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Roman Wituła

RAMANUJAN TYPE TRIGONOMETRIC FORMULAE

Abstract. In the paper, new Ramanujan type trigonometric formulae for arguments $2\pi/7$ and $2\pi/9$ are presented.

1. Introduction

This paper presents some new Ramanujan type trigonometric identities in the spirit of his original identities (see [1]):

$$\left(\cos\frac{2\pi}{7}\right)^{1/3}+\left(\cos\frac{4\pi}{7}\right)^{1/3}+\left(\cos\frac{8\pi}{7}\right)^{1/3}=\left(\frac{5-3\sqrt[3]{7}}{2}\right)^{1/3}, \tag{1.1}$$

$$\left(\cos\frac{2\pi}{9}\right)^{1/3}+\left(\cos\frac{4\pi}{9}\right)^{1/3}+\left(\cos\frac{8\pi}{9}\right)^{1/3}=\left(\frac{3\sqrt[3]{9}-6}{2}\right)^{1/3}. \tag{1.2}$$

It is worth to mention that Wituła and Słota already discussed such kind of identities in papers [7] and [9]. The main reason of taking an interest in this matter was an intention of applying the, so called, quasi-Fibonacci numbers (see [6, 8, 10]) for generating the Ramanujan type identities. It seems that this research succeeded. For example, in paper [9] the following formulae were received:

$$\begin{aligned}\sqrt[3]{\frac{\cos\alpha}{\cos 2\alpha}}\,\big(2\cos\alpha\big)^k+\sqrt[3]{\frac{\cos 2\alpha}{\cos 4\alpha}}\,\big(2\cos 2\alpha\big)^k+\sqrt[3]{\frac{\cos 4\alpha}{\cos\alpha}}\,\big(2\cos 4\alpha\big)^k\\ =\sqrt[3]{\frac{\cos\alpha}{\cos 2\alpha}}\,\big(2\cos 2\alpha\big)^{k+1}+\sqrt[3]{\frac{\cos 2\alpha}{\cos 4\alpha}}\,\big(2\cos 4\alpha\big)^{k+1}\\ +\sqrt[3]{\frac{\cos 4\alpha}{\cos\alpha}}\,\big(2\cos\alpha\big)^{k+1}=\sqrt[3]{7}\,\psi_k,\end{aligned} \tag{1.3}$$

where $\alpha=\frac{2\pi}{7}$, $\psi_0=-1$, $\psi_1=0$, $\psi_2=-3$ and

$$\psi_{k+3}+\psi_{k+2}-2\,\psi_{k+1}-\psi_k=0, \qquad k\in\mathbb{Z};$$

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and

$$(1.4)\qquad \sqrt[3]{\frac{\cos\alpha}{\cos 4\alpha}}\,\big(2\,\cos\alpha\big)^{k}+\sqrt[3]{\frac{\cos 2\alpha}{\cos\alpha}}\,\big(2\,\cos 2\alpha\big)^{k}+\sqrt[3]{\frac{\cos 4\alpha}{\cos 2\alpha}}\,\big(2\,\cos 4\alpha\big)^{k}$$
$$=\sqrt[3]{\frac{\cos 2\alpha}{\cos\alpha}}\,\big(2\,\cos\alpha\big)^{k+1}+\sqrt[3]{\frac{\cos 4\alpha}{\cos 2\alpha}}\,\big(2\,\cos 2\alpha\big)^{k+1}$$
$$+\sqrt[3]{\frac{\cos\alpha}{\cos 4\alpha}}\,\big(2\,\cos 4\alpha\big)^{k+1}=\sqrt[3]{49}\,\varphi_k,$$

where $\varphi_0=0$, $\varphi_1=-1$, $\varphi_2=1$ and

$$\varphi_{k+3}+\varphi_{k+2}-2\,\varphi_{k+1}-\varphi_k=0,\qquad k\in\mathbb{Z}.$$

Equivalents of the above formulae for the angle $\beta=\frac{2\pi}{9}$ are presented in the current work (see formulae (2.1) and (2.2)).

Moreover, V. Shevelev in the context of works [4], [7] and [9] distinguished the Ramanujan cubic polynomials (shortly RCP), i.e. real cubic polynomials

$$(1.5)\qquad x^3+px^2+qx+r,\qquad r\neq 0,$$

having real roots ξ_1,ξ_2,ξ_3 and satisfying the condition

$$(1.6)\qquad p\sqrt[3]{r}+3\sqrt[3]{r^2}+q=0.$$

Then we can note that two crucial identities hold: (Ramanujan type, see [4, 9])

$$(1.7)\qquad \sqrt[3]{\xi_1}+\sqrt[3]{\xi_2}+\sqrt[3]{\xi_3}=\sqrt[3]{-p-6\sqrt[3]{r}+3\sqrt[3]{9r-pq}}$$

and (Shevelev type, see [3, 4])

$$(1.8)\qquad \sqrt[3]{\frac{\xi_1}{\xi_2}}+\sqrt[3]{\frac{\xi_2}{\xi_1}}+\sqrt[3]{\frac{\xi_1}{\xi_3}}+\sqrt[3]{\frac{\xi_3}{\xi_1}}+\sqrt[3]{\frac{\xi_2}{\xi_3}}+\sqrt[3]{\frac{\xi_3}{\xi_2}}=\sqrt[3]{\frac{pq}{r}-9}.$$

Wituła, continuing Shevelev's research (see [11, 13]), distinguished the next class of Ramanujan cubic polynomials of the second kind (shortly RCP2), defined as the real cubic polynomials of the form (1.5), having real roots and satisfying the condition

$$(1.9)\qquad p^3r+27r^2+q^3=0$$

(every term in this sum is cube of the corresponding term in the sum (1.6)). For example, polynomial $f(z)=z^3+3z^2-3\sqrt[3]{2}z+1$ is the RCP2 and, simultaneously, is not RCP. Roots ξ_1,ξ_2,ξ_3 of $f(z)$ satisfy the following conditions (see [13]):

$$\sqrt[3]{\xi_1}+\sqrt[3]{\xi_2}+\sqrt[3]{\xi_3}=0$$

and

$$\sqrt[3]{\frac{\xi_1}{\xi_2}}+\sqrt[3]{\frac{\xi_2}{\xi_1}}+\sqrt[3]{\frac{\xi_1}{\xi_3}}+\sqrt[3]{\frac{\xi_3}{\xi_1}}+\sqrt[3]{\frac{\xi_2}{\xi_3}}+\sqrt[3]{\frac{\xi_3}{\xi_2}}=-3.$$

In the figure (1) Venn diagram for the sets of RCP's and RCP2's is given. Let us notice, that RCP's and RCP2's share many similar properties.

