DEMONSTRATIO MATHEMATICA
Vol. XLV No 4 2012

10.1515/dema-2013-0415

Bishwambhar Roy

UNIFICATION OF ALMOST REGULAR, ALMOST NORMAL AND MILDLY NORMAL TOPOLOGICAL SPACES

Abstract. In this paper, a new kind of sets called regular μ-generalized closed (briefly $r\mu g$-closed) sets are introduced and studied in a topological space. Some of their properties are investigated. Finally, some characterizations of almost μ-regular, almost μ-normal and mildly μ-normal spaces have been given.

1. Introduction

In 1970, the concept of generalized closed sets in a topological space was introduced by N. Levine [12] in order to extend many of the important properties of closed sets to a larger family. After that, the concept of generalized closed sets has been investigated by many mathematicians because the notion of generalized closed sets is a natural generalization of closed sets (see [12, 13, 22] for details). It is also well known that separation axioms are one of the basic subjects of study in general topology and in several branches of mathematics. In literature, separation axioms have been studied by different mathematicians. In 1973, Singal et al. introduced the concept of almost regular [27], almost normal [28] and mildly normal [29] spaces. Recently, Ekici, Malghan, Navalagi, Noiri and Park [8, 15, 17, 20, 23, 24] continued the study of several weaker forms of separation axioms, while different forms of continuity have been studied in [2]. The aim of this paper is to unify such types of existing spaces by using the concept of generalized topology introduced by Á. Császár.

Throughout this paper (X, τ) always means a topological space on which no separation axioms are assumed unless explicitly stated. The closure and interior of a set $A(\subseteq X)$ is denoted by clA and $intA$ respectively. A subset A is said to be regular open (resp. regular closed) if $A = intclA$ (resp.

The author acknowledges the financial support from UGC, New Delhi.

2000 *Mathematics Subject Classification*: 54D15, 54A05, 54C08.

Key words and phrases: μ-open set, $r\mu g$-closed set, almost μ-regular space, almost μ-normal space, mildly μ-normal space.

$A = clintA$). The collection of all regular open (regular closed) sets in a topological space (X, τ) is denoted by $RO(X)$ (resp. $RC(X)$). The δ-closure [30] of a subset A of X is denoted by $cl_\delta A$ and defined by $cl_\delta A = \{x : A \cap U \neq \varnothing$, for each $U \in RO(X)$ with $x \in U\}$. Let (X, τ) be a space and $A \subseteq X$. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is called ω-closed [10] if it contains all its condensation points. The complement of an ω-closed set is called ω-open. It is well known that the family of all ω-open subsets of a space (X, τ), denoted by τ_ω, forms a topology on X finer than τ. A subset A of a space X is said to be preopen [16] (resp. semi-open [11], δ-preopen [25], α-open [14]) if $A \subseteq intclA$ (resp. $A \subseteq clintA$, $A \subseteq intcl_\delta A$, $A \subseteq intclintA$). The family of all preopen (resp. semiopen, δ-preopen, α-open) sets in a space X is denoted by (resp. $PO(X)$, $SO(X)$, $\delta PO(X)$, $\alpha O(X)$).

We recall some notions defined in [4]. Let X be a non-empty set, $expX$ denotes the power set of X. We call a class $\mu \subseteq expX$ a generalized topology [4], (briefly, GT) if $\varnothing \in \mu$ and union of elements of μ belongs to μ. A set X, with a GT μ on it, is said to be a generalized topological space (briefly, GTS) and is denoted by (X, μ). For a GTS (X, μ), the elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A; and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A (see [4, 5, 26] for details). Obviously in a topological space (X, τ), if one takes τ as the GT, then c_μ becomes equivalent to the usual closure operator. Similarly, c_μ becomes pcl, scl, pcl_δ, cl_α if μ stands for $PO(X)$, $SO(X)$, $\delta PO(X)$, $\alpha O(X)$ respectively.

It is easy to observe that i_μ and c_μ are idempotent and monotonic, where $\gamma : expX \to expX$ is said to be idempotent iff $A \subseteq B \subseteq X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic iff $\gamma(A) \subseteq \gamma(B)$. It is also well known from [5, 6] that if μ is a GT on X and $A \subseteq X$, $x \in X$, then $x \in c_\mu(A)$ iff $x \in M \in \mu \Rightarrow M \cap A \neq \varnothing$ and $c_\mu(X \setminus A) = X \setminus i_\mu(A)$.

We recall the following definitions to be used in sequel.

