. ∎

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Received September 7, 2010; revised version March 27, 2011.