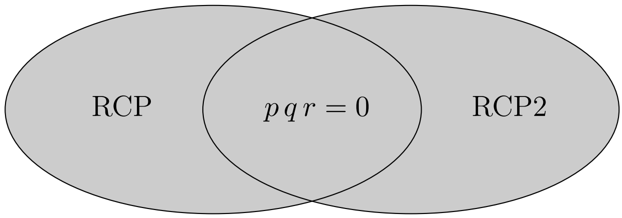

Fig. 1. Venn diagram for the sets of RCP's and RCP2's

Now let us resume the contents of the current paper. In Section 2, the equivalents of formulae (2) and (3) from paper [9] for the angle $2\,\pi/9$ are presented, whereas, the initial values for those recurrence identities are generated in Section 5. In Section 3 we give few more trigonometric identities for the angle $2\,\pi/7$, essentially completing the set of identities from work [9]. Moreover, in Section 4, the generalizations of some Berndt, Zhang and Liu formulae from the paper [2] are presented.

We note that all the identities are related, just as in [9], where formula (10) from [9] was applied to the sum of the cubic roots of the roots of some special polynomials of the third degree, discussed by Wituła and Słota in [7]. Some detailed calculations have been omitted in the paper.

2. The argument $\frac{2\,\pi}{9}$

We remind in this moment that notation β will be consistently used for $\frac{2\,\pi}{9}$.

First let us discuss identities that are equivalent to identities (2.1) and (2.2) from [9]:

$$\sqrt[3]{\frac{\cos(\beta)}{\cos(2\beta)}}\big(2\,\cos(\beta)\big)^n+\sqrt[3]{\frac{\cos(2\beta)}{\cos(4\beta)}}\big(2\,\cos(2\beta)\big)^n + \sqrt[3]{\frac{\cos(4\beta)}{\cos(\beta)}}\big(2\,\cos(4\beta)\big)^n \tag{2.1}$$

$$= -\left(\sqrt[3]{\frac{\cos(\beta)}{\cos(2\beta)}}\,\big(2\cos(2\beta)\big)^{n+1} + \sqrt[3]{\frac{\cos(2\beta)}{\cos(4\beta)}}\,\big(2\cos(4\beta)\big)^{n+1} + \sqrt[3]{\frac{\cos(4\beta)}{\cos(\beta)}}\,\big(2\cos(\beta)\big)^{n+1}\right)$$

$$= -\left(\sqrt[3]{2\cos(\beta)\,\big(2\cos(2\beta)\big)^{3n+2}} + \sqrt[3]{2\cos(2\beta)\,\big(2\cos(4\beta)\big)^{3n+2}} + \sqrt[3]{2\cos(4\beta)\,\big(2\cos(\beta)\big)^{3n+2}}\right) = \sqrt[3]{3}\,\Psi_n,$$

where $\Psi_0 = 0$, $\Psi_1 = 3$, $\Psi_2 = 0$ and

$$\Psi_{n+3} - 3\,\Psi_{n+1} + \Psi_n = 0, \qquad n \in \mathbb{Z};$$

(2.2) $$\sqrt[3]{\frac{\cos(\beta)}{\cos(4\beta)}}\,\big(2\cos(\beta)\big)^{n} + \sqrt[3]{\frac{\cos(2\beta)}{\cos(\beta)}}\,\big(2\cos(2\beta)\big)^{n} + \sqrt[3]{\frac{\cos(4\beta)}{\cos(2\beta)}}\,\big(2\cos(4\beta)\big)^{n}$$

$$= -\left(\sqrt[3]{\frac{\cos(2\beta)}{\cos(\beta)}}\,\big(2\cos(\beta)\big)^{n+1} + \sqrt[3]{\frac{\cos(4\beta)}{\cos(2\beta)}}\,\big(2\cos(2\beta)\big)^{n+1} + \sqrt[3]{\frac{\cos(\beta)}{\cos(4\beta)}}\,\big(2\cos(4\beta)\big)^{n+1}\right)$$

$$= -\left(\sqrt[3]{2\cos(2\beta)\,\big(2\cos(\beta)\big)^{3n+2}} + \sqrt[3]{2\cos(4\beta)\,\big(2\cos(2\beta)\big)^{3n+2}} + \sqrt[3]{2\cos(\beta)\,\big(2\cos(4\beta)\big)^{3n+2}}\right) = \sqrt[3]{9}\,\Phi_n,$$

where $\Phi_0 = -1$, $\Phi_1 = 1$, $\Phi_2 = -4$ and

$$\Phi_{n+3} - 3\,\Phi_{n+1} + \Phi_n = 0, \qquad n \in \mathbb{Z}.$$

Proof. We note that

(2.3) $$\mathbb{X}^3 - 3\,\mathbb{X} + 1 = \prod_{k=0}^{2}\Big(\mathbb{X} - 2\cos\big(2^k\beta\big)\Big)$$

(it is easy to calculate, see also [14]). Since it is generating function for (2.1) and (2.2) so the rest of the proof reduces to checking whether (2.1) and (2.2) hold true for the initial values $n = 0, 1, 2$. It will be presented in Section 5. ∎

We note that (1.3) and (1.4), as well as (2.1) and (2.2) from above, all equalities for $n = 0$, include the Shevelev's formulae [3]:

$$\sum_{k=0}^{2}\left(\sqrt[3]{\frac{\cos(2^k\alpha)}{\cos(2^{k+1}\alpha)}}+\sqrt[3]{\frac{\cos(2^k\alpha)}{\cos(2^{k+2}\alpha)}}\right)=-\sqrt[3]{7}$$

and

$$\sum_{k=0}^{2}\left(\sqrt[3]{\frac{\cos(2^k\beta)}{\cos(2^{k+1}\beta)}}+\sqrt[3]{\frac{\cos(2^k\beta)}{\cos(2^{k+2}\beta)}}\right)=-\sqrt[3]{9},$$

respectively. Moreover, using Remark 1 from [9] we deduce the following relation

$$S_n=\Big(\frac{\cos(\beta)}{\cos(2\beta)}\Big)^{n/3}+\Big(\frac{\cos(2\beta)}{\cos(4\beta)}\Big)^{n/3}+\Big(\frac{\cos(4\beta)}{\cos(\beta)}\Big)^{n/3}, \tag{2.4}$$

where $S_0=3$, $S_1=0$, $S_2=2\sqrt[3]{9}$. We have also

$$S_{n+3}=\sqrt[3]{9}\,S_{n+1}+S_n. \tag{2.5}$$

On the other hand, from (2.5) we obtain

$$S_n=x_n+\sqrt[3]{9}\,y_n+\sqrt[3]{81}\,z_n, \tag{2.6}$$

where

$$\begin{aligned} &x_0=3,\quad y_0=z_0=0,\\ &x_1=y_1=z_1=0,\\ &x_2=z_2=0,\quad y_2=2, \end{aligned}$$

and, we have

$$\begin{aligned} x_{n+3}&=x_n+9\,z_{n+1},\\ y_{n+3}&=y_n+x_{n+1},\\ z_{n+3}&=z_n+y_{n+1}. \end{aligned}$$

