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GENERALIZED JORDAN LEFT DERIVATIONS IN RINGS WITH INVOLUTION

Abstract. In the present paper we study generalized left derivations on Lie ideals of rings with involution. Some of our results extend other ones proven previously just for the action of generalized left derivations on the whole ring. Furthermore, we prove that every generalized Jordan left derivation on a 2-torsion free $*$-prime ring with involution is a generalized left derivation.

1. Introduction

Let R denote an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. Recall that a ring R is prime if $aRb = 0$ implies that either $a = 0$ or $b = 0$. A ring with involution $(R, *)$ is called $*$-prime if $aRb = aRb^* = 0$ yields $a = 0$ or $b = 0$. Note that every prime ring having an involution $*$ is $*$-prime but the converse is in general not true. For example, if R^o denotes the opposite ring of a prime ring R, then $R \times R^o$ equipped with the exchange involution $*_{ex}$, defined by $*_{ex}(x, y) = (y, x)$, is $*_{ex}$-prime but not prime. This example shows that every prime ring can be injected in a $*$-prime ring and from this point of view $*$-prime rings constitute a more general class of prime rings.

An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$. A Lie ideal U which satisfies $U^* = U$ is called a $*$-Lie ideal. If U is a Lie (resp. $*$-Lie) ideal of R, then U is called a square closed Lie (resp. $*$-Lie) ideal if $u^2 \in U$ for all $u \in U$. The fact that $uv + vu = (u + v)^2 - u^2 - v^2 \in U$ together with $[u, v] \in U$, implies that $2uv \in U$ for all $u, v \in U$.

An additive mapping $d : R \to R$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all pairs $x, y \in R$. An additive mapping $F : R \to R$ is said to be a left

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derivation (resp. Jordan left derivation) if $F(xy) = xF(y) + yF(x)$ (resp. $F(x^2) = 2xF(x)$) holds for all pairs $x, y \in R$. Clearly, every left derivation on a ring R is a Jordan left derivation but the converse need not be true in general. In [1] it is proved that a Jordan left derivation on a 2-torsion free prime ring is a left derivation. This result has been extended in [4], to rings with involution, by showing that a Jordan left derivation on a 2-torsion free $*$-prime ring is a left derivation. Further in [2], it is proved that if R is a 2-torsion free prime ring and $\delta : R \to R$ is an additive mapping such that $\delta(u^2) = 2u\delta(u)$ for all u in a square closed Lie ideal U of R, then either $U \subseteq Z(R)$ or $\delta(U) = 0$. The author together with S. Salhi [4] extended this result to $*$-prime rings. An additive mapping $F : R \to R$ is called a generalized derivation (resp. generalized Jordan derivation) if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ (resp. $F(x^2) = F(x)x + xd(x)$) holds for all pairs $x, y \in R$. Following [3], an additive mapping $g : R \to R$ is called a generalized left derivation (resp. generalized Jordan left derivation), if there exists a Jordan left derivation $d : R \to R$ such that $g(xy) = xg(y) + yd(x)$ (resp. $g(x^2) = xg(x) + xd(x)$) holds for all $x, y \in R$. It is obvious to see that every generalized left derivation on a ring R is a generalized Jordan left derivation. But the converse need not be true in general. In this paper we extend some known results for generalized left derivations on prime rings to Lie ideals of rings with involution and we prove that every generalized Jordan left derivation on a 2-torsion free $*$-prime ring is a generalized left derivation.

Throughout, $(R, *)$ will be a 2-*torsion free* ring with involution and $Sa_*(R) := \{r \in R / r^* = \pm r\}$ being the set of symmetric and skew symmetric elements.

2. Generalized left derivations on Lie ideals

Throughout the paper, we will make extensive use of the basic commutator identities $[x, yz] = y[x, z] + [x, y]z$, $[xy, z] = x[y, z] + [x, z]y$ and we shall require the following lemmas.

Lemma 1. ([4], Lemma 4) *If $U \not\subseteq Z(R)$ is a $*$-Lie ideal of a 2-torsion free $*$-prime ring R and $a, b \in R$ such that $aUb = a^*Ub = 0$, then $a = 0$ or $b = 0$.*

Lemma 2. ([5], Lemma 2.3) *Let $0 \neq U$ be a $*$-Lie ideal of a 2-torsion free $*$-prime ring R. If $[U, U] = 0$, then $U \subseteq Z(R)$.*

Let U be a square closed $*$-Lie ideal of R and $d : R \to R$ be an additive mapping such that $d(u^2) = 2ud(u)$ for all $u \in U$.

Theorem 1. *Let $g : R \to R$ be an additive mapping such that $g(uv) = ug(v) + vd(u)$ for all $u, v \in U$. If R is $*$-prime, then either $d(U) = 0$ or $U \subseteq Z(R)$.*

Proof. Suppose that $U \not\subseteq Z(R)$. From $g(uv) = ug(v) + vd(u)$ it follows that

$$g(u^2v) = u^2g(v) + 2vud(u) \text{ for all } u, v \in U. \tag{1}$$

On the other hand

$$2g(u^2v) = g(u(2uv)) = 2ug(uv) + 2uvd(u) = 2u^2g(v) + 4uvd(u).$$

Since R is 2-torsion free, we find that

$$g(u^2v) = u^2g(v) + 2uvd(u) \text{ for all } u, v \in U. \tag{2}$$

Comparing (1) and (2), we obtain $2[u, v]d(u) = 0$ for all $u, v \in U$.