Definition 1.1. A subset A of a space (X, τ) is called

(a) generalized closed (briefly, g-closed) [12] if $clA \subseteq U$ whenever $A \subseteq U$ and $U \in \tau$;
(b) regular generalized closed (briefly, rg-closed) [22] if $clA \subseteq U$ whenever $A \subseteq U \in RO(X)$;
(c) generalized preregular closed [9] (briefly, gpr-closed), or preregular generalized closed [19] if $pclA \subseteq U$ whenever $A \subseteq U \in RO(X)$;
(d) $r\alpha g$-closed [20] if $cl_\alpha A \subseteq U$ whenever $A \subseteq U \in RO(X)$;

(e) $g\delta pr$-closed [8] if $pcl_\delta A \subseteq U$ whenever $A \subseteq U \in RO(X)$;
(f) $rg\omega$-closed [1] if $cl_\omega(A) \subseteq U$ whenever $A \subseteq U \in RO(X)$.

2. Properties of $r\mu g$-closed sets

Definition 2.1. Let μ be a GT on a topological space (X, τ). Then $A \subseteq X$ is called a regular μ-generalized closed set or simply an $r\mu g$-closed set (resp. $g\mu$-closed set [21]) if $c_\mu(A) \subseteq U$ whenever $A \subseteq U \in RO(X)$ (resp. $A \subseteq U \in \tau$). The complement of an $r\mu g$-closed set (resp. $g\mu$-closed set) is called an $r\mu g$-open (resp. $g\mu$-open [21]) set.

Remark 2.2. Let μ be a GT on a topological space (X, τ). Then we have the following relation between $r\mu g$-closed sets and other known sets :

$$\mu\text{-closed set} \Rightarrow g\mu\text{-closed set} \Rightarrow r\mu g\text{-closed set}.$$

The next example shows that none of the implications is reversible.

Example 2.3. Consider the topological space (X, τ), where $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{b\}, \{c\}, \{b, c\}\}$. Let $\mu = \{\varnothing, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ be a GT on the space X. Then it is easy to check that $\{b, c\}$ is $r\mu g$-closed in (X, τ) but neither a $g\mu$-closed set nor a μ-closed set.

Remark 2.4. Obviously if on a space (X, τ) one takes the GT $\mu = \tau$, then $r\mu g$-closed sets become equivalent to rg-closed sets [9, 22]. Similarly, $r\mu g$-closed sets become gpr-closed sets [17, 19, 23], $r\alpha g$-closed sets [20], $g\delta pr$-closed sets [8], $rg\omega$-closed sets [1] if the role of μ is taken to stand for $PO(X)$, $\alpha O(X)$, $\delta PO(X)$, τ_ω respectively.

The next two examples show that union (intersection) of two $r\mu g$-closed sets is not in general an $r\mu g$-closed set.

Example 2.5. **(a)** Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. Then (X, τ) is a topological space with $RO(X) = \{\varnothing, X, \{a\}, \{b, c\}\}$. Consider the GT μ on the space X as $\{\varnothing, X, \{a, b\}, \{a, c\}\}$. Then $\{b\}$ and $\{c\}$ are two $r\mu g$-closed sets but their union $\{b, c\}$ is not $r\mu g$-closed.

(b) Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ) is a topological space such that $RO(X) = \{\varnothing, X, \{a\}, \{b\}\}$. Consider the GT $\mu = \{\varnothing, X, \{a, b\}, \{a, c\}\}$ on X. Then $\{a, c\}$ and $\{a, b\}$ are two $r\mu g$-closed sets on X but their intersection $\{a\}$ is not an $r\mu g$-closed set in X.

Theorem 2.6. *Let μ be a GT on a topological space (X, τ). Let $A \subseteq X$ be an $r\mu g$-closed subset of X. Then $c_\mu(A) \setminus A$ does not contain any non-empty regular closed set.*

Proof. Let F be a regular closed subset of (X, τ) such that $F \subseteq c_\mu(A) \setminus A$. Then $F \subseteq X \setminus A$ and hence $A \subseteq X \setminus F \in RO(X)$. Since A is $r\mu g$-closed, $c_\mu(A) \subseteq X \setminus F$ and hence $F \subseteq X \setminus c_\mu(A)$. So $F \subseteq c_\mu(A) \cap (X \setminus c_\mu(A)) = \varnothing$. ∎

That the converse of the above theorem is false, is shown by the next example.

Example 2.7. Let $X = \{a, b, c, d\}$ and $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then (X, τ) is a topological space. Consider the GT $\mu = \{\varnothing, X, \{a\}, \{a, d\}, \{b, c\}, \{a, b, c\}\}$ on X. Then $RO(X) = \{\varnothing, X, \{a\}, \{b\}\}$. Consider $A = \{a\}$. Then $c_\mu(A) \setminus A = \{a, d\} \setminus \{a\} = \{d\}$ does not contain any non-empty regular closed set. But A is not $r\mu g$-closed.

Theorem 2.8. *Let μ be a GT on a topological space (X, τ). Then a subset A is $r\mu g$-open iff $F \subseteq i_\mu(A)$ whenever F is a regular closed subset such that $F \subseteq A$.*

Proof. Let A be an $r\mu g$-open subset of X and F be a regular closed subset of X such that $F \subseteq A$. Then $X \setminus A$ is an $r\mu g$-closed set and $X \setminus A \subseteq X \setminus F \in RO(X)$. So $c_\mu(X \setminus A) = X \setminus i_\mu(A) \subseteq X \setminus F$. Thus $F \subseteq i_\mu(A)$.