Moreover, one can deduce the following relation:

$$S_n^*=\Big(\sqrt[3]{\frac{\cos(\beta)}{\cos(2\beta)}}\,\big(2\cos(\beta)\big)^2\Big)^n+\Big(\sqrt[3]{\frac{\cos(2\beta)}{\cos(4\beta)}}\,\big(2\cos(2\beta)\big)^2\Big)^n+ \\ +\Big(\sqrt[3]{\frac{\cos(4\beta)}{\cos(\beta)}}\,\big(2\cos(4\beta)\big)^2\Big)^n, \tag{2.7}$$

where $S_0^*=3$, $S_1^*=0$, $S_2^*=14\sqrt[3]{9}$. Furthermore

$$S_{n+3}^*=7\sqrt[3]{9}\,S_{n+1}^*+S_n^*. \tag{2.8}$$

Likewise, the following relation can be generated

$$S_n^*=x_n^*+\sqrt[3]{9}\,y_n^*+\sqrt[3]{81}\,z_n^*, \tag{2.9}$$

where

$$x_0^* = 3, \quad y_0^* = z_0^* = 0,$$
$$x_1^* = y_1^* = z_1^* = 0,$$
$$x_2^* = z_2^* = 0, \quad y_2^* = 14,$$

and, by (2.8), we have

$$x_{n+3}^* = x_n^* + 63\, z_{n+1}^*,$$
$$y_{n+3}^* = y_n^* + 7\, x_{n+1}^*,$$
$$z_{n+3}^* = z_n^* + 7\, y_{n+1}^*.$$

Let us present one more identity derived by using Lemma 5.4 (see also the equation (10) from [9]):

$$\frac{\sqrt[3]{2}}{\sqrt{3}}\left(\sqrt[3]{\sin(2\,\beta)} - \sqrt[3]{\sin(\beta)} - \sqrt[3]{\sin(4\,\beta)}\right) = \sqrt[3]{\frac{2}{\sqrt[3]{3}} + \sqrt[3]{1-\sqrt[3]{9}} + \sqrt[3]{2-\sqrt[3]{9}}},$$

since

$$\prod_{k=0}^{2}\left(\mathbb{X} - (-1)^k\, 2\,\sin\left(2^k \beta\right)\right) = \mathbb{X}^3 - 3\,\mathbb{X} + \sqrt{3}.$$

3. The argument $\dfrac{2\,\pi}{7}$

3.1. The first identity. The notation α will be consistently used for $\frac{2\pi}{7}$.

The following identity holds

$$\text{(3.1)}\quad \sin^n(\alpha)\sqrt[3]{\frac{\sin(4\alpha)}{\sin(\alpha)}} + \sin^n(2\alpha)\sqrt[3]{\frac{\sin(\alpha)}{\sin(2\alpha)}} + \sin^n(4\alpha)\sqrt[3]{\frac{\sin(2\,\alpha)}{\sin(4\alpha)}} = a_n\sqrt[3]{4-3\sqrt[3]{7}} + b_n\sqrt[3]{11-3\sqrt[3]{49}},$$

where $a_0 = 1$, $b_0 = 0$, $a_1 = -\sqrt[6]{7}/2$, $b_1 = 0$, $a_2 = 0$, $b_2 = \sqrt[3]{7}/4$ (see [7]), and

$$\text{(3.2)}\qquad x_{n+1} = \sqrt{7}\left(x_n - x_{n-2}\right),$$

for every $x \in \{a, b\}$, $n = 2, 3, 4, \ldots$. We note that

$$\text{(3.3)}\qquad b_n = \frac{\sqrt[3]{7}}{4}\left(\sqrt[4]{7}\right)^{3+(-1)^n}\gamma_n,$$

where $\gamma_0 = \gamma_1 = 0$, $\gamma_2 = 1/7$,

$$\text{(3.4)}\qquad \gamma_{n+1} = \left(\sqrt{7}\right)^{1+(-1)^n}\left(\gamma_n - \gamma_{n-2}\right),$$

and γ_n, $n = 6, 7, 8, \ldots$, are all integers (see Table 1). Moreover, let us remind that (see [7, 14]):

$$\mathbb{X}^3 - \sqrt{7}\,\mathbb{X}^2 + \sqrt{7} = \prod_{k=0}^{2} \big(\mathbb{X} - 2\,\sin(2^k\,\alpha)\big),$$

which implies the relation (3.2).

3.2. The second identity. We have the following identity

$$\begin{aligned}\csc^n(2\alpha)\,\sqrt[3]{2\cos(\alpha)} + \csc^n(4\alpha)\,\sqrt[3]{2\cos(2\alpha)} + \csc^n(\alpha)\,\sqrt[3]{2\cos(4\alpha)} \\ = c_n\,\sqrt[3]{5 - 3\,\sqrt[3]{7}} + d_n\,\sqrt[3]{2 + 3\,\sqrt[3]{49}},\end{aligned} \tag{3.5}$$

where $c_0 = 1$, $d_0 = 0$, $c_1 = -2/\sqrt[6]{7}$, $d_1 = 0$, $c_2 = 0$, $d_2 = -4/\sqrt[3]{7}$, and

$$x_{n+2} = x_n - \frac{\sqrt{7}}{7}\,x_{n-1}, \tag{3.6}$$

for every $x \in \{c, d\}$, $n = 1, 2, 3, \ldots$.

On the other hand, by (4.32) from [7] we have

$$\begin{aligned}\Big(-\frac{\sqrt{7}}{2}\Big)^n \Big(\csc^n(\alpha)\,\sqrt[3]{2\cos(4\alpha)} + \csc^n(2\alpha)\,\sqrt[3]{2\cos(2\alpha)} \\ +\csc^n(4\alpha)\,\sqrt[3]{2\cos(\alpha)}\,\Big) \\ = \sqrt[3]{w^*_{3n} + 6\,7^n - \frac{3}{\sqrt[3]{2}}\Big(\sqrt[3]{\mathcal{S} + \sqrt{\mathcal{T}}} + \sqrt[3]{\mathcal{S} - \sqrt{\mathcal{T}}}\,\Big)},\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}\mathcal{S} &= (-1)^{n-1}\,y_{3n-1}\,\big(7^{3n/2}\,w^*_{3n} + 6\,7^{5n/2}\big) - 6\,7^{2n}\,w^*_{3n} - 9\,7^{3n},\\ \mathcal{T} &= 7^{3n}\,(w^*_{3n})^2\,y^2_{3n-1} - 4\,(-\sqrt{7})^{9n}\,y^3_{3n-1} - 4\,7^{3n}\,(w^*_{3n})^3\\ &\quad + 18\,(-7\,\sqrt{7})^{3n}\,w^*_{3n}\,y_{3n-1} - 27\,7^{6n},\end{aligned}$$

where

$$w^*_{n+3} - 3\,w^*_{n+1} - w^*_n = z_{2n+1} + z_{2n-1} - z^2_n - z^2_{n-1}, \tag{3.8}$$

$$z_{n+6} - 7\,z_{n+4} + 14\,z_{n+2} - 7\,z_n = 0, \tag{3.9}$$

$$y_n = z_{n+2} - 3\,z_n, \tag{3.10}$$

for $n \in \mathbb{N}$ and $z_0 = y_0 = \sqrt{7}$, $z_1 = 7$ and $w^*_0 = -1$ (see Tables 3 and 4 in [7]).