Once again using the fact that R is 2-torsion free, we conclude that

$$[u, v]d(u) = 0 \text{ for all } u, v \in U. \tag{3}$$

Substituting $2vw$ for v in (3) and using (3), we get $[u, v]wd(u) = 0$ for all $w, u, v \in U$ and therefore

$$[u, v]Ud(u) = 0 \text{ for all } u, v \in U. \tag{4}$$

From the above relation, it follows, according to Lemma 1, that

$$d(y) = 0 \text{ or } [y, U] = 0 \text{ for all } y \in U \cap Sa_*(R). \tag{5}$$

Let $u \in U$; since $u - u^* \in U \cap Sa_*(R)$, then $d(u - u^*) = 0$ or $[u - u^*, U] = 0$.

If $[u - u^*, U] = 0$, then $[u, v] = [u^*, v]$ for all $v \in U$ and (4) assures that $[u, v]^*Ud(u) = 0$, whence it follows, applying again Lemma 1, that $d(u) = 0$ or $[u, U] = 0$.

If $d(u - u^*) = 0$; replacing u by u^* in (4) we find that $[v, u]^*Ud(u) = 0$, which leads to $d(u) = 0$ or $[u, U] = 0$. Consequently, $d(u) = 0$ or $[u, U] = 0$ for all $u \in U$. Set $U_1 = \{u \in U/d(u) = 0\}$ and $U_2 = \{u \in U/[u, U] = 0\}$; clearly U_1 and U_2 are additive subgroups of U such that $U_1 \cup U_2 = U$ and thus $U = U_1$ or $U = U_2$. As $U \not\subseteq Z(R)$, then Lemma 2 forces $U = U_1$ and therefore $d(U) = 0$. ∎

The following example proves the necessity of the $*$-primeness hypothesis in Theorem 1.

Example 1. Let $R = A \oplus \mathbb{Q}[X]$ where A is a noncommutative prime ring with involution τ and $\mathbb{Q}[X]$ with the identity map as its involution. It is obvious that $* : R \to R$ defined by $(a, P(X))^* = (\tau(a), P(X))$ is an involution. Moreover, R is a semiprime ring which is not $*$-prime. On the other hand, if we define: $G(a, P(X)) = (a, P'(X))$ and $D(a, P(X)) = (0, P'(X))$ then G is a generalized left derivation using the Jordan left derivation D but $D(R) \neq 0$ and R is not commutative.

Corollary 1. *Let R be a 2-torsion free $*$-prime ring. If R admits a generalized left derivation with associated Jordan left derivation d, then either $d = 0$ or R is commutative.*

If J is a Jordan left derivation on R, then J is a left derivation by [4]. Hence, J is a generalized left derivation with associated Jordan left derivation J. We therefore have the commutativity criterion:

Corollary 2. *Let R be a 2-torsion free $*$-prime ring. If R admits a nonzero Jordan left derivation, then R is commutative.*

Theorem 2. *Let R be a 2-torsion free prime ring and U a square closed Lie ideal of R. If $g, d : R \to R$ are additive mappings such that $g(uv) = ug(v) + vd(u)$ and $d(u^2) = 2ud(u)$ for all $u, v \in U$, then either $d(U) = 0$ or $U \subseteq Z(R)$.*

Proof. Let $\mathcal{R} = R \times R^0$ and $W = U \times U$. Clearly W is a square closed $*_{ex}$-Lie ideal of $\mathcal{R}$. Moreover, if we define $D : \mathcal{R} \to \mathcal{R}$ by

$$D(x, y) = (d(x), 0) \text{ for all } x, y \in R,$$

then D is an additive mapping of $\mathcal{R}$ such that $D(w^2) = 2wD(w)$ for all $w \in W$. On the other hand, if we consider $G : \mathcal{R} \to \mathcal{R}$, where $G(x, y) = (g(x), y)$ for all $x, y \in R$, then G is an additive mapping of $\mathcal{R}$ satisfying $G(uv) = uG(v) + vD(u)$ for all $u, v \in W$. Since $\mathcal{R}$ is $*_{ex}$-prime, applying Theorem 1 it follows that $D(W) = 0$ or $W \subseteq Z(\mathcal{R})$. Accordingly, $d(U) = 0$ or $U \subseteq Z(R)$. ∎

Corollary 3. ([3], Proposition 3.1) *Let R be a 2-torsion free prime ring. If R admits a generalized left derivation with associated Jordan left derivation d, then either $d = 0$ or R is commutative.*

3. Generalized Jordan left derivations

To prove the main theorem of this section, we need the following lemma.