Conversely, let $F \subseteq i_\mu(A)$ whenever F is regular closed such that $F \subseteq A$. Let $X \setminus A \subseteq U$ where $U \in RO(X)$. Then $X \setminus U \subseteq A$ and $X \setminus U$ is regular closed. By the assumption, $X \setminus U \subseteq i_\mu(A)$ and hence $c_\mu(X \setminus A) = X \setminus i_\mu(A) \subseteq U$. Hence $X \setminus A$ is $r\mu g$-closed and hence A is $r\mu g$-open. ∎

Theorem 2.9. *Let μ be a GT on a topological space (X, τ) and A be an $r\mu g$-closed subset of X. If $B \subseteq X$ be such that $A \subseteq B \subseteq c_\mu(A)$, then B is also an $r\mu g$-closed set.*

Proof. Let A be an $r\mu g$-closed set and $B \subseteq U \in RO(X)$. Then $A \subseteq U \in RO(X)$ and hence $c_\mu(A) \subseteq U$. Thus by monotonicity and idempotent property of c_μ we have $c_\mu(B) \subseteq U$, showing B to be $r\mu g$-closed. ∎

Theorem 2.10. *Let (X, τ) be a topological space and μ be a GT on X. If A is an $r\mu g$-closed subset of X, then $c_\mu(A) \setminus A$ is $r\mu g$-open.*

Proof. Let A be an $r\mu g$-closed subset of (X, τ) and F be a regular closed subset such that $F \subseteq c_\mu(A) \setminus A$, so by Theorem 2.6, $F = \varnothing$ and thus $F \subseteq i_\mu(c_\mu(A) \setminus A)$. So by Theorem 2.8, $c_\mu(A) \setminus A$ is $r\mu g$-open. ∎

Example 2.11. Consider Example 2.7 once again. If we take $A = \{a\}$ then $c_\mu(A) \setminus A = \{d\}$ is $r\mu g$-closed but A is not $r\mu g$-closed.

Definition 2.12. Let μ be a GT on a topological space (X, τ). Then (X, τ) is said to be $r\mu g$-$T_{1/2}$ if every $r\mu g$-closed set in (X, τ) is μ-closed.

Theorem 2.13. *Let μ be a GT on a topological space (X, τ). Then the following are equivalent :*

(i) *(X, τ) is $r\mu g$-$T_{1/2}$.*
(ii) *Every singleton is either regular closed or μ-open.*

Proof. (i)⇒(ii): Suppose $\{x\}$ is not regular closed for some $x \in X$. Then $X \setminus \{x\}$ is not regular open and hence X is the only regular open set containing $X \setminus \{x\}$. Thus $X \setminus \{x\}$ is $r\mu g$-closed. Hence $X \setminus \{x\}$ is μ-closed (by (i)). Thus $\{x\}$ is μ-open.

(ii)⇒(i): Let A be any $r\mu g$-closed subset of (X, τ) and $x \in c_\mu(A)$. We have to show that $x \in A$. If $\{x\}$ is regular closed and $x \notin A$, then $x \in c_\mu(A) \setminus A$. Thus $c_\mu(A) \setminus A$ contains a non-empty regular closed set $\{x\}$, a contradiction to Theorem 2.6. So $x \in A$. Again if $\{x\}$ is μ-open, then since $x \in c_\mu(A)$, it follows that $x \in A$. So in the both cases $x \in A$. Thus A is μ-closed. ∎

Remark 2.14. Let μ be a GT on a space (X, τ). Then every $r\mu g$-$T_{1/2}$ space reduces to doore space [7] (resp. preregular $T_{1/2}$ [9], δp-regular $T_{1/2}$ [8], $rg\omega$-$T_{1/2}$) if one takes μ to be τ (resp. $PO(X)$, $\delta PO(X)$, τ_ω).

Theorem 2.15. *Let μ be a GT on a topological space (X, τ). Then the following are equivalent :*

(i) *Every regular open set of X is μ-closed.*
(ii) *Every subset of X is $r\mu g$-closed.*

Proof. (i)⇒(ii): Let $A \subseteq U \in RO(X)$. Then by (i) U is μ-closed and so $c_\mu(A) \subseteq c_\mu(U) = U$. Thus A is $r\mu g$-closed.