We note that

$$z_n = \big(2\,\sin(\tfrac{2\pi}{7})\big)^{n+1} + \big(2\,\sin(\tfrac{4\pi}{7})\big)^{n+1} + \big(2\,\sin(\tfrac{8\pi}{7})\big)^{n+1}, \tag{3.11}$$

$$y_n = 2\,\sin(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n + 2\,\sin(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n + 2\,\sin(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n, \tag{3.12}$$

$$w_n^* = 2\,\cos(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n + 2\,\cos(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n + 2\,\cos(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n, \tag{3.13}$$

(see A079309 [5] for the sequence $\{z_{2n}/\sqrt{7}\}$).

3.3. The next identities. Moreover, by using formula (4.10) from [7] we get

$$\sum_{k=0}^{2} \sqrt[3]{2\cos(2^k\alpha)}\,\big(2\,\sin(2^k\alpha)\big)^n \tag{3.14}$$
$$= \sqrt[6]{7^n}\,\sqrt[3]{A_n\,\sqrt[3]{49} + B_n\,\sqrt[3]{7} + C_n}$$
$$= a_n\,\sqrt[3]{5 - 3\,\sqrt[3]{7}} + b_n\,\sqrt[3]{5 + 3\,\sqrt[3]{7} - 3\,\sqrt[3]{49}}$$
$$= a_n^*\,\sqrt[3]{5 + 3\,\sqrt[3]{7} - 3\,\sqrt[3]{49}} + b_n^*\,\sqrt[3]{\big(2 - \sqrt[3]{7}\big)^2} + c_n^*\,\sqrt[3]{\big(4 - 3\,\sqrt[3]{7}\big)^2},$$

where

$$\begin{array}{lll} a_0 = 1, & a_1 = -\sqrt[6]{7}, & a_2 = 0, \\ b_0 = 0, & b_1 = 0, & b_2 = \sqrt[3]{7}, \\ a_1^* = \sqrt[3]{7}, & a_2^* = 0, & a_3^* = 0, \\ b_1^* = 0, & b_2^* = -\sqrt[3]{2}\,\sqrt{7}, & b_3^* = 0, \\ c_1^* = 0, & c_2^* = 0, & c_3^* = -\sqrt[3]{49}, \end{array}$$

and

$$x_{n+3} - \sqrt{7}\,x_{n+2} + \sqrt{7}\,x_n = 0, \tag{3.15}$$

for every $n \in \mathbb{Z}$ and $x \in \{a, b, a^*, b^*, c^*\}$;

$$A_0 = A_1 = 0,\ \ A_2 = -3,\ \ A_{n+3} - A_{n+2} - 2\,A_{n+1} + A_n = 0,\ \ n \in \mathbb{Z}, \tag{3.16}$$

$$B_0 = -3,\ \ B_1 = B_2 = 3,\ \ B_{n+3} - 2\,B_{n+2} - B_{n+1} + B_n = 0,\ \ n \in \mathbb{Z}, \tag{3.17}$$

$$C_n = (\sqrt{7})^{-n}\,u_{3n} + 6\,(-1)^n,\ \ \ n \in \mathbb{Z}, \tag{3.18}$$

and finally

$$u_0 = -1,\ \ u_1 = \sqrt{7},\ \ u_2 = 0,\ \ u_{n+3} - \sqrt{7}\,u_{n+2} + \sqrt{7}\,u_n = 0,\ \ n \in \mathbb{Z}. \tag{3.19}$$

Additionally, we note that for every $n \in \mathbb{Z}$ we have

$$u_n = \sum_{k=0}^{2} 2\,\cos(2^k\alpha)\,\big(2\,\sin(2^k\alpha)\big)^n. \tag{3.20}$$

Remark 3.1. Furthermore, we get the following formula

$$\sum_{k=0}^{2} \big(2\,\sin(2^k\alpha) + \sqrt[6]{7}\big)\sqrt[3]{2\,\cos(2^k\alpha)} = 0. \tag{3.21}$$

By formula (4.11) from [7] we receive

$$\begin{aligned} &\sqrt[3]{2\cos(2\alpha)}\big(2\,\sin(\alpha)\big)^n + \sqrt[3]{2\cos(\alpha)}\big(2\,\sin(4\alpha)\big)^n \\ &+ \sqrt[3]{2\cos(4\alpha)}\big(2\,\sin(2\alpha)\big)^n = -\sqrt[6]{7^n}\,\sqrt[3]{A_n\,\sqrt[3]{49} + B_n\,\sqrt[3]{7} + C_n} \\ &= a_n\,\sqrt[3]{5 - 3\,\sqrt[3]{7}} + b_n\,\sqrt[3]{3\sqrt{7}\,\big(3 + (1+\sqrt[3]{7})^2\big)} + c_n\,\sqrt[3]{63\,\big(1+\sqrt[3]{7}\big)}, \end{aligned} \tag{3.22}$$

where

$$\begin{array}{lll} a_0 = 1, & a_1 = 0, & a_2 = 0, \\ b_0 = 0, & b_1 = -1, & b_2 = 0, \\ c_0 = 0, & c_1 = 0, & c_2 = -1, \end{array}$$

and

$$x_{n+3} - \sqrt{7}\,x_{n+2} + \sqrt{7}\,x_n = 0, \tag{3.23}$$

for $n \in \mathbb{Z}$ and $x \in \{a, b, c\}$;

$$A_0 = 0,\ A_1 = 3,\ A_2 = 0,\ A_{n+3} - A_{n+2} - 2A_{n+1} - A_n = 0,\ \ n \in \mathbb{Z}, \tag{3.24}$$

$$B_0 = 3,\ B_1 = 6,\ B_2 = 9,\ B_{n+3} - 2B_{n+2} - B_{n+1} + B_n = 0,\ \ n \in \mathbb{Z}, \tag{3.25}$$

$$C_n = -(\sqrt{7})^{-n} v_{3n} - 6(-1)^n, \quad n \in \mathbb{Z}, \tag{3.26}$$

and where

$$\begin{aligned} &v_0 = -1, \quad v_1 = -2\sqrt{7}, \quad v_2 = -7, \\ &v_{n+3} - \sqrt{7}\,v_{n+2} + \sqrt{7}\,v_n = 0, \quad n \in \mathbb{Z}. \end{aligned} \tag{3.27}$$

Let us note that for every $n \in \mathbb{Z}$ we have

$$\begin{aligned} v_n = 2\,\cos(2\alpha)\,\big(2\,\sin(\alpha)\big)^n + 2\,\cos(\alpha)\,\big(2\,\sin(4\alpha)\big)^n& \\ + 2\,\cos(4\alpha)\,\big(2\,\sin(2\alpha)\big)^n&. \end{aligned} \tag{3.28}$$