Lemma 3. ([3], Lemma 2.2) *Let R be a 2-torsion free ring and G be a generalized Jordan left derivation with associated Jordan left derivation δ. Then*

(i) $G(xy + yx) = xG(y) + yG(x) + x\delta(y) + y\delta(x)$ *for all* $x, y \in R$,
(ii) $G(xyx) = xyG(x) + 2xy\delta(x) + x^2\delta(y) - yx\delta(x)$ *for all* $x, y \in R$,
(iii) $G(xyz + zyx) = xyG(z) + zyG(x) + 2xy\delta(z) + 2zy\delta(x) + xz\delta(y) + zx\delta(y) - yx\delta(z) - yz\delta(x)$ *for all* $x, y, z \in R$.

We shall also use the fact that if R is $*$-prime then R is semiprime. Indeed, if $aRa = 0$ with $a \in R$, then $aRaRa^* = 0$ and $aR(aRa^*)^* = 0$. Thus, the $*$-primeness of R yields $a = 0$ or $aRa^* = 0$. If $aRa^* = 0$; since $aRa = 0$, once again using the $*$-primeness of R we get $a = 0$. Accordingly R is semiprime.

Theorem 3. *Let $(R,*)$ be a 2-torsion free ring with involution. If R is $*$-prime, then every generalized Jordan left derivation of R is a generalized left derivation of R.*

Proof. Let G be a generalized Jordan left derivation with associated Jordan left derivation d. In view of Corollary 2, we have $d = 0$ or R is commutative.

If $d = 0$, then $G(x^2) = xG(x)$ and thus G is a Jordan left multiplier. Since a $*$-prime ring is semiprime, applying Proposition 1.4 of [8], we deduce that G is a left multiplier and a fortiori a generalized left derivation.

Now assume that R is commutative; since d is a Jordan left derivation and R is $*$-prime, then d is a left derivation by [4]. Applying (i) of Lemma 3, we get

$$G(zxy + xyz) = zG(xy) + xyG(z) + zxd(y) + zyd(x) + xyd(z). \tag{6}$$

Since R is commutative, (6) together with (iii) of Lemma 3 yield $z(G(xy) - xG(y) - yd(x)) = 0$ for all $y, z \in R$ and therefore

$$R(G(xy) - xG(y) - yd(x)) = 0 \text{ for all } x, y \in R. \tag{7}$$

Set $J(x,y) = G(xy) - xG(y) - yd(x)$, from the above relation we get

$$J(x,y)RJ(x,y) = 0 \quad \text{and} \quad (J(x,y))^*RJ(x,y) = 0,$$

whence it follows, by the $*$-primeness of R, that $J(x,y) = 0$. Accordingly, $G(xy) = xG(y) + yd(x)$ for all $x, y \in R$ and therefore G is a left derivation. ■

The following example demonstrates R to be $*$-prime is essential in the hypothesis of Theorem 3.

Example 2. Let $A = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \text{ such that } x, y \in S \right\}$ where S is a non-commutative ring in which $s^2 = 0$ for all $s \in S$. Define g, d from A into itself by

$$g\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad d\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}.$$

It is easy to check that g is a generalized Jordan left derivation with associated Jordan left derivation d. If we set $\mathcal{R} = A \times A^o$ and $r = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}$ where $0 \neq s \in S$, then $u = (r, 0)$, $v = (0, r)$ are nonzero elements in $\mathcal{R}$ such that $u\mathcal{R}v = 0 = u\mathcal{R}v^{*_{\text{ex}}}$ and therefore $\mathcal{R}$ is a non $*_{\text{ex}}$-prime ring. Furthermore, if we define

$$G: \mathcal{R} \to \mathcal{R}, \quad (x,y) \mapsto (g(x), y) \qquad \text{and} \qquad D: \mathcal{R} \to \mathcal{R}, \quad (x,y) \mapsto (d(x), 0)$$

then G is a generalized Jordan left derivation with associated Jordan left derivation D. Moreover, G is not a generalized left derivation on $\mathcal{R}$. Indeed, let $a = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} \in A$, where $s, t \in S$ with $st \neq 0$. If we set $x = (a, 0)$ and $y = (b, 0)$, then one can easily verify that $G(xy) \neq xG(y) + yG(x)$. Hence, in Theorem 3 the hypothesis of $*$-primeness is crucial.

Corollary 4. ([3], Theorem 3.2) *Every generalized Jordan left derivation on a 2-torsion free prime ring is a generalized left derivation.*

Proof. Let g be a generalized Jordan left derivation with associated Jordan left derivation d and let $\mathcal{R} = R \times R^0$. Let us define $G, D : \mathcal{R} \to \mathcal{R}$ by

$$G(x, y) = (g(x), y) \ \text{ and } \ D(x, y) = (d(x), 0) \ \text{ for all } \ x, y \in R.$$

It is clear that G is a generalized Jordan left derivation with associated Jordan left derivation D. Since $\mathcal{R}$ is $*_{\text{ex}}$-prime, applying Theorem 3 it follows that G is a left derivation of $\mathcal{R}$. Let $x, y \in R$, from $G((x, 0)(y, 0)) = (x, 0)G((y, 0)) + (y, 0)G((x, 0))$ it follows that $g(xy) = xg(y) + yg(x)$ for all $x, y \in R$ and therefore g is a left derivation of R. ∎

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