(ii)⇒(i): Let $U \in RO(X)$. Then by (ii), U is $r\mu g$-closed and hence $c_\mu(U) \subseteq U$, showing U to be a μ-closed set. ∎

Theorem 2.16. *Let μ be a GT on a topological space (X, τ). If A be $r\mu g$-open then $U = X$ whenever U is regular open and $i_\mu(A) \cup (X \setminus A) \subseteq U$.*

Proof. Let $U \in RO(X)$ and $i_\mu(A) \cup (X \setminus A) \subseteq U$ for an $r\mu g$-open set A. Then $X \setminus U \subseteq [X \setminus i_\mu(A)] \cap A$, i.e., $X \setminus U \subseteq c_\mu(X \setminus A) \setminus (X \setminus A)$. Since $X \setminus A$ is $r\mu g$-closed by Theorem 2.6, $X \setminus U = \varnothing$ and hence $U = X$. ∎

The converse of the theorem above is not always true as shown by the following example.

Example 2.17. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then (X, τ) is a topological space. Consider the GT $\mu = \tau$. Let $A = \{b, c, d\}$. Then X is the only regular open set containing $i_\mu(A) \cup (X \setminus A)$ but A is not $r\mu g$-open in X.

3. Almost μ-regular, almost μ-normal and mildly μ-normal spaces

Definition 3.1. Let (X, τ) be a topological space and μ be a GT on X. Then (X, τ) is said to be almost μ-regular if for each regular closed set F of X and each $x \notin F$, there exist disjoint μ-open sets U and V such that $x \in U$, $F \subseteq V$.

Remark 3.2. Let μ be a GT on a space (X, τ). Then every almost μ-regular space reduces to an almost regular [27] (resp. almost p-regular [15]) space if one takes μ to be τ (resp. $PO(X)$).

Theorem 3.3. *Let μ be a GT on a topological space (X, τ). Then the following statements are equivalent :*

(i) *X is almost μ-regular.*
(ii) *For each $x \in X$ and each $U \in RO(X)$ with $x \in U$ there exists $V \in \mu$ such that $x \in V \subseteq c_\mu(V) \subseteq U$.*
(iii) *For each regular closed set F of X, $\cap\{c_\mu(V) : F \subseteq V \in \mu\} = F$.*
(iv) *For each $A \subseteq X$ and each $U \in RO(X)$ with $A \cap U \neq \varnothing$, there exists $V \in \mu$ such that $A \cap V \neq \varnothing$ and $c_\mu(V) \subseteq U$.*
(v) *For each non-empty subset A of X and each regular closed subset F of X with $A \cap F = \varnothing$, there exist $V, W \in \mu$ such that $A \cap V \neq \varnothing$, $F \subseteq W$ and $W \cap V = \varnothing$.*
(vi) *For each regular closed set F and $x \notin F$, there exist $U \in \mu$ and an $r\mu g$-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \varnothing$.*
(vii) *For each $A \subseteq X$ and each regular closed set F with $A \cap F = \varnothing$, there exist $U \in \mu$ and an $r\mu g$-open set V such that $A \cap U \neq \varnothing$, $F \subseteq V$ and $U \cap V = \varnothing$.*

Proof. (i)⇒(ii): Let $U \in RO(X)$ with $x \in U$. Then $x \notin X \setminus U \in RC(X)$. Thus by (i), there exist disjoint $G, V \in \mu$ such that $x \in V$, $X \setminus U \subseteq G$. So, $x \in V \subseteq c_\mu(V) \subseteq c_\mu(X \setminus G) = X \setminus G \subseteq U$.

(ii)⇒(iii): Let $X \setminus F \in RO(X)$ and $x \in X \setminus F$. Then by (ii), there exists $U \in \mu$ such that $x \in U \subseteq c_\mu(U) \subseteq X \setminus F$. So $F \subseteq X \setminus c_\mu(U) = V$ (say) $\in \mu$ and $U \cap V = \varnothing$. Then $x \notin c_\mu(V)$. Thus $F \supseteq \cap\{c_\mu(V) : F \subseteq V \in \mu\}$.

(iii)⇒(iv): Let A be a subset of X and $U \in RO(X)$ be such that $A \cap U \neq \varnothing$. Let $x \in A \cap U$. Then $x \notin X \setminus U$. Hence by (iii), there exists $W \in \mu$ such that $X \setminus U \subseteq W$ and $x \notin c_\mu(W)$. Put $V = X \setminus c_\mu(W)$. Then $V \in \mu$ contains x and hence $A \cap V \neq \varnothing$. Now $V \subseteq X \setminus W$, so $c_\mu(V) \subseteq X \setminus W \subseteq U$.

(iv)⇒(v): Let F be a set as in the hypothesis of (v). Then $X \setminus F \in RO(X)$ with $A \cap (X \setminus F) \neq \varnothing$ and hence by (iv), there exists $V \in \mu$ such that $A \cap V \neq \varnothing$ and $c_\mu(V) \subseteq X \setminus F$. If we put $W = X \setminus c_\mu(V)$, then $W \in \mu$, $F \subseteq W$ and $W \cap V = \varnothing$.