By formula (4.12) from [7] we obtain

$$(3.29)\quad \sqrt[3]{2\cos(4\alpha)}\,\big(2\,\sin(\alpha)\big)^n + \sqrt[3]{2\cos(\alpha)}\,\big(2\,\sin(2\alpha)\big)^n + \sqrt[3]{2\cos(2\alpha)}\,\big(2\,\sin(4\alpha)\big)^n = \sqrt[6]{7^n}\,\sqrt[3]{A_n\,\sqrt[3]{49} - B_n\,\sqrt[3]{7} + C_n}$$
$$= a_n\,\sqrt[3]{5 - 3\,\sqrt[3]{7}} + b_n\,\sqrt[3]{\sqrt{7}\,\big(-5 - 3\,\sqrt[3]{7} + 3\,\sqrt[3]{49}\big)} + c_n\,\sqrt[3]{21\,\big(2 - \sqrt[3]{7}\big)^2}$$
$$= a_n^*\,\sqrt[3]{21\,\big(2 - \sqrt[3]{7}\big)^2} + b_n^*\,\sqrt{7}\,\sqrt[3]{\big(3\,\sqrt[3]{7} - 4\big)^2} + c_n^*\,\sqrt[3]{147\,\big((2\,\sqrt[3]{7} - 5)^2 + \sqrt[3]{7}\big)},$$

where

$$\begin{array}{lll} a_0 = 1, & a_1 = 0, & a_2 = 0, \\ b_0 = 0, & b_1 = 1, & b_2 = 0, \\ c_0 = 0, & c_1 = 0, & c_2 = 1, \\ a_2^* = 1, & a_3^* = 0, & a_4^* = 0, \\ b_2^* = 0, & b_3^* = 1, & b_4^* = 0, \\ c_2^* = 0, & c_3^* = 0, & c_4^* = 1, \end{array}$$

and

$$(3.30)\qquad x_{n+3} - \sqrt{7}\,x_{n+2} + \sqrt{7}\,x_n = 0,$$

for every $n \in \mathbb{Z}$ and $x \in \{a, b, c, a^*, b^*, c^*\}$;

$$(3.31)\quad A_0 = 0,\ \ A_1 = A_2 = 3,\ \ A_{n+3} - A_{n+2} - 2\,A_{n+1} + A_n = 0,\ \ n \in \mathbb{Z},$$
$$(3.32)\quad B_0 = B_1 = 3,\ \ B_2 = 12,\ \ B_{n+3} - 2B_{n+2} - B_{n+1} + B_n = 0,\ \ n \in \mathbb{Z},$$
$$(3.33)\quad C_n = (\sqrt{7})^{-n}\,w_{3n} + 6\,(-1)^n,\ \ n \in \mathbb{Z},$$

and

$$(3.34)\quad w_0 = -1,\ \ w_1 = w_2 = 0,\ \ w_{n+3} - \sqrt{7}\,w_{n+2} + \sqrt{7}\,w_n = 0,\ \ n \in \mathbb{Z}.$$

Let us note that for every $n \in \mathbb{Z}$ we have

$$(3.35)\quad w_n = 2\,\cos(4\alpha)\,\big(2\,\sin(\alpha)\big)^n + 2\,\cos(\alpha)\,\big(2\,\sin(2\alpha)\big)^n + 2\,\cos(2\alpha)\,\big(2\,\sin(4\alpha)\big)^n.$$

Remark 3.2. Multiplying (36) by (51) (from [9]) we get the following equality

$$\Big(\big(2\,\cos(\alpha)\big)^{-2/3}+\big(2\,\cos(2\alpha)\big)^{-2/3}+\big(2\,\cos(4\alpha)\big)^{-2/3}\Big) \times\Big(2\cos(\alpha)\big(2\,\cos(4\alpha)\big)^{-1/3}+2\cos(2\alpha)\big(2\,\cos(\alpha)\big)^{-1/3} + 2\cos(4\alpha)\big(2\,\cos(2\alpha)\big)^{-1/3}\Big)=3, \tag{3.36}$$

i.e.,

$$\sqrt[3]{\frac{\cos(\alpha)}{\cos(4\alpha)}}+\sqrt[3]{\frac{\cos(2\alpha)}{\cos(\alpha)}}+\sqrt[3]{\frac{\cos(4\alpha)}{\cos(2\alpha)}}+\frac{\cos(2\alpha)}{\cos(\alpha)}+\frac{\cos(4\alpha)}{\cos(2\alpha)} +\frac{\cos(\alpha)}{\cos(4\alpha)}+2\cos(4\alpha)\sqrt[3]{\frac{\cos(4\alpha)}{\cos(\alpha)}}+2\cos(\alpha)\sqrt[3]{\frac{\cos(\alpha)}{\cos(2\alpha)}} +2\cos(2\alpha)\sqrt[3]{\frac{\cos(2\alpha)}{\cos(4\alpha)}}=3, \tag{3.37}$$

which, by (37) and (41) from [9], is equivalent to the equality

$$\frac{\cos(2\alpha)}{\cos(\alpha)}+\frac{\cos(4\alpha)}{\cos(2\alpha)}+\frac{\cos(\alpha)}{\cos(4\alpha)}=3. \tag{3.38}$$

Similarly, multiplying (48) by (49) from [9] we get

$$\Big(\frac{\sqrt[3]{\cos(\alpha)}}{\cos(2\alpha)}+\frac{\sqrt[3]{\cos(2\alpha)}}{\cos(4\alpha)}+\frac{\sqrt[3]{\cos(4\alpha)}}{\cos(\alpha)}\Big) \times\Big(\sqrt[3]{\frac{\cos(\alpha)}{\cos^2(4\alpha)}}+\sqrt[3]{\frac{\cos(2\alpha)}{\cos^2(\alpha)}}+\sqrt[3]{\frac{\cos(4\alpha)}{\cos^2(2\alpha)}}\Big)=12, \tag{3.39}$$

which, by (37) and (41) from [9], is equivalent to

$$\operatorname{scs}^2(\alpha)+\operatorname{scs}^2(2\alpha)+\operatorname{scs}^2(4\alpha)=24. \tag{3.40}$$

4. Generalizations of some Berndt-Zhang and Liu trigonometric identities

Now we will present the generalizations of equations (1.1)–(1.5) discussed in [2] (the elementary proof of the identities shall be given in [15]).

Let us set

$$\kappa_n=\frac{2^{2n+1}}{\sqrt7}\sum_{k=0}^{2}\sin\left(2^k\,\alpha\right)\,\sin^{2n}\left(2^{k+1}\,\alpha\right) =\frac{2^{2n+1}}{\sqrt7}\sum_{k=0}^{2}\sin\left(2^{k+2}\,\alpha\right)\,\sin^{2n}\left(2^{k}\,\alpha\right), \tag{4.1}$$

$$\lambda_n = -\frac{2^{2n-1}}{\sqrt{7}} \sum_{k=0}^{2} \csc\left(2^{k+1}\alpha\right) \sin^{2n}\left(2^k\alpha\right), \tag{4.2}$$

$$\tau_n = \frac{2^{2n-1}}{\sqrt{7}} \sum_{k=0}^{2} \csc\left(2^{k+2}\alpha\right) \sin^{2n}\left(2^k\alpha\right). \tag{4.3}$$

Then we have

$$\begin{aligned} &\kappa_0 = 1, && \kappa_1 = 2, && \kappa_2 = 7,\\ &\lambda_0 = 0, && \lambda_1 = 1, && \lambda_2 = 5,\\ &\tau_0 = 0, && \tau_1 = 0, && \tau_2 = 1, \end{aligned}$$

and

$$x_{n+3} - 7\,x_{n+2} + 14\,x_{n+1} - 7\,x_n = 0, \tag{4.4}$$

for every $x \in \{\kappa, \lambda, \tau\}$ and $n \in \mathbb{N}_0$.

Furthermore, let us set

$$\begin{aligned} \mu_n &= \frac{2^{2n+1}}{\sqrt{3}} \sum_{k=0}^{2} (-1)^k \sin\left(2^k\beta\right) \sin^{2n}\left(2^{k+1}\beta\right) \\ &= \frac{2^{2n+1}}{\sqrt{3}} \sum_{k=0}^{2} (-1)^k \sin\left(2^{k+2}\beta\right) \sin^{2n}\left(2^k\beta\right), \end{aligned} \tag{4.5}$$

$$\nu_n = -\frac{2^{2n-1}}{\sqrt{3}} \sum_{k=0}^{2} (-1)^k \csc\left(2^{k+1}\beta\right) \sin^{2n}\left(2^k\beta\right), \tag{4.6}$$

$$\xi_n = \frac{2^{2n-1}}{\sqrt{3}} \sum_{k=0}^{2} (-1)^k \csc\left(2^{k+2}\beta\right) \sin^{2n}\left(2^k\beta\right). \tag{4.7}$$

Then we have

$$\begin{aligned} &\mu_0 = 0, && \mu_1 = 3, && \mu_2 = 12,\\ &\nu_0 = 1, && \nu_1 = 3, && \nu_2 = 12,\\ &\xi_0 = 1, && \xi_1 = 3, && \xi_2 = 9, \end{aligned}$$

and

$$x_{n+3} - 6\,x_{n+2} + 9\,x_{n+1} - 3\,x_n = 0, \tag{4.8}$$

for every $x \in \{\mu, \nu, \xi\}$ and $n \in \mathbb{N}_0$.

5. Discussion of the initial values for (2.1) and (2.2)

Let us set

$$f_n = 2^{n+1}\Big(\cos(\beta)\big(\cos(2\beta)\big)^n + \cos(2\beta)\big(\cos(4\beta)\big)^n + \\ + \cos(4\beta)\big(\cos(\beta)\big)^n\Big), \tag{5.1}$$

$$g_n = 2^{n+1}\Big(\cos(4\beta)\big(\cos(2\beta)\big)^n + \cos(\beta)\big(\cos(4\beta)\big)^n \\ + \cos(2\beta)\big(\cos(\beta)\big)^n\Big), \tag{5.2}$$

$$h_n = 2^{n+1}\Big(\big(\cos(\beta)\big)^{n+1} + \big(\cos(2\beta)\big)^{n+1} + \big(\cos(4\beta)\big)^{n+1}\Big), \tag{5.3}$$

for every $n = 0, 1, 2, \ldots$.

Lemma 5.1. *We have*

$$f_0 = g_0 = h_0 = 0, \quad f_1 = g_1 = -3 \quad \text{and} \quad h_1 = 6,$$

$$\begin{cases} f_{n+1} = f_n - h_{n-1}, \\ g_{n+1} = h_n - h_{n-1}, \\ h_{n+1} = g_n + 2\,h_{n-1}, \end{cases}$$

for every $n \in \mathbb{N}$. Elements of any of the following three sequences: $\{f_n\}_{n=0}^{\infty}$, $\{g_n\}_{n=0}^{\infty}$ and $\{h_n\}_{n=0}^{\infty}$ satisfy the same recurrence relation

$$x_{n+3} - 3\,x_{n+1} + x_n = 0, \qquad n = 0, 1, \ldots \tag{5.4}$$

Proof. We have

$$h_{n+1} = g_n + 2\,h_{n-1} = h_{n-1} - h_{n-2} + 2\,h_{n-1},$$

i.e.,

$$h_{n+1} - 3\,h_{n-1} + h_{n-2} = 0.$$

Similar relation holds for the sequence $\{g_n\}_{n=0}^{\infty}$ since

$$g_3 - 3\,g_1 + g_0 = h_2 - h_1 + 9 = g_1 + 2\,h_0 - 6 + 9 = 0,$$

$$g_4 - 3\,g_2 + g_1 = h_3 - h_2 - 3\,(h_1 - h_0) - 3 = g_2 + 2\,h_1 - 18 = \\ = 3\,h_1 - h_0 - 18 = 0$$

and

$$g_{n+1} = h_n - h_{n-1}, \qquad n \in \mathbb{N}.$$

Moreover, we find

$$f_2 = f_1 - h_0 = -3, \qquad f_3 = f_2 - h_1 = -9,$$
$$f_3 - 3\,f_1 + f_0 = 0,$$

and the induction step runs as follows

$$f_{n+4} - 3\,f_{n+2} + f_{n+1} = f_{n+3} - h_{n+2} - 3\,\big(f_{n+1} - h_n\big) + f_n - h_{n-1} = 0. \;\blacksquare$$

The first twelve elements of the sequences $\{f_n\}_{n=0}^{\infty}$, $\{g_n\}_{n=0}^{\infty}$ and $\{h_n\}_{n=0}^{\infty}$ are given in Table 1.

Furthermore, let us set

$$\begin{aligned}(5.5)\quad a_n = {} & \Big(4\,\cos(\beta)\,\cos(2\beta)\Big)^n + \Big(4\,\cos(\beta)\,\cos(4\beta)\Big)^n \\ & + \Big(4\,\cos(2\beta)\,\cos(4\beta)\Big)^n,\end{aligned}$$

$$\begin{aligned}(5.6)\quad b_n = {} & 2\,\cos(\beta)\,\Big(4\,\cos(\beta)\,\cos(2\beta)\Big)^n + 2\,\cos(4\beta)\,\Big(4\,\cos(\beta)\,\cos(4\beta)\Big)^n \\ & + 2\,\cos(2\beta)\,\Big(4\,\cos(2\beta)\,\cos(4\beta)\Big)^n,\end{aligned}$$

$$\begin{aligned}(5.7)\quad c_n = {} & 2\,\cos(\beta)\,\Big(4\,\cos(2\beta)\,\cos(4\beta)\Big)^n + 2\,\cos(2\beta)\,\Big(4\,\cos(\beta)\,\cos(4\beta)\Big)^n \\ & + 2\,\cos(4\beta)\,\Big(4\,\cos(\beta)\,\cos(2\beta)\Big)^n,\end{aligned}$$

$$\begin{aligned}(5.8)\quad d_n = {} & 2\,\cos(\beta)\,\Big(4\,\cos(\beta)\,\cos(4\beta)\Big)^n + 2\,\cos(2\beta)\,\Big(4\,\cos(\beta)\,\cos(2\beta)\Big)^n \\ & + 2\,\cos(4\beta)\,\Big(4\,\cos(2\beta)\,\cos(4\beta)\Big)^n,\end{aligned}$$

for every $n = 0, 1, 2, \ldots$.