(v)⇒(i): Let F be a regular closed set such that $x \notin F$. Then $F \cap \{x\} = \varnothing$. Thus by (v), there exist $U, V \in \mu$ such that $x \in U$, $F \subseteq V$ and $U \cap V = \varnothing$.

(i)⇒(vi): Trivial in view of Remark 2.2.

(vi)⇒(vii): Let $A \subseteq X$ and F be a regular closed set with $A \cap F = \varnothing$. Then for $a \in A$, $a \notin F$ and hence by (vi), there exist $U \in \mu$ and an $r\mu g$-open set V such that $a \in U$, $F \subseteq V$ and $U \cap V = \varnothing$. So $A \cap U \neq \varnothing$, $F \subseteq V$ and $U \cap V = \varnothing$.

(vii)⇒(i): Let $x \notin F$ where F is regular closed in X. Since $\{x\} \cap F = \varnothing$, by (vii) there exist $U \in \mu$ and an $r\mu g$-open set W such that $x \in U$, $F \subseteq W$ and $U \cap W = \varnothing$. Then $F \subseteq i_\mu(W) = V$ (say) $\in \mu$ (by Theorem 2.8) and hence $V \cap U = \varnothing$.

Note 3.4. If in a topological space (X, τ) we take $\mu = \alpha O(X)$ then an almost μ-regular space reduces to an almost regular space [20].

Definition 3.5. Let μ be a GT on a topological space (X, τ). Then (X, τ) is said to be almost μ-normal if for each closed set A and each regular closed set B of X with $A \cap B = \varnothing$, there exist two disjoint μ-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Remark 3.6. Let μ be a GT on a space (X, τ). Then an almost μ-normal space reduces to an almost normal [28] (resp. almost p-normal [17, 23], almost δp-normal [8]) space if one takes μ to be τ (resp. $PO(X)$, $\delta PO(X)$).

Theorem 3.7. *Let μ be a GT on a topological space (X, τ). Then the following statements are equivalent :*

(i) *X is almost μ-normal.*

(ii) *For any closed set A and any regular closed set B of X with $A \cap B = \varnothing$, there exist disjoint $g\mu$-open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$.*

(iii) *For each closed set A and each regular open set B containing A, there exists a $g\mu$-open set V of X such that $A \subseteq V \subseteq c_\mu(V) \subseteq B$.*

(iv) *For each rg-closed set A and each regular open set B containing A, there exists a $g\mu$-open set V of X such that $clA \subseteq V \subseteq c_\mu(V) \subseteq B$.*

(v) *For each rg-closed set A and each regular open set B containing A, there exists a μ-open set V of X such that $clA \subseteq V \subseteq c_\mu(V) \subseteq B$.*

(vi) *For each g-closed set A and each regular open set B containing A, there exists a μ-open set V such that $cl(A) \subseteq V \subseteq c_\mu(V) \subseteq B$.*

(vii) *For each g-closed set A and each regular open set B containing A, there exists a $g\mu$-open set V such that $cl(A) \subseteq V \subseteq c_\mu(V) \subseteq B$.*

Proof. (i)⇒(ii): Obvious by Remark 2.2.

(ii)⇒(iii): Let A be a closed set and B be a regular open set containing A. Then $A \cap (X \setminus B) = \varnothing$, where A is closed and $X \setminus B$ is regular closed. So by (ii) there exist disjoint $g\mu$-open sets V and W such that $A \subseteq V$ and

$X \setminus B \subseteq W$. Thus by Remark 2.2 and Theorem 2.8, $X \setminus B \subseteq i_\mu(W)$ and $V \cap i_\mu(W) = \varnothing$. Hence $c_\mu(V) \cap i_\mu(W) = \varnothing$ and hence $A \subseteq V \subseteq c_\mu(V) \subseteq X \setminus i_\mu(W) \subseteq B$.

(iii)⇒(iv): Let A be rg-closed and B be a regular open set containing A. Then $clA \subseteq B$. The rest follows from (iii).

(iv)⇒(v): This follows from (iv) and the fact that a subset A is $g\mu$-open iff $F \subseteq i_\mu(A)$ whenever $F \subseteq A$ and F is closed [21].

(v)⇒(vi): Follows from (v) and the fact that every g-closed set is an rg-closed set [8].

(vi)⇒(vii): Trivial by Remark 2.2.

(vii)⇒(i): Let A be any closed set and B be a regular closed set such that $A \cap B = \varnothing$. Then $X \setminus B$ is a regular open set containing A where A is g-closed (as every closed set is g-closed [8]). So there exists a $g\mu$-open set G of X such that $clA \subseteq G \subseteq c_\mu(G) \subseteq X \setminus B$. Put $U = i_\mu(G)$ and $V = X \setminus c_\mu(G)$. Then U and V are two disjoint μ-open subsets of X such that $clA \subseteq U$ (as G is $g\mu$-open [21]), i.e., $A \subseteq U$ and $B \subseteq V$. Hence X is almost μ-normal. ■

Note 3.8. If in a topological space (X, τ) we take $\mu = \alpha O(X)$, then an almost μ-normal space reduces to an almost normal space.