Lemma 5.2. *The following relations are satisfied*

$$(5.9)\qquad \begin{cases} a_{n+1} = \frac{1}{2}\left(h_n^2 - h_{2n+1}\right) = b_n - a_n, \\ b_{n+1} = 2\,a_n - b_n + d_n, \\ c_{n+1} = -a_n, \\ d_{n+1} = -a_n + b_n - d_n, \end{cases}$$

for every $n = 0, 1, \ldots$. Additionally, all four sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$ and $\{d_n\}_{n=0}^{\infty}$ satisfy the recurrence relation of the form

$$(5.10)\qquad x_{n+3} + 3\,x_{n+2} - x_n = 0, \qquad n \in \mathbb{N}.$$

Proof. The relations (5.9) from simple trigonometric considerations follow, for example:

$$\begin{aligned} b_{n+1} &= 8\,\cos^2(\beta)\,\cos(2\,\beta)\,\big(4\,\cos(\beta)\,\cos(2\,\beta)\big)^n \\ &\quad + 8\,\cos(\beta)\,\cos^2(4\,\beta)\,\big(4\,\cos(\beta)\,\cos(4\,\beta)\big)^n + \ldots \\ &= \big(4\,\cos(2\,\beta) + 4\,\cos^2(2\,\beta)\big)\,\big(4\,\cos(\beta)\,\cos(2\,\beta)\big)^n \\ &\quad + \big(4\,\cos(\beta) + 4\,\cos^2(\beta)\big)\,\big(4\,\cos(\beta)\,\cos(4\,\beta)\big)^n + \ldots \end{aligned}$$

$$= \big(4\,\cos(2\,\beta) + 2 + 2\,\cos(4\,\beta)\big)\,\big(4\,\cos(\beta)\,\cos(2\,\beta)\big)^n + \big(4\,\cos(\beta) + 2 + 2\,\cos(2\,\beta)\big)\,\big(4\,\cos(\beta)\,\cos(4\,\beta)\big)^n + \dots$$
$$\stackrel{(2.3)}{=} \big(2\,\cos(2\,\beta) + 2 - 2\,\cos(\beta)\big)\,\big(4\,\cos(\beta)\,\cos(2\,\beta)\big)^n + \big(2\,\cos(\beta) + 2 - 2\,\cos(4\,\beta)\big)\,\big(4\,\cos(\beta)\,\cos(4\,\beta)\big)^n + \dots$$
$$= 2\,a_n - b_n + d_n.$$

From (5.9) (more precisely from the first to the last identity of the system of equation (5.9)) it can be deduced the relations

$$b_n = a_{n+1} + a_n, \tag{5.11}$$
$$a_{n+2} + a_{n+1} = 2\,a_n - a_{n+1} - a_n + d_n,$$

i.e.,

$$d_n = a_{n+2} + 2\,a_{n+1} - a_n, \tag{5.12}$$
$$d_{n+1} + d_n = -a_n + b_n = a_{n+1} \tag{5.13}$$

and at last

$$a_{n+3} + 2\,a_{n+2} - a_{n+1} + a_{n+2} + 2\,a_{n+1} - a_n = a_{n+1},$$

i.e.,

$$a_{n+3} + 3\,a_{n+2} - a_n = 0. \tag{5.14}$$

Hence and from identities: $c_{n+1} = -a_n$, (5.11) and (5.12) the relation (5.10) follows. ∎

Theorem 5.3. *The following decompositions of polynomials hold*

$$\big(\mathbb{X} - \big(2\,\cos(\beta)\big)^n\big)\,\big(\mathbb{X} - \big(2\,\cos(2\beta)\big)^n\big)\,\big(\mathbb{X} - \big(2\,\cos(4\beta)\big)^n\big) = \mathbb{X}^3 - h_{n-1}\,\mathbb{X}^2 + a_n\,\mathbb{X} + (-1)^{n+1},$$

$$\big(\mathbb{X} - 2\,\cos(\beta)\,\big(2\,\cos(2\beta)\big)^n\big)\,\big(\mathbb{X} - 2\,\cos(2\beta)\,\big(2\,\cos(4\beta)\big)^n\big) \times \big(\mathbb{X} - 2\,\cos(4\beta)\,\big(2\,\cos(\beta)\big)^n\big) = \mathbb{X}^3 - f_n\,\mathbb{X}^2 + (c_n - a_n)\,\mathbb{X} + (-1)^n = \mathbb{X}^3 - f_n\,\mathbb{X}^2 - b_{n-1}\,\mathbb{X} + (-1)^n, \tag{5.15}$$

$$\big(\mathbb{X} - 2\,\cos(\beta)\,\big(2\,\cos(4\beta)\big)^n\big)\,\big(\mathbb{X} - 2\,\cos(2\beta)\,\big(2\,\cos(\beta)\big)^n\big) \times \big(\mathbb{X} - 2\,\cos(4\beta)\,\big(2\,\cos(2\beta)\big)^n\big) = \mathbb{X}^3 - g_n\,\mathbb{X}^2 + (d_n - a_n)\,\mathbb{X} + (-1)^n = \mathbb{X}^3 - g_n\,\mathbb{X}^2 - d_{n-1}\,\mathbb{X} + (-1)^n, \tag{5.16}$$

$$\big(\mathbb{X} - 2\,\cos(\beta)\,\big(4\,\cos(\beta)\,\cos(2\beta)\big)^n\big)\,\big(\mathbb{X} - 2\,\cos(4\beta)\,\big(4\,\cos(\beta)\,\cos(4\beta)\big)^n\big) \times \big(\mathbb{X} - 2\,\cos(2\beta)\,\big(4\,\cos(2\beta)\,\cos(4\beta)\big)^n\big) = \mathbb{X}^3 - b_n\,\mathbb{X}^2 + (-1)^n\,(f_n - h_{n-1})\,\mathbb{X} + 1,$$

$$\big(\mathbb{X} - 2\,\cos(\beta)\,\big(4\,\cos(2\beta)\,\cos(4\beta)\big)^n\big)\,\big(\mathbb{X} - 2\,\cos(2\beta)\,\big(4\,\cos(\beta)\,\cos(4\beta)\big)^n\big)$$
$$\times\,\big(\mathbb{X} - 2\,\cos(4\beta)\,\big(4\,\cos(\beta)\,\cos(2\beta)\big)^n\big)$$
$$= \mathbb{X}^3 - c_n\,\mathbb{X}^2 + (-1)^n\,(g_n - h_{n-1})\,\mathbb{X} + 1,$$

$$\big(\mathbb{X} - 2\,\cos(\beta)\,\big(4\,\cos(\beta)\,\cos(4\beta)\big)^n\big)\,\big(\mathbb{X} - 2\,\cos(2\beta)\,\big(4\,\cos(\beta)\,\cos(2\beta)\big)^n\big)$$
$$\times\,\big(\mathbb{X} - 2\,\cos(4\beta)\,\big(4\,\cos(2\beta)\,\cos(4\beta)\big)^n\big)$$
$$= \mathbb{X}^3 - d_n\,\mathbb{X}^2 + (-1)^n\,(h_n - h_{n-1})\,\mathbb{X} + 1.$$