Definition 3.9. Let μ be a GT on a topological space (X, τ). Then (X, τ) is said to be mildly μ-normal if for any two disjoint regular closed sets A and B there exist two disjoint μ-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Remark 3.10. Let μ be a GT on a space (X, τ). Then a mildly μ-normal space reduces to a mildly normal [29] (resp. mildly p-normal [17, 23], mildly δp-normal [8]) if one takes μ to be τ (resp. $PO(X)$, $\delta PO(X)$).

Theorem 3.11. *Let μ be a GT on a topological space (X, τ). Then the following are equivalent:*

(i) *X is mildly μ-normal.*

(ii) *For any disjoint $H, K \in RC(X)$, there exist $g\mu$-open sets U and V such that $H \subseteq U$ and $K \subseteq V$.*

(iii) *For $H, K \in RC(X)$ with $H \cap K = \varnothing$, there exist disjoint $r\mu g$-open sets U and V such that $H \subseteq U$ and $K \subseteq V$.*

(iv) *For any $H \in RC(X)$ and any $V \in RO(X)$ with $H \subseteq V$, there exists an $r\mu g$-open set U of X such that $H \subseteq U \subseteq c_\mu(U) \subseteq V$.*

(v) *For any $H \in RC(X)$ and any $V \in RO(X)$ with $H \subseteq V$, there exists a μ-open set U of X such that $H \subseteq U \subseteq c_\mu(U) \subseteq V$.*

Proof. (i)⇒(ii): Follows from Remark 2.2.

(ii)⇒(iii): Follows from Remark 2.2.

(iii)⇒(iv): Let $H \in RC(X)$ and $V \in RO(X)$ be such that $H \subseteq V$. Then by (iii) there exist disjoint $r\mu g$-open sets U and W such that $H \subseteq U$ and $X \setminus V \subseteq W$. Thus by Theorem 2.8, $X \setminus V \subseteq i_\mu(W)$ and $U \cap i_\mu(W) = \varnothing$. So $c_\mu(U) \cap i_\mu(W) = \varnothing$ and hence $H \subseteq U \subseteq c_\mu(U) \subseteq X \setminus i_\mu(W) \subseteq V$.

(iv)⇒(v): Let $H \in RC(X)$ and $V \in RO(X)$ be such that $H \subseteq V$. Thus by (iv) there exists an $r\mu g$-open set G of X such that $H \subseteq G \subseteq c_\mu(G) \subseteq V$. Since $H \in RC(X)$, by Theorem 2.8, $H \subseteq i_\mu(G) = U$ (say). Hence $U \in \mu$ and $H \subseteq U \subseteq c_\mu(U) \subseteq c_\mu(G) \subseteq V$.

(v)⇒(i): Let $H, K \in RC(X)$ be such that $H \cap K = \varnothing$. Then $X \setminus K \in RO(X)$ with $H \subseteq X \setminus K$. Thus by (v) there exists a μ-open set U of X such that $H \subseteq U \subseteq c_\mu(U) \subseteq X \setminus K$. Put $V = X \setminus c_\mu(U)$. Then U and V are disjoint μ-open sets such that $H \subseteq U$ and $K \subseteq V$. ∎

Note 3.12. If in a topological space (X, τ) we take $\mu = \alpha O(X)$, then a mildly μ-normal space reduces to a mildly normal space.

4. Preservation theorems

Definition 4.1. Let μ and λ be two GT's on two topological spaces (X, τ) and (Y, σ) respectively. A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be

(i) an R map [3] if $f^{-1}(V)$ is regular open in X for every regular open set V in Y.

(ii) rc-preserving [18] if $f(F)$ a is regular closed in Y for each regular closed F in X.

(iii) (μ, λ)-open [21] if $f(U)$ is λ-open in Y for each μ-open subset U of X.

(iv) μ-$r\mu g$-continuous if $f^{-1}(F)$ is $r\mu g$-closed in X for each λ-closed set F in Y.

(v) μ-$r\mu g$-closed if $f(F)$ is $r\lambda g$-closed in Y for each μ-closed set F of X.

Theorem 4.2. *Let μ and λ be two GT's on two topological spaces (X, τ) and (Y, σ) respectively. A surjective mapping $f : (X, \tau) \to (Y, \sigma)$ is μ-$r\mu g$-closed iff for each subset B of Y and each $U \in \mu$ containing $f^{-1}(B)$ there exists an $r\lambda g$-open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Proof. Suppose f is μ-$r\mu g$-closed, B be a subset of Y and U $(\in \mu)$ contains $f^{-1}(B)$. Put $V = Y \setminus f(X \setminus U)$. Then V is an $r\lambda g$-open set of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

Conversely, let F be a μ-closed set in X. Then $f^{-1}(Y\backslash f(F)) \subseteq X\backslash F \in \mu$. Thus there exists an $r\lambda g$-open set V of Y such that $Y \setminus f(F) \subseteq V$ and $f^{-1}(V) \subseteq X \setminus F$. Therefore, we have $f(F) \supseteq Y \setminus V$ and $F \subseteq f^{-1}(Y \setminus V)$. Hence we obtain that $f(F)$ is $r\lambda g$-closed in Y. Thus f is μ-$r\mu g$-closed. ■

Theorem 4.3. *Let μ and λ be two GT's on two topological spaces (X, τ) and (Y, σ) respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be a surjective (μ, λ)-open μ-$r\mu g$-closed R map. If X is almost μ-regular, then Y is almost λ-regular.*

Proof. Let $F \in RC(Y)$ and $y \in Y \setminus F$. Then $f^{-1}(y)$ and $f^{-1}(F)$ are disjoint. Since f is an R map, $f^{-1}(F)$ is regular closed in X. For each $x \in f^{-1}(y)$, there exist disjoint μ-open sets U and V of X such that $x \in U$ and $f^{-1}(F) \subseteq V$. Since f is (μ, λ)-open, we have $y = f(x) \in f(U) \in \lambda$. Since f is μ-$r\mu g$-closed, by Theorem 4.2, there exists an $r\lambda g$-open set W of Y such that $F \subseteq W$ and $f^{-1}(W) \subseteq V$. Since $f(U)$ and W are disjoint, by Theorem 3.3, we obtain Y is almost λ-regular. ■

Theorem 4.4. *Let μ and λ be two GT's on two topological spaces (X, τ) and (Y, σ) respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be a surjective μ-$r\mu g$-closed R-map. If X is a mildly μ-normal space, then Y is also a mildly λ-normal space.*

Proof. Let A and B be two disjoint regular closed sets in Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint regular closed sets in X. Since X is mildly μ-normal, there exist disjoint μ-open sets U and V of X such $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Then by Theorem 4.2, there exist $r\lambda g$-open sets K and L of Y such that $A \subseteq K$, $B \subseteq L$, $f^{-1}(K) \subseteq U$ and $f^{-1}(L) \subseteq V$. Since U and V are disjoint, so are K and L. By Theorem 2.8, then it follows that $A \subseteq i_\lambda(K)$, $B \subseteq i_\lambda(H)$ and $i_\lambda(K) \cap i_\lambda(H) = \varnothing$. This shows that Y is mildly λ-normal. ■

Theorem 4.5. *Let μ and λ be two GT's on two topological spaces (X, τ) and (Y, σ) respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be a μ-$r\mu g$-continuous rc-preserving injection. If Y is a mildly λ-normal space, then X is mildly μ-normal.*

Proof. Let A and B be any disjoint regular closed sets of X. Since f is an rc-preserving injection, $f(A)$ and $f(B)$ are disjoint regular closed sets of Y. By mildly normality of Y, there exist disjoint λ-open sets U and V of Y such that $f(A) \subseteq U$ and $f(B) \subseteq V$. Since f is μ-$r\mu g$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint $r\mu g$-open sets containing A and B respectively. Hence by Theorem 3.11, X is mildly μ-normal. ■

Conclusion: If we take the GT to be τ, $PO(X)$, $\alpha O(X)$, $\delta PO(X)$ suitably, then we get corresponding versions of the results as listed in the table below.

GT	τ	$PO(X)$	$\alpha O(X)$	$\delta PO(X)$	τ_ω
Th. 2.6	*Th. 3.6* [22]	*Th. 3.27* [9]		*Th. 24* [8]	*Th. 2.6* [1]
Th. 2.8	*Th. 4.2* [22]	*Th. 4.3* [9]	*Lem. 4.12* [20]	*Th. 15* [8]	*Th. 2.7* [1]
Th. 2.9	*Th. 3.10* [22]	*Th. 3.21* [9]		*Th. 29* [8]	*Th. 2.12* [1]
Th. 2.10		*Th. 4.6* [9]		*Th. 27* [8]	*Th. 2.13* [1]
Th. 2.13	*Th. 3.1(4)* [7]	*Th. 5.4* [9]		*Th. 40* [8]	*Th. 3.2* [1]
Th. 2.16	*Th. 4.4* [22]	*Th. 2.11* [23]		*Th. 26* [8]	
Th. 3.3	*Th. 4.8* [19]	*Th. 4.2* [19]	*Th. 5.2* [20]		
Th. 3.7		*Th. 3.2* [23]		*Th. 59* [8]	
Th. 3.11	*Th. 3.3* [18]	*Th. 3.4* [23]	*Th. 4.13* [20]	*Th. 60* [8]	
Th. 4.2	*Th. 4.17* [18]	*Th. 3.8* [19]	*Th. 3.8* [20]	*Lem. 62* [8]	
Th. 4.4		*Th. 5.9(b)* [23]	*Th. 4.17* [20]	*Th. 64* [8]	
Th. 4.5		*Th. 5.11(b)* [23]		*Th. 67* [8]	

References

[1] A. Al-Omari, M. S. Md Noorani, *Regular generalized ω-closed sets*, Internat. J. Math. Math. Sci. (2007), 1–11.