Proof. The respective formulas can be easily deduced from definitions of all sequences: $\{f_n\}$–$\{h_n\}$, $\{a_n\}$–$\{d_n\}$ and Lemmas 5.1 and 5.2. ∎

The following result finishes the preparatory investigations.

Lemma 5.4. *Let $f(z) \in \mathbb{R}[z]$ and $f(z) = z^3 + p\,z^2 + q\,z + r = (z - \xi_1)\,(z - \xi_2)\,(z - \xi_3)$. Suppose that $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$. Then we have*

$$\sqrt[3]{A} = \sqrt[3]{\xi_1} + \sqrt[3]{\xi_2} + \sqrt[3]{\xi_3} = \sqrt[3]{-p - 6\,\sqrt[3]{r} - \frac{3}{\sqrt[3]{2}}\Big(\sqrt[3]{\mathcal{S} + \sqrt{\mathcal{T}}} + \sqrt[3]{\mathcal{S} - \sqrt{\mathcal{T}}}\Big)}, \tag{5.17}$$

where

$$\mathcal{S} := p\,q + 6\,q\,\sqrt[3]{r} + 6\,p\,\sqrt[3]{r^2} + 9\,r,$$
$$\mathcal{T} := p^2\,q^2 - 4\,q^3 - 4\,p^3\,r + 18\,p\,q\,r - 27\,r^2.$$

Moreover, if $\mathcal{T} \geq 0$ then we can assume that all the roots appearing here are real.

Proof. See Section 3 of the paper [9]. ∎

Now let us describe how the initial values of recurrence sequences (2.1) and (2.2) could be generated.

The value of Ψ_0 follows from (5.15) for $n = 2$ (then from Table 1 we obtain $p = 3$, $q = -6$, $r = 1$) and from Lemma 5.4 (then we obtain $\mathcal{S} = -27$, $\mathcal{T} = 27^2$). The value of Φ_0 follows from (5.16) for $n = 2$ (then by Table 1 we have $p = -6$, $q = 3$, $r = 1$) and from Lemma 5.4 (then we deduce $\mathcal{S} = -27$, $\mathcal{T} = 3^6$).

The value of Ψ_1 follows from (5.15) for $n = 5$ (then from Table 1 we obtain $p = 24$, $q = 129$ and $r = -1$) and from Lemma 5.4 (then we deduce $\mathcal{S} = 27 \cdot 91$, $\mathcal{T} = 3^6 \cdot 37^2$, $\mathcal{S} \pm \sqrt{\mathcal{T}} = \left\{\begin{smallmatrix} 12^3 \cdot 2 \\ 3^6 \cdot 2 \end{smallmatrix}\right)$. The value of Φ_1 follows from (5.16) for $n = 5$ (then from Table 1 we get $p = 33$, $q = -105$ and $r = -1$) and from Lemma 5.4 (then we deduce $\mathcal{S} = -27 \cdot 98$, $\mathcal{T} = 6^6 \cdot 19^2$, $\mathcal{S} \pm \sqrt{\mathcal{T}} = \left\{\begin{smallmatrix} 3^6 \cdot 2 \\ -15^3 \cdot 2 \end{smallmatrix}\right)$.

The value of Ψ_2 follows from (5.15) for $n = 8$ (then from Table 1 we obtain $p = 3$, $q = -3084$ and $r = 1$) and from Lemma 5.4 (then we deduce $\mathcal{S} = -27 \cdot 1027$, $\mathcal{T} = 3^4 \cdot 38073^2$, $\mathcal{S} \pm \sqrt{\mathcal{T}} = \left\{ {-2\cdot(3\cdot19)^3 \atop 2^4\cdot3^9} \right)$. At last, the value of Φ_2 can be obtained from (5.16) for $n = 8$ (then from Table 1 we get $p = -249$, $q = 2514$ and $r = 1$) and from Lemma 5.4 (then we deduce $\mathcal{S} = -27 \cdot 22681$, $\mathcal{T} = 2^6 \cdot 19^2 \cdot 1117^2$, $\sqrt[3]{\mathcal{S} \pm \sqrt{\mathcal{T}}} = \left\{ {-3\cdot28\cdot\sqrt[3]{2} \atop -27\cdot\sqrt[3]{2}} \right)$.

Table 1. The first twelve values of some recurrent sequences discussed in the paper

n	0	1	2	3	4	5	6	7	8	9	10	11
Ψ_n	0	3	0	9	−3	27	−18	84	−81	270	−327	891
Φ_n	−1	1	−4	4	−13	16	−43	61	−145	226	−496	823
γ_n	0	0	1/7	1/7	1	6/7	5	4	22	17	91	69
κ_n	1	2	7	28	112	441	1715	6615	25382	97069	370440	1411788
λ_n	0	1	5	21	84	329	1274	4900	18767	71687	273371	1041348
τ_n	0	0	1	7	35	154	637	2548	9996	38759	149205	571781
μ_n	0	3	12	45	171	657	2538	9828	38097	147744	573075	2223045
ν_n	1	3	12	48	189	738	2871	11151	43281	167940	651564	2527767
ξ_n	1	3	9	30	108	405	1548	5967	23085	89451	346842	1345248
f_n	0	−3	−3	−9	−6	−24	−9	−66	−3	−189	57	−564
g_n	0	−3	6	−9	21	−33	72	−120	249	−432	867	−1545
h_n	0	6	−3	18	−15	57	−63	186	−246	621	−924	2109
a_n	3	−3	9	−24	69	−198	570	−1641	4725	−13605	39174	−112797
b_n	0	6	−15	45	−129	372	−1071	3084	−8880	25569	−73623	211989
c_n	0	−3	3	−9	24	−69	198	−570	1641	−4725	13605	−39174
d_n	0	−3	12	−36	105	−303	873	−2514	7239	−20844	60018	−172815

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INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY OF TECHNOLOGY
Kaszubska 23
GLIWICE 44-100, POLAND
E-mail: roman.witula@polsl.pl

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