[2] Y. Beceren, T. Noiri, *Some functions defined by semi-open and β-open sets*, Chaos Solitons Fractals 36(5) (2008), 1225–1231.

[3] D. Carnahan, *Some properties related to compactness in topological spaces*, Ph.D thesis, University of Arkansas, 1973.

[4] Á. Császár, *Generalized topology, generalized continuity*, Acta Math. Hungar. 96 (2002), 351–357.

[5] Á. Császár, *Generalized open sets in generalized topologies*, Acta Math. Hungar. 106 (2005), 53–66.

[6] Á. Császár, *δ- and θ-modifications of generalized topologies*, Acta Math. Hungar. 120(3) (2008), 275–279.

[7] J. Dontchev, M. Ganster, T. Noiri, *Unified operation approach of generalized closed sets via topological ideals*, Math. Japon. 49 (1999), 395–401.

[8] E. Ekici, T. Noiri, *On a generalization of normal, almost normal and mildly normal spaces II*, Filomat 20(2) (2006), 67–80.

[9] Y. Gnanambal, *On generalized regular closed sets in topological spaces*, Indian J. Pure Appl. Math. 28(3) (1997), 351–360.

[10] H. Z. Hdeib, *ω-closed mappings*, Rev. Colombiana Mat. 16(1-2) (1982), 65–78.

[11] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly 70 (1963), 36–41.

[12] N. Levine, *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo 19(2) (1970), 89–96.

[13] H. Maki, J. Umehara, T. Noiri, *Every topological space is pre-$T_{1/2}$*, Mem. Fac. Sci. Kochi Univ. Ser. A Math. 17 (1996), 33–42.

[14] H. Maki, R. Devi, K. Balachandran, *Generalized α-closed sets in topology*, Bull. Fukuoka Univ. Ed. III 42 (1993), 13–21.

[15] S. R. Malghan, G. B. Navalagi, *Almost p-regular, p-completely regular and almost p-completely completely regular spaces*, Bull. Math. Soc. Sci. Math. R. S. Roumanie 34(82) (1990), 417–326.
[16] A. S. Mashhour, M. E. Abd El-Monsef, S. N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt 53 (1982), 47–53.
[17] G. B. Navalagi, *p-normal, almost p-normal and midly p-normal spaces*, Topology Atlas, (Preprint).
[18] T. Noiri, *Mildly normal spaces and some functions*, Kyungpook Math. J. 36 (1996), 183–190.
[19] T. Noiri, *Almost p-regular spaces and some functions*, Acta Math. Hungar. 79 (1998), 207–216.
[20] T. Noiri, *Almost αg-closed functions and separation axioms*, Acta Math. Hungar. 82(3) (1999), 193–205.
[21] T. Noiri, B. Roy, *Unification of generalized open sets on topological spaces*, Acta Math. Hungar. (accepted and to appear).
[22] N. Palaniappan, K. Chandrasekhara Rao, *Regular generalized closed sets*, Kyungpook Math. J. 33(2) (1993), 211–219.
[23] J. H. Park, *Almost p-normal, mildly p-normal spaces and some functions*, Chaos Solitons Fractals 18 (2003), 267–274.
[24] J. K. Park, J. H. Park, *Mildly generalized closed sets, almost normal and mildly normal spaces*, Chaos Solitons Fractals 20 (2004), 1103–1111.
[25] S. Raychaudhuri, M. N. Mukherjee, *On δ-almost continuity and δ-preopen sets*, Bull. Inst. Math. Acad. Sinica 21 (1993), 357–366.
[26] B. Roy, *On generalization of R_0 and R_1 spaces*, Acta Math. Hungar. 127 (2010), 291–300.
[27] M. K. Singal, S. P. Arya, *On almost-regular spaces*, Glasnik Mat. 4(24) (1969), 89–99.
[28] M. K. Singal, S. P. Arya, *Almost normal, almost completely regular spaces*, Glasnik Mat. 5(25) (1970), 141–152.
[29] M. K. Singal, A. R. Singal, *Mildly normal spaces*, Kyungpook Math. J. 13 (1973), 27–31.
[30] N. V. Veličko, *H-closed topological spaces*, Mat. Sb. 70 (1966), 98–112.

DEPARTMENT OF MATHEMATICS
WOMEN'S CHRISTIAN COLLEGE
6, Greek Church Row
KOLKATA-700 026
INDIA
E-mail: bishwambhar_roy@yahoo.co.in

Received June 12, 